UNIQUENESS OF SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. We consider the stochastic differential equation

$$dx(t) = dW(t) + f(t, x(t))dt, \qquad x(0) = x_0$$

for $t \geq 0$, where $x(t) \in \mathbb{R}^d$, W is a standard d-dimensional Brownian motion, and f is a bounded Borel function from $[0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ to $\mathbb{R}^d$. We show that, for almost all Brownian paths $W(t)$, there is a unique $x(t)$ satisfying this equation.

1. Introduction

In this paper we consider the stochastic differential equation

$$dx(t) = dW(t) + f(t, x(t))dt, \qquad x(0) = x_0$$

for $t \geq 0$, where $x(t) \in \mathbb{R}^d$, W is a standard d-dimensional Brownian motion, and f is a bounded Borel function from $[0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ to $\mathbb{R}^d$. Without loss of generality we suppose $x_0 = 0$ and then we can write the equation as

$$x(t) = W(t) + \int_0^t f(s, x(s))ds, \qquad t \geq 0 \tag{1}$$

It follows from a theorem of Veretennikov [4] that (1) has a unique strong solution, i.e. there is a unique process $x(t)$, adapted to the filtration of the Brownian motion, satisfying (1). Veretennikov in fact proved this for a more general equation. Here we consider a different question, posed by N. V. Krylov [2]: we choose a Brownian path W and ask whether (1) has a unique solution for that particular path. The main result of this paper is the following affirmative answer:

Theorem 1.1. *For almost every Brownian path W, there is a unique continuous $x : [0, \infty) \to \mathbb{R}^d$ satisfying (1).*

This theorem can also be regarded as a uniqueness theorem for a random ODE: writing $x(t) = W(t) + u(t)$, the theorem states that for almost all choices of W, the differential equation $\frac{du}{dt} = f(t, W(t) + u(t))$ with $u(0) = 0$ has a unique solution.

In Section 4, we give an application of this theorem to convergence of numerical approximations to (1).

Idea of proof of theorem. The theorem is trivial when f is Lipschitz in x, and the idea of

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the proof is essentially to find some substitute for a Lipschitz condition. The proof splits into two parts, the first (section 2) being the derivation of an estimate which acts as a substitute for the Lipschitz condition, and the second (section 3) being the application of this estimate to prove the theorem. We start with a reduction to a slightly simpler problem.

A reduction. It will be convenient to suppose $|f(t,x)| \leq 1$ everywhere, which we can by scaling. Then it will suffice to prove uniqueness of a solution on [0,1], as we can then repeat to get uniqueness on [1,2] and so on.

So we work on [0,1], let X be the space of continuous functions $x : [0,1] \to \mathbb{R}^d$ with $x(0) = 0$, and let P_W be the law of $\mathbb{R}^d$-valued Brownian motion on [0,1], which can be regarded as a probability measure on X. Now we apply the Girsanov theorem (see [3]): define $\phi(x) = \exp\{\int_0^1 f(t,x(t))dx(t) - \frac{1}{2}\int_0^1 f(t,x(t))^2 dt\}$, which is well-defined for P_W almost all $x \in X$, and define a measure μ on X by $d\mu = \phi dP_W$. Then if $x \in X$ is chosen at random with law μ, the path $W \in X$ defined by

$$W(t) = x(t) - \int_0^t f(s,x(s))ds \tag{2}$$

is a Brownian motion, i.e. W has law P_W.

For a particular choice of x, and with W defined by (2), x will be the unique solution of (1) provided the only solution of

$$u(t) = \int_0^t \{f(s,x(s)+u(s)) - f(s,x(s))\}ds \tag{3}$$

in X is $u = 0$. So, to prove the theorem it suffices to show that, for μ-a.a. x, (3) has no non-trivial solution, since for such x, with W defined by (2) no other x can satisfy (2).

But μ is absolutely continuous w.r.t. P_W, so it suffices to show that, for P_W-a.a. x, (3) has no non-trivial solution. In other words, it suffices to show that, if W is a Brownian motion then with probability 1 there is no non-trivial solution $u \in X$ of

$$u(t) = \int_0^t \{f(s,W(s)+u(s)) - f(s,W(s))\}ds \tag{4}$$

We prove this in section 3.

Remark. Our proof does not make use of the existence of a strong solution. It is tempting to try to prove the theorem by measure-theoretic arguments based on the strong solution and Girsanov's theorem. Define $T : X \to X$ by

$$Tx(t) = x(t) - \int_0^t f(s,x(s))ds$$

The strong solution gives a measurable map $S : E \to F$ where E and F are Borel subsets of X with $P_W(E) = P_W(F) = 1$, such that $T \circ S$ is the identity on E, and F is the range of S. It follows that T is (1-1) on F and for any $W \in E$ there is a unique solution of (1) in F. But we need a solution which is unique in X and to achieve this we need to show that $T(X \backslash F)$ is a P_W-null set, and this seems to be a significant obstacle.

Our proof is quite complicated and it seems reasonable to hope that it can be simplified. In particularly one might expect a simpler proof of Proposition 2.2. This seems to be nontrivial even for $p = 2$. The bound for $p = 2$ follows from the first part of Lemma 2.5 (with $t_0 = 0$ and $r = 0$) and I do not know an essentially simpler proof.

In one dimension, in the case when $f(t, x)$ depends only on x, a different and shorter proof of Theorem 1.1 can be given, using local time, but it is not clear how to extend it to $d > 1$.

