

Neighbourhood Theorems

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(ω_0 , M)

$$U \subset \overset{\circ}{T} \quad NT \hookrightarrow MT : T$$

(T, M) Moise's Trick: given ω_0 in $\Omega^1(M)$ s.t. ω_0 is exact on T . A

Q: Suppose there is a homotopy $\omega_t \in \Omega^1(W)$

$t \in [0, 1]$. Is there an isotopy $\phi_t : W \rightarrow W$ s.t.

$$\text{L}_{\dot{\phi}_t} \omega_t^* (\omega_0) = \omega_t, \forall t? \quad (\text{if } \omega_t \text{ exact})$$

Exact cobordism: $\partial M = \emptyset$ \Rightarrow ω_t is exact

A: Not always. It suffices to find a t -dependent

vec. field v_t s.t. $\phi_t = e^{tL_{v_t}}$

i.e. $\dot{\phi}_t(x) = v_t(\phi_t(x)), \forall x \in W$.

~~bleif weiter dran~~ \Rightarrow ω_t ist für $t \neq 0$ nicht exakt

Differentiate: we can write $\omega_t = \omega_0 + \alpha_t$ s.t. α_t has

Sympl. $\dot{\phi}_t^* \omega_0 = \omega_t$ \Rightarrow $\dot{\alpha}_t$ is exact

$$\Rightarrow L_{v_t} \omega_t = \omega_t \text{ from } R = \left(\frac{d}{dt} \right) T$$

$(Y \times \mathbb{R})$: is exact cobordism, then

Prop: Solutions to $L_{v_t} \omega_t = \dot{\alpha}_t$ for exact homotopy
of sympl. forms.

$$\text{Let } \omega_t = \omega_0 + d\alpha_t$$

be a homotopy of sympl. forms on W and

$$v_t = L_{\omega_t}^{-1} (\dot{\alpha}_t) \quad (\text{i.e., } L_{v_t} \omega_t = \dot{\alpha}_t)$$

$$\text{Then } L_{v_t} \omega_t = \dot{\alpha}_t$$

Given an exact cobordism, we can glue paths

$$\text{Pf: } L_{v_t} \omega_t = d(L_{v_t} \omega_t) = d(\dot{\alpha}_t) = \dot{\alpha}_t$$

Symplectic neighbourhood theorems

(W, ω) symp. mfd.

Prop (Thurston): If $v: X \rightarrow W$ is a sympl. vec. bundle, then \exists sympl. structure $\tilde{\omega}$ s.t. $\tilde{\omega}|_W = \omega$ and $\tilde{\omega}|_{\text{fibre}}$ coincides with the linear sympl. struc. near the 0 -section.

Pf: By def'n, \exists closed 2-form $\eta = \omega|_{\text{fibre}}$ s.t. $\eta|_{\text{fibre}}$ defines the linear sympl. structure.

Let $\tilde{\omega} = \eta + v^* \omega$. This is sympl. in a nbhd of 0 -section.

Weinstein's thm: Any isotropic immersion $(L \rightarrow (W, \omega))$

extends to an isosymplectic immersion $(J(TL) \rightarrow (T(W, \omega)))$ for J an

Pf: Suppose $\dim L = k$. \exists transverse isotropic plane field Θ s.t. $L \oplus \Theta$ is symplectic. Moreover, the space of such k -planes is contractible.

\Rightarrow the immersion $L \rightarrow W$ extends in a

homotopically unique way to an immersion $J(L) \rightarrow W$.

Then $v^* \omega = \omega_0$ on the 0 -sections of $T^* L \rightarrow L$

and $v^* \omega = \omega_0 + d\alpha$ where α vanishes on L .

Apply Moser's trick.

Symplectic nbhd thm: (from gauge theory A.)

Let $f: (V, \omega_V) \rightarrow (W, \omega_W)$ be an isosympl. immersion, and

$E \rightarrow V$ sympl. vector bundle with the fibre over every point $x \in V$ is $df_p(TV)^{\perp}_{\omega_W}$.

Then f extends to an isosymplectic immersion

$$(OpV, \omega_E) \rightarrow (W, \omega_0)$$

open nbhd of
0-section.

Pf: By definition, \exists an immersion $f: OpV \rightarrow W$
extending $\phi|_{OpV}$ s.t.

$$f^*\omega_W = \omega_E|_V$$

on the 0-section of $E \rightarrow V$ and s.t.

$$f^*\omega_W - \omega_E = d\alpha$$

Apply Moser's trick.

$$(M, \xi = \ker \alpha)$$

minimising isotropic part: metric invariant!

Contact Neighbourhood Thm: on a 3d contact

$(M, \xi = \ker \alpha)$: contact mfld. with respect to α

$d\alpha$ is a symplectic structure on ξ if

$$L \subset (M; \xi)$$
 isotropic submfld. i.e. $T_L \xi \subset (TL)^{\perp_{d\alpha}}$

in chart $W \hookrightarrow M$ we have $L \cap W = L \cap W$

Rank: The symplectic structure $d\alpha|_L$ does not

depend on the choice of α up to "conformality"

i.e. if $f\alpha$ is another contact form then

$$d(f\alpha)|_{\xi} = f d\alpha|_{\xi}.$$

Defn: A conformal sympl. normal bundle $CSN(L)$

$$:= (T_L)^+ / TL \text{ w.r.t. } (W, \omega) \hookrightarrow (V, \omega_V)$$

Splitting of the ^{normal} tangent bundle: \oplus

$$NL \oplus \xi^{\perp} \oplus CSN(L)$$

isomorphism: no of charts $\{$ refl

trivialised by Reeb vec. field.

Consequences of J-holomorphic spheres on a compact symplectic manifold

Thm: Let $(M_i, \xi_i)_{i=0,1}$ be two contact manifolds

with closed isotropic submfds L_i . Suppose there exists an isom. $\Phi: CSN(L_0) \rightarrow CSN(L_1)$ covering a diffeo $\Psi: L_0 \rightarrow L_1$. Then Φ extends to a contactomorphism

$$\begin{array}{ccc} \text{G = P} & \Psi: Op(L_0) \rightarrow Op(L_1) & \text{noncompact} \\ \text{is a local isom} & \uparrow & \uparrow \\ N(L_0) & N(L_1) & \end{array}$$

s.t. $d\Psi|_{CSN(L_0)}$ and Φ are bundle homotopies.

Cor: Diffeomorphic Legendrian submfds admit contactomorphic mfdhs.

Ex: $(M^3, \xi) \supset S^1$ Legendrian

$$\alpha = \cos\theta dx - \sin\theta dy$$

$\Rightarrow \alpha$ is smooth

but group S^1 is only homeo to itself up to ± 1

so α is not smoothly isotopic to α' for $\alpha' \in S^1$

so S^1 is not smoothly isotopic to itself

Finally: $C\mathbb{P} \rightarrow M$ has fibre S^1 (cf p22)

so there is a subspace C^1 converging on

S^1 in C^1 compact mfdhs

\Rightarrow $C^1 \hookrightarrow C^1$ is imprecise

Some results about J-hol curves

Removal of singularities

Given $u: B_2 \setminus 0 \rightarrow M$, $E(u) = 0$, then this extends to a smooth J-hol map $u: B_2 \rightarrow M$.