2. The basic estimate

This section is devoted to the proof of the following:

Proposition 2.1. *Let g be a Borel function on $[0,1] \times \mathbb{R}^d$ with $|g(s,z)| \leq 1$ everywhere. For any even positive integer p and $x \in \mathbb{R}^d$, we have*

$$\mathbb{E}\left(\int_0^1 \{g(t, W(t) + x) - g(t, W(t))\}dt\right)^p \leq C^p (p/2)!|x|^p$$

where C is an absolute constant, $|x|$ denotes the usual Euclidean norm and $W(t)$ is a standard d-dimensional Brownian motion with $W(0) = 0$,

This will be deduced from the following one-dimensional version:

Proposition 2.2. *Let g be a compactly supported smooth function on $[0,1] \times \mathbb{R}$ with $|g(s,z)| \leq 1$ everywhere and g' bounded (where the prime denotes differentiation w.r.t. the second variable). For any even positive integer p, we have*

$$\mathbb{E}\left(\int_0^1 g'(t, W(t))dt\right)^p \leq C^p (p/2)!$$

where C is an absolute constant, and here $W(t)$ is one-dimensional Brownian motion with $W(0) = 0$.

Proof. We start by observing that the LHS can be written as

$$p! \int_{0<t_1<\cdots<t_p<1} \mathbb{E} \prod_{j=1}^p g'(t_j, W(t_j)) dt_1 \cdots dt_p$$

and using the joint distribution of $W(t_1), \cdots, W(t_p)$ this can be expressed as

$$p! \int_{0<t_1<\cdots<t_p<1} \int_{\mathbb{R}^p} \prod_{j=1}^p \{g'(t_j, z_j) E(t_j - t_{j-1}, z_j - z_{j-1})\} dz_1 \cdots dz_p dt_1 \cdots dt_p$$

where $E(t,z) = (2\pi t)^{-1/2} e^{-z^2/2t}$ and here $t_0 = 0$, $z_0 = 0$.

We introduce the notation

$$J_k(t_0, z_0) = \int_{t_0<t_1<\cdots<t_k<1} \int_{\mathbb{R}^k} \prod_{j=1}^k \{g'(t_j, z_j) E(t_j - t_{j-1}, z_j - z_{j-1})\} dz_1 \cdots dz_k dt_1 \cdots dt_k$$

and we shall show that $J_p(0,0) \leq C^p/\Gamma(\frac{p}{2}+1)$; Proposition 2.2 will then follow since $p! \leq 2^p((p/2)!)^2$.

In order to estimate J_k we use integration by parts to shift the derivatives to the exponential terms. We introduce some notation to handle the resulting terms - we define $B(t,z) = E'(t,z)$ and $D(t,z) = E''(t,z)$ (where again primes denote differentiation w.r.t. the second variable).

If $S = S_1 \cdots S_k$ is a word in the alphabet $\{E, B, D\}$ then we define

$$I_S(t_0, z_0) = \int_{t_0 < t_1 < \cdots < t_k < 1} \int_{\mathbb{R}^d} \prod_{j=1}^{k} \{g(t_j, z_j) S_j(t_j - t_{j-1}, z_j - z_{j-1})\} dz_1 \cdots dz_k dt_1 \cdots dt_k$$

In fact, only certain words in $\{E, B, D\}$ will be required: we say a word is *allowed* if, when all B's are removed from the word, a word of the form $(ED)^r = EDED \cdots ED$, $r \geq 0$, is left. The allowed words of length k correspond to the subsets of $\{1, 2, \cdots, k\}$ having an even number of members (namely the set of positions occupied by E and D in the word). Hence the number of allowed words of length k is the number of such subsets of $\{1, 2, \cdots, k\}$, namely 2^{k-1}.

We shall show that

$$J_k(t_0, z_0) = \sum_{j=1}^{2^{k-1}} \pm I_{S^{(j)}}(t_0, z_0) \tag{5}$$

where each $S^{(j)}$ is an allowed word of length k (in fact each allowed word of length k appears exactly once in this sum, but we do not need this fact). The proof will then be completed by obtaining a bound for I_S.

We prove (5) by induction on k. So, assuming (5) for J_k, we have

$$\begin{aligned} J_{k+1}(t_0, z_0) &= \int_{t_0}^{1} dt_1 \int g'(t_1, z_1) E(t_1 - t_0, z_1 - z_0) J_k(t_1, z_1) dz_1 \\ &= -\int_{t_0}^{1} dt_1 \int g(t_1, z_1) B(t_1 - t_0, z_1 - z_0) J_k(t_1, z_1) dz_1 \\ &\quad - \int_{t_0}^{1} \int g(t_1, z_1) E(t_1 - t_0, z_1 - z_0) J_k'(t_1, z_1) dz_1 \end{aligned}$$

Now we observe that, if S is an allowed string then $I_S' = -I_{\tilde{S}}$ where $\tilde{S}$ is defined as BS^* if $S = ES^*$ and as DS^* if $S = BS^*$ (note that $\tilde{S}$ is not an allowed string). Applying this to (5) we find $J_k'(t_0, z_0) = \sum_{j=1}^{2^{k-1}-1} \mp I_{\tilde{S}^j}(t_0, z_0)$ and then we obtain

$$J_{k+1}(t_0, z_0) = \mp \sum_{j=1}^{2^{k-1}-1} I_{BS^j}(t_0, z_0) \pm \sum_{j=1}^{2^{k-1}-1} I_{E\tilde{S}^j}(t_0, z_0)$$

Noting that, if S is an allowed string, BS and $E\tilde{S}$ are also allowed, this completes the inductive proof of (5).

We now proceed to the estimation of $I_S(t_0, z_0)$, when S is an allowed string. We start with some preliminary lemmas.

Lemma 2.3. *There is a constant C such that, if ϕ and h are real-valued Borel functions on $[0,1]\times\mathbb{R}$ with $|\phi(t,y)| \le e^{-y^2/3t}$ and $|h(t,y)| \le 1$ everywhere, then*

$$\left|\int_{1/2}^{1} dt \int_{t/2}^{t} ds \int_{\mathbb{R}}\int_{\mathbb{R}} \phi(s,z)h(t,y)D(t-s,y-z)dydz\right| \le C$$

Proof. Denote the above integral by I. For $l \in \mathbb{Z}$, let χ_l be the characteristic function of the interval $[l, l+1)$ and define $\phi_l(s,y) = \phi(s,y)\chi_l(y)$, and similarly h_l. Let I_{lm} denote the integral I with ϕ, h replaced by ϕ_l, h_m. Then we have $I = \sum_{l,m\in\mathbb{Z}} I_{lm}$. Let $C_1, C_2, \cdots$ denote positive absolute constants.

Now if $|l-m| = k \ge 2$ then for $z \in [l, l+1)$ and $y \in [m, m+1)$ we have $|z-y| \ge k-1$ and then it follows easily that

$$|D(t-s, y-z)| \le C_1 e^{-(k-2)^2/4}$$

and hence $I_{lm} \le C_2 e^{-l^2/8} e^{-(k-2)^2/4}$ from which we deduce

$$\sum_{|l-m|\ge 2} |I_{lm}| \le C_3$$

Now suppose $|l-m| \le 1$. We use $\hat{\phi}_l(s,u)$ for the Fourier transform in the second variable, and similarly $\hat{h}_m$. We note that $\int \hat{\phi}_l(s,u)^2 du = \int \phi_l(s,z)^2 dz \le C_4 e^{-|l|^2/6}$ for $0 \le s \le 1$ and similarly $\int \hat{h}_m(t,u)^2 du \le 1$. We have

$$I_{lm} = \int_{1/2}^{1} dt \int_{t/2}^{t} ds \int_{\mathbb{R}} \hat{\phi}_l(s,u)\hat{h}_m(t,-u)e^{-(t-s)|u|^2/2}u^2 du$$

Applying $ab \le \frac{1}{2}(a^2c + b^2c^{-1}$ with $a = \hat{\phi}_l(s,u)$, $b = \hat{h}_m(t,-u)$ and $c = e^{l^2/12}$, we deduce that

$$\begin{aligned}|I_{lm}| \le& \int_{1/2}^{1} dt \int_{t/2}^{t} ds \int_{\mathbb{R}} \hat{\phi}_l(s,u)^2 e^{l^2/12} u^2 e^{-(t-s)u^2/2} du \\ &+ \int_{1/2}^{1} dt \int_{t/2}^{t} ds \int_{\mathbb{R}} \hat{h}_m(-t,u)^2 e^{-l^2/12} u^2 e^{-(t-s)u^2/2} du\end{aligned}$$

In the first integral we integrate first w.r.t. t and obtain the bound const.$e^{-l^2/12}$ for the integral. We get a similar bound for the second integral (integrating w.r.t. s first), and hence

$$|I_{lm}| \le C_5 e^{-l^2/12}$$

Summing over l and m such that $|l-m| \le 1$, we obtain

$$\sum_{|l-m|\le 1} |I_{lm}| \le C_6$$

which completes the proof. □

Corollary 2.4. *There is an absolute constant C such that if g and h are Borel functions on $[0,1]\times\mathbb{R}$ bounded by 1 everywhere then*

$$\left|\int_{1/2}^1 dt\int_{t/2}^t ds\int_{\mathbb{R}^2} g(s,z)E(s,z)h(t,y)D(t-s,y-z)dydz\right|\le C$$

and

$$\left|\int_{1/2}^1 dt\int_{t/2}^t ds\int_{\mathbb{R}^2} g(s,z)B(s,z)h(t,y)D(t-s,y-z)dydz\right|\le C$$

Proof. These follow easily from Lemma (2.3), the second using the easily verified fact that $|B(s,z)|\le Cs^{-1/2}(e^{-z^2/3s})$. □

We note that $\int_{\mathbb{R}} E(t,z)dz=1$, and we have the bounds

$$\int_{\mathbb{R}}|B(t,z)|dz\le C_0t^{-1/2},\qquad \int_{\mathbb{R}}|D(t,z)|dz\le C_0t^{-1} \tag{6}$$

where C_0 is an absolute constant.

Lemma 2.5. *There is an absolute constant C such that if g and h are Borel functions on $[0,1]\times\mathbb{R}$ bounded by 1 everywhere, and $r\ge 0$ then*

$$\left|\int_{t_0}^1 dt\int_{t_0}^t ds\int_{\mathbb{R}^2} g(s,z)E(s-t_0,z)h(t,y)D(t-s,y-z)(1-t)^rdydz\right|\le C(1+r)^{-1}(1-t_0)^{r+1}$$

and

$$\left|\int_{t_0}^1 dt\int_{t_0}^t ds\int_{\mathbb{R}^2} g(s,z)B(s-t_0,z)h(t,y)D(t-s,y-z)(1-t)^rdydz\right|\le C(1+r)^{-1/2}(1-t_0)^{r+\frac{1}{2}}$$

Proof. Again, we let $C_1,\cdots$ be absolute constants. By using the change of variables $t'=(t-t_0)/(1-t_0)$, $s'=(s-t_0)/(1-t_0)$, $y'=y(1-t_0)^{-1/2}$, it suffices to prove these estimates when $t_0=0$. To do this, we start by scaling the first part of Corollary 2.4, and get

$$\left|\int_{2^{-k-1}}^{2^{-k}} dt\int_{t/2}^t ds\int_{\mathbb{R}^2} g(s,z)E(s,z)h(t,y)D(t-s,y-z)(1-t)^rdydz\right|\le C_1(1-2^{-k-1})^r2^{-k}$$

for $k=0,1,2\cdots$ and then by summing over k, we get

$$\left|\int_0^1 dt\int_{t/2}^t ds\int_{\mathbb{R}^2} g(s,z)A(s,z)h(t,y)D(t-s,y-z)(1-t)^rdydz\right|\le C_2(1+r)^{-1}$$

Moreover, from the bounds (6) we have

$$\left|\int_0^1 dt\int_0^{t/2} ds\int_{\mathbb{R}^2} g(s,z)E(s,z)h(t,y)D(t-s,y-z)(1-t)^rdydz\right|\le$$

$$\le C_3\int_0^1 dt\int_0^{t/2}(t-s)^{-1}(1-t)^rds\le C_4(1+r)^{-1}$$

and combining these bounds gives the first result. Similarly, by scaling the second part of Corollary 2.4, we get

$$\left|\int_{2^{-k-1}}^{2^{-k}} dt \int_{t/2}^{t} ds \int_{\mathbb{R}^2} g(s,z)B(s,z)h(t,y)D(t-s,y-z)(1-t)^r dydz\right| \le C_5(1-2^{-k-1})^r 2^{-k/2}$$

for $k=0,1,2\cdots$ and then by summing over k, we get

$$\left|\int_{0}^{1} dt \int_{t/2}^{t} ds \int_{\mathbb{R}^2} g(s,z)B(s,z)h(t,y)D(t-s,y-z)(1-t)^r dydz\right| \le C_6(1+r)^{-1/2}$$

Moreover, from the bounds (6) we have

$$\left|\int_{0}^{1} dt \int_{0}^{t/2} ds \int_{\mathbb{R}^2} g(s,z)B(s,z)h(t,y)D(t-s,y-z)(1-t)^r dydz\right| \le$$
$$\le C_0 \int_0^1 dt \int_0^{t/2} (t-s)^{-1}(1-t)^r ds \le C_7(1+r)^{-1/2}$$

which give the second result. □

We can now complete the proof of Proposition 2.2 by obtaining the required bound for $I_S(t_0,z_0)$. Again we use $C_1, C_2, \cdots$ for absolute constants. We shall show that, for a suitable choice of M, we have for any allowed string S of length k

$$|I_S(t_0,z_0)| \le \frac{M^k}{\Gamma(\frac{k}{2}+1)}(1-t_0)^{k/2} \tag{7}$$

We shall prove (7) by induction on k, provided M is chosen large enough. The case $k=0$ is immediate, so assume $k>0$ and that (7) holds for all allowed strings of length less than k. Then there are three cases: (1) $S=BS'$ where S' has length $k-1$; (2) $S=EDS'$ where S' has length $k-2$; (3) $S=EB^mDS'$ where $m\ge 1$ and S' has length $k-m-2$. In each case S' is an allowed string. We consider the three cases separately.

Case 1. In this case we have

$$\begin{aligned}|I_S(t_0,z_0)| &= \left|\int_{t_0}^1 dt_1 \int_{\mathbb{R}} B(t_1-t_0,z_1-z_0)g(t_1,z_1)I_{S'}(t_1,z_1)dz_1\right| \\ &\le \frac{M^{k-1}}{\Gamma(\frac{k+1}{2})}\int_{t_0}^1 (1-t_1)^{(k-1)/2}dt_1 \int_{\mathbb{R}} |B(t_1-t_0,z_1-z_0)|dz_1 \\ &\le \frac{C_1M^{k-1}}{\Gamma(\frac{k+1}{2})}\int_{t_0}^1 (1-t_1)^{(k-1)/2}(t_1-t_0)^{-1/2}dt_1 \\ &= C_1\sqrt{\pi}M^{k-1}(1-t_0)^{k/2}/\Gamma\left(\frac{k}{2}+1\right)\end{aligned}$$

where we have used the inductive hypothesis to bound $I_{S'}$, and then the bound (6). (7) then follows if M is large enough.

Case 2. Now we have

$$I_S(t_0,z_0)=\int_{t_0}^1 dt_1\int_{t_1}^1 dt_2\int_{\mathbb{R}^2} g(t_1,z_1)g(t_2,z_2)E(t_1-t_0,z_1-z_0)D(t_2-t_1,z_2-z_1)I_{S'}(t_2,z_2)dz_1dz_2$$

We set $h(t,z)=g(t,z)I_{S'}(t,z)(1-t)^{1-\frac{k}{2}}$ so that $\|h\|_\infty\le M^{k-2}/\Gamma(k/2)$ by the inductive hypothesis, and then from the first part of Lemma 2.5 we deduce that

$$|I_S(t_0,z_0)|\le\frac{C_2M^{k-2}(1-t_0)^{k/2}}{k\Gamma(k/2)}$$

and (7) follows if M is large enough.

Case 3. In this case have

$$I_S(t_0,z_0)=\int_{t_0<t_1<\cdots<t_{m+2}<1} dt_1\cdots dt_{m+2}\int_{\mathbb{R}^{m+2}}\left(\prod_{j=1}^{m+2} g(t_j,z_j)\right)E(t_1-t_0,z_1-z_0)\times$$
$$\times\prod_{j=2}^{m+1} B(t_j-t_{j-1},z_j-z_{j-1})D(t_{m+2}-t_{m+1},z_{m+2}-z_{m+1})I_{S'}(t_{m+2},z_{m+2})dz_1\cdots dz_{m+2}$$

Now let $h(t,z)=g(t,z)I_{S'}(t,z)(1-t_{m+2})^{(2+m-k)/2}$, so that by the inductive hypothesis on S' we have $\|h\|_\infty\le M^{k-m-2}/\Gamma(\frac{k-m}{2})$. Then, writing

$$\Omega(t,z)=\int_t^1 dt_{m+1}\int_{t_{m+1}}^1 dt_{M+2}\int_{\mathbb{R}^2} g(t_{m+1},z_{m+1})h(t_{m+2},z_{m+2})(1-t_{m+2})^{(k-m-2)/2}\times$$
$$\times B(t_{m+1}-t,z_{m+1}-z)D(t_{m+2}-t_{m+1},z_{m+2}-z_{m+1})dz_{m+1}dz_{m+2}$$

we find from Lemma 2 that

$$|\Omega(t,z)|\le C_3(k-m)^{-1/2}M^{k-m-2}(1-t)^{(k-m-1)/2}/\Gamma\left(\frac{k-m}{2}\right)$$

Using this in

$$I_S(t_0,z_0)=\int_{t_0<t_1<\cdots<t_m<1} dt_1\cdots dt_m\int_{\mathbb{R}^m}\left(\prod_{j=1}^{m} g(t_j,z_j)\right)E(t_1-t_0,z_1-z_0)\times$$
$$\times\prod_{j=2}^{m} B(t_j-t_{j-1},z_j-z_{j-1})\Omega(t_m,z_m)dz_1\cdots dz_m$$

and using the bounds (6) we find

$$|I_S(t_0,z_0)|\le C_4^{m+1}(k-m)^{-1/2}\frac{M^{k-m-2}}{\Gamma(\frac{k-m}{2})}\int_{t_0<t_1<\cdots<t_m<1}(t_2-t_1)^{-1/2}\cdots$$
$$\cdots(t_m-t_{m-1})^{-1/2}(1-t_m)^{(k-m-1)/2}dt_1\cdots dt_m$$
$$=C_4^{m+1}(k-m)^{-1/2}\frac{M^{k-m-2}\pi^{(m-1)/2}\Gamma(\frac{k-m+1}{2})}{\Gamma(\frac{k-m}{2})\Gamma(\frac{k}{2}+1)}(1-t_0)^{k/2}$$

from which again (7) follows, provided M is large enough. Putting (7) with $t_0 = 0$, $z_0 = 0$ and $k = p$ in (5) completes the proof of Proposition 2.2. $\square$

Proof of Proposition 2.1. We first note that it suffices to prove it for $d = 1$. To see this let g, W, x be as in the statement of Proposition 2.1. By a rotation of coordinates we can suppose $x = (\alpha, 0, \cdots, 0)$. Then for fixed Brownian paths $W_2, \cdots, W_d$ we can define h on $[0,1] \times \mathbb{R}$ by $h(t,u) = g(t, u, W_2(t), \cdots, W_d(t))$ and the $d = 1$ case of the Proposition gives

$$\mathbb{E}\left(\int_0^1 \{h(t, W_1(t) + \alpha) - h(t, W_1(t))\}dt\right)^p \leq C^p (p/2)! |\alpha|^p$$

and then the required result follows by averaging over $W_2, \cdots, W_d$.

So we suppose $d = 1$. Given a Borel function g on $[0,1] \times \mathbb{R}$ with $|g| \leq 1$ we can find a sequence of compactly supprted smooth functions g_n with $|g_n| \leq 1$, converging to g a.e. on $[0,1] \times \mathbb{R}$. Then $g_n(t, W(t)) \to g(t, W(t))$ a.s. for a.a. $t \in [0,1]$, and the same for $g_n(t, W(t) + x)$, so by Fatou's lemma it suffices to prove the proposition for smooth g. But then we have $g(t, W(t)+x) - g(t, W(t)) = \int_0^x g'(t, W(t)+u)du$ and we can apply Proposition 2.2 and Minkowsi's inequality to conclude the proof of Proposition 2.1.

What we in fact need is a scaled version of Proposition 2.1 for subintervals of [0,1]. For $s \geq 0$ we denote by $\mathcal{F}_s$ the σ-field generated by $\{W(\tau) : 0 < \tau < s\}$. Then we can state the required result:

Corollary 2.6. *Let g be a Borel function on $[0,1] \times \mathbb{R}^d$ with $|g| \leq 1$ everywhere. Let $0 \leq s \leq a < b \leq 1$ and let $\rho(x) = \int_a^b \{g(W(t) + x) - g(W(t))\}dt$. Then for $x \in \mathbb{R}^d$ and $\lambda > 0$ we have*

$$\mathbb{P}(|\rho(x)| \geq \lambda l^{1/2} |x| \; |\mathcal{F}_s) \leq 2e^{-\lambda^2/(2C^2)}$$

where $l = b - a$ and C is the constant in Proposition 2.1.

Proof. First assume $s = a = 0$, $b = 1$. Let $\alpha = (2C^2|x|^2)^{-1}$. Then

$$\mathbb{E}(e^{\alpha\rho(x)^2}) = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} \mathbb{E}(\rho(x)^{2k}) \leq \sum_{k=0}^{\infty} \alpha^k C^{2k} |x|^{2k} = 2$$

and so $\mathbb{P}(|\rho(x)| \geq \lambda|x|) = \mathbb{P}(e^{\alpha\rho(x)^2} \geq e^{\alpha\lambda^2|x|^2}) \leq 2e^{-\alpha\lambda^2|x|^2} = 2e^{-\lambda^2/(2C^2)}$.

For the general case, let $\tilde{W}(t) = l^{-1/2}\{W(a+tl) - W(a)\}$, so that $\tilde{W}$ is a standard Brownian motion, and let $h(x) = g(W(a) + x)$. Then $\rho(x) = l^{1/2} \int_0^1 \{h(\tilde{W}(t) + x) - h(\tilde{W}(t))\}dt$ and it follows from the first part that $\mathbb{P}(|\rho(x)| \geq \lambda l^{1/2}|x| \; |\mathcal{F}_a) \leq 2e^{-\lambda^2/(2C^2)}$. The required result then follows by taking conditional expectation w.r.t. $\mathcal{F}_s$. $\square$

We note that the unconditional bound

$$\mathbb{P}(|\rho(x)| \geq \lambda l^{1/2}|x|) \leq 2e^{-\lambda^2/(2C^2)}$$

follows by taking $s = 0$. Also in the same way we obtain, for any even $p \in \mathbb{N}$,

$$\mathbb{E}(\rho(x)^p | \mathcal{F}_s) \leq C^p l^{p/2} (p/2)! |x|^p \tag{8}$$

The following lemma will also be needed:

Lemma 2.7. *If $p > 1 + \frac{d}{2}$ there is a constant $c(p,d)$ such that is $g \in L^p([0,1] \times \mathbb{R}^d)$ then*

$$\mathbb{E}\left(\int_0^1 g(t, W(t))dt\right)^2 \le c(p,d)\|g\|_p^2$$

Proof. We have

$$\mathbb{E}\left(\int_0^1 g(t, W(t))dt\right)^2 = 2\int_0^1 dt \int_0^t ds \int_{\mathbb{R}^{2d}} g(s,\zeta)g(t,z)E(s,\zeta)E(t-s, z-\zeta)d\zeta dz$$

Now, if $q = \frac{p}{p-1}$ then $\int E(t,z)^q dz = O(t^{-(q-1)d/2})$ and $p > 1 + \frac{d}{2}$ implies $(q-1)d/2 < 1$, so the result follows from Hölder's inequality. ☐

3. Proof of Theorem

We now apply Corollary 2.6 and Lemma 2.7 to the proof of the theorem. First we give a brief sketch of the proof.

Outline of proof. The proof is motivated by the elementary case when f is Lipschitz in the second variable. In this case, if $I = [a,b]$ is a subinterval of [0,1] and u is a solution of (4) satisfying

$$|u(t)| \le \alpha, \qquad t \in I \tag{9}$$

and $\beta = |u(a)|$, then we deduce from (9) that $|u(t)| \le \alpha' = \beta + L|I|\alpha$ for $t \in I$, where L is the Lipschitz constant, i.e. (9) holds with α replaced by α'. If $L|I| < 1$ it follows that (9) holds with $\alpha = (1 - L|I|)^{-1}\beta$, and of course if $\beta = 0$ this gives $u = 0$ on I.

We try to copy this argument using Corollary 2.6 as a substitute for a Lipschitz condition. There are two difficulties: first, Corollary 2.6 is a statement about probabilities and we need an 'almost sure' version, and in doing so we lose something; second, in Corollary 2.6, x is a constant, whereas we are dealing with a function u depending on t. The way round the second problem is to approximate u by a sequence of step functions u_l and then use

$$\begin{aligned}(10)\quad &\int_I \{f(W(t) + u(t)) - f(W(t))\}dt = \lim_{l\to\infty} \int_I \{f(W(t) + u_l(t)) - f(W(t))\}dt \\ &= \int_I \{f(W(t) + u_n(t)) - f(W(t))\}dt + \sum_{l=n}^{\infty} \int_I \{f(W(t) + u_{l+1}(t)) - f(W(t) + u_l(t))\}dt\end{aligned}$$

where u_n is constant on the interval I, and then to apply the 'almost sure' form of the proposition to each interval of constancy of the terms on the right. Again, we lose something in doing this, but, as it turns out, we still have good enough estimates to prove the theorem. In fact, we need two versions of the 'almost sure' (nearly) Lipschitz condition, the first to estimate $\int\{f(W(t) + u_n(t)) - f(W(t))\}dt$ and the second to estimate $\int\{f(W(t) + u_{l+1}(t)) - f(W(t) + u_l(t))\}dt$. We also need a third estimate, for sums of integrals of the second type.

The two versions of the 'almost sure' nearly-Lipschitz condition are conditions (11) and (12) below, and the third estimate is (20). In Lemmas 3.1, 3.2, 3.5 and 3.6 it is shown that these conditions indeed hold almost surely. Lemmas 3.3 and 3.4 establish a technical condition (15) needed to justify the passage to the limit as $l \to \infty$ (which is not trivial when f is not continuous). With these preliminaries the above programme is carried out in Lemma 3.7. The analogue of (9) above is (25). We no longer immediately get $\alpha = 0$ when $\beta = 0$, but we get a good enough bound to prove the uniqueness of the solution to (1), for any W satisfying (11,12,15,20).

We now turn to the details.

For any $n \geq 0$ we can divide [0,1] into 2^n intervals $I_{nk} = [k2^{-n}, (k+1)2^{-n}]$, $k = 0, 1, 2, \cdots, 2^n - 1$. We shall also consider dyadic decompositions of $\mathbb{R}^d$, and say $x \in \mathbb{R}^d$ is a *dyadic* point if each component of x is rational with denominator a power of 2. Let $Q = \{x \in \mathbb{R}^d : \|x\| \leq 1\}$, where $\|x\|$ denotes the supremum norm $\max_{1 \leq j \leq d} |x_j|$. We also introduce the notation

$$\sigma_{nk}(x) = \int_{I_{nk}} \{g(W(t) + x) - g(W(t))\} dt$$

and

$$\rho_{nk}(x, y) = \sigma_{nk}(x) - \sigma_{nk}(y) = \int_{I_{nk}} \{g(W(t) + x) - g(W(t) + y)\} dt$$

Then we can state:

Lemma 3.1. *Let g be a real function on $[0,1] \times \mathbb{R}^d$ with $|g(t,z)| \leq 1$ everywhere. Then with probability 1 we can find $C > 0$ so that*

$$|\rho_{nk}(x, y)| \leq C \left\{ n^{1/2} + \left(\log^+ \frac{1}{|x - y|} \right)^{1/2} \right\} 2^{-n/2} |x - y| \tag{11}$$

for all dyadic $x, y \in Q$ and all choices of integers n, k with $n > 0$ and $0 \leq k \leq 2^n - 1$.

Proof. Let us say that two dyadic points $x, y \in \mathbb{R}^d$ are *dyadic neighbours* if for some integer $m \geq 0$ we have $\|x - y\| = 2^{-m}$ and $2^{-m}x, 2^{-m}y \in \mathbb{Z}^d$. Then using the Corollary 2.6 we have, for any such pair $x, y \in Q$ and any n, k that

$$\mathbb{P}\left(|\rho_{nk}(x, y)| \geq \lambda (n^{1/2} + m^{1/2}) 2^{-m-n/2} \right) \leq C_1 e^{-C_2 \lambda^2 (n+m)}$$

and by summing over all possible choices of n, k, m, x, y we find that the probability that

$$|\rho_{nk}(x, y)| \geq \lambda (n^{1/2} + m^{1/2}) 2^{-m-n/2}$$

for some choice of I_{nk} and dyadic neighbours $x, y \in Q$ is not more than $\sum_{n=1}^{\infty} \sum_{m=0}^{\infty} 2^n 3^d 2^{d(m+3)} C_1 e^{-C_2 \lambda^2 (1+m+n)}$ which approaches 0 as $\lambda \to \infty$.

It follows that, given $\epsilon > 0$, we can find $\lambda(\epsilon)$ such that, with probability $> 1 - \epsilon$, we have

$$|\rho_{nk}(x, y)| < \lambda (1 + n^{1/2} + m^{1/2}) 2^{-m-n/2}$$

for all choices of n, k and dyadic neighbours in Q.

Next let x, y be any two dyadic points in Q, and let m be the smallest non-negative integer such that $\|x - y\| < 2^{-m}$. For $r \geq m$, choose x_r to minimise $\|x - x_r\|$ subject to $2^r x_r \in \mathbb{Z}^d$, and y_r similarly. Then $\|x_m - y_m\| = 2^{-m}$ or 0, and for $r \geq m$, $\|x_r - x_{r+1}\| = 2^{-r-1}$ or 0. So x_m, y_m are dyadic neighbours or equal, and the same applies to x_r, x_{r+1} and y_r, y_{r+1}. Then we have

$$\rho_{nk}(x,y) = \rho_{nk}(x_m, y_m) + \sum_{r=m}^{\infty} \rho_{nk}(x_r, x_m) + \sum_{r=m}^{\infty} \rho_{nk}(y_m, y_r)$$

(note that the sums are actually finite, since x, y are dyadic, so that $x = x_r$ and $y = y_r$ for large r). Then applying the above bounds for the case of dyadic neighbours to each term, we get the desired result. $\square$

Next we prove a similar estimate for σ_{nk}, which is analogous to the Law of the Iterated Logarithm for Brownian motion.

Lemma 3.2. *With probability 1 there is a constant $C > 0$ such that for all $n \in \mathbb{N}$, $k \in \{0, 1, \cdots, 2^n - 1\}$ and dyadic $x \in Q$ we have*

$$|\sigma_{nk}(x)| \leq C n^{1/2} 2^{-n/2} (|x| + 2^{-2^n}) \tag{12}$$

Proof. For any integer $r \geq 0$ we let $Q_r = \{x \in \mathbb{R}^d : \|x\| \leq 2^{-r}\}$. Then if $m \geq r$ the number of pairs (x, y) of dyadic neighbours in Q_r with $\|x - y\| = 2^{-m}$ is $\leq (9 \times 2^{m-r})^d$ and for each such pair we have

$$\mathbb{P}(|\rho_{nk}(x,y)| \geq \lambda(n^{1/2} + \sqrt{m-r}) 2^{-m-n/2}) \leq C_1 e^{-C_2 \lambda^2 (n+m-r)} \leq C_1 2^{2d(r-m)} e^{-C_2 \lambda^2} e^{-n}$$

for λ large. By summing over n, $1 \leq r \leq 2^n$ and $m \geq r$ and all pairs (x, y), we deduce that, with probability $\geq 1 - C_3 e^{-C_4 \lambda^2}$, we have $\rho_{nk}(x, y) \leq \lambda(n^{1/2} + \sqrt{m-r}) 2^{-r-n/2}$ for $n \in \mathbb{N}$, $1 \leq r \leq n$ and $m \geq r$ and all pairs (x, y) of dyadic neighbours in Q_r with $\|x - y\| = 2^{-m}$, and then, by an argument similar to Lemma 3.1, we get for all n and $1 \leq r \leq n$ that $\sigma_{nk}(x) \leq C_5 \lambda n^{1/2} 2^{-r-n/2}$ for all dyadic $x \in Q_r$. The required result follows. $\square$

The next two lemmas are used to justify the passage to the limit $l \to \infty$ in (10).

Let Φ denote the set of Q-valued functions u on [0,1] satisfying $|u(s) - u(t)| \leq |s - t|$, $s, t \in [0, 1]$, and let Φ_n denote the set of Q-valued functions on [0,1] which are constant on each I_{nk} and satisfy $|u(k2^{-n}) - u(l2^{-n})| \leq |k - l| 2^{-n}$. Then let $\Phi^* = \Phi \cup \cup_n \Phi_n$.

Lemma 3.3. *Given $\epsilon > 0$, we can find $\eta > 0$ such that if $U \subset (0, 1) \times \mathbb{R}^d$ is open with $|U| < \eta$, then, with probability $\geq 1 - \epsilon$, we have $\int_0^1 \chi_U(t, W(t) + u(t)) dt \leq \epsilon$ for all $u \in \Phi^*$.*

Proof. Fix $\epsilon > 0$. By Lemma 3.1 we can find K such that, for any Borel function ϕ on $[0, 1] \times \mathbb{R}^d$ with $|\phi| \leq 1$ everywhere we have with probability $> 1 - \epsilon/2$ that

$$\int_{I_{kn}} \{\phi(W(t) + x) - \phi(W(t) + y)\} dt \leq K n^{1/2} 2^{-3n/2} \tag{13}$$

for all pairs of dyadic points x, y in Q and all choices of n, k. Then we choose m such that $4K \sum_{n=m}^{\infty} n^{1/2} 2^{-n/2} < \epsilon$. Let Ω be a finite set of dyadic points of Q such that every $x \in Q$ is within distance 2^{-m} of some point of Ω.

Provided δ is chosen small enough, any bounded Borel function ϕ on $[0,1] \times \mathbb{R}^d$ with $\|\phi\|_{L^p([0,1]\times\mathbb{R}^d)} < \delta$ will satisfy

$$\mathbb{P}\left(\left|\int_{I_{mk}} \phi(t, W(t)+x)dt\right| \geq 2^{-m}\epsilon/4\right) < \frac{\epsilon}{2^{m+1}\#(\Omega)}$$

for each k, x. Then the probability that

$$\left|\int_{I_{mk}} \phi(t, W(t)+x)dt\right| < 2^{-m}\epsilon/4 \text{ for every } k \in \{0,1,\cdots,2^m-1\}, \quad x \in \Omega \tag{14}$$

is at least $1-\epsilon/2$.

Now let $\eta = \delta^p$, and suppose U is open with $m(U) < \eta$. Let (ϕ_r) be an increasing sequence of continuous non-negative functions on $[0,1] \times \mathbb{R}^d$, converging pointwise to χ_U. Note that then $\|\phi_r\|_{L^p([0,1]\times\mathbb{R}^d} < \delta$. For each r define events A_r: (14) holds for $\phi = \phi_r$ and B_r: (13) holds for $\phi = \phi_r$. Then $\mathbb{P}(A_r) \geq 1-\epsilon/2$ and $\mathbb{P}(B_r) \geq 1-\epsilon/2$. Also, when A_r and B_r both hold, we have $\int_{I_{km}} \phi_r(t, W(t)+x)dt < 2^{-m}\epsilon/2$ for all x such that $|x| \leq 2$.

Now let $u \in \Phi^*$. For each $n \geq m$ choose $u_n \in \Phi_n$ taking a constant dyadic value within 2^{-n} of $u(k2^{-n})$ on I_{nk} for $k = 0,1,\cdots,2^n-1$. Now if A_r and B_r hold then $\int_0^1 \phi_r(t, W(t)+u_m(t))dt \leq \epsilon/2$ and

$$\left|\int_0^1 \{\phi_r(t, W(t)+u_n(t)) - \phi_r(t, W(t)+u_{n+1}(t))\}dt\right| \leq Kn^{1/2}2^{-n/2}$$

from which it follows that $\int_0^1 \phi_r(t, W(t)+u(t))dt < \epsilon$. So if we define the event Q_r : $\int_0^1 \phi_r(t, W(t)+u(t))dt \leq \epsilon$ for all $u \in \phi$, then we have $\mathbb{P}(Q_r) \geq 1-\epsilon$. But since $\phi_{r+1} \geq \phi_r$ we have $Q_{r+1} \subseteq Q_r$, and it follows that with probability $\geq 1-\epsilon$ we have Q_r for all r, from which the result follows, since $\int_0^1 \phi_r(t, W(t)+u(t))dt \to \int_0^1 \chi_U(t, W(t)+u(t))dt$ by the bounded convergence theorem. ☐

Lemma 3.4. *If g is a bounded Borel function on $[0,1]\times\mathbb{R}^d$, then, with probability 1, whenever (u_n) is a sequence in Φ^* converging pointwise to a limit $u \in \Phi^*$, we have*

$$\int_0^1 g(t, W(t)+u_n(t))dt \to \int_0^1 g(t, W(t)+u(t))dt \tag{15}$$

Proof. Given $\epsilon > 0$, let η be as in Lemma 3.3, and let h be a bounded continuous function on $[0,1]\times\mathbb{R}^d$ such that $g = h$ outside an open set U with $m(U) < \eta$. With probability $\geq 1-\epsilon$, the conclusion of Lemma 3.3 holds, which means that for any convergent sequence (u_n) in Φ we have $\int_0^1 \mathbb{I}_U(t, W(t)+u_n(t))dt \leq \epsilon$, and the same for the limit $u(t)$, so, if M is an upper bound for $|g-h|$, we have the bound $\left|\int\{g(t, W(t)+u_n(t)) - h(t, W(t)+u_n(t))\}dt\right| \leq M\epsilon$, and the same for x in place of u_n. Also, since h is continuous, $\int_0^1 h(t, W(t)+u_n(t))dt \to \int_0^1 h(t, W(t)+u(t)dt$. It follows that, for n large enough, $\left|\int_0^1 g(t, W(t)+u_n(t))dt - \int_0^1 g(t, W(t)+u(t)dt\right| < (2M+1)\epsilon$, and, since this holds for any $\epsilon > 0$, the result follows. ☐

Note that Lemma 3.4 implies that $\rho_{nk}(x,y)$ and $\rho_{nk}(x)$ are continuous, so that the estimates of Lemmas 3.1 and 3.2 will hold for all $x,y\in Q$.

We also need a stronger bound for sums of ρ_{nk} terms than that given by the bounds for individual terms in Lemma 3.1, and the next two lemmas provide this. They are motivated by the idea that any solution of (4) should satisfy the approximate equation $u((k+1)2^{-n})\approx u(k2^{-n})+\sigma_{nk}(u(k2^{-n}))$ which suggests that on a short time interval a solution can be approximated by an 'Euler scheme' $x_{k+1}=x_k+\sigma_{nk}(x_k)$.

Lemma 3.5. *Given even $p\geq 2$ we can find $C>0$ such that, for any choice of $n,r\in\mathbb{N}$ with $r\leq 2^{n/2}$, $k\in\{0,1,\cdots,2^n-r\}$ and $x_0\in Q$, if we define $x_1,\cdots,x_r$ by the recurrence relation $x_{q+1}=x_q+\sigma_{n,k+q}(x_q)$, then*

$$\mathbb{P}\left(\sum_{q=1}^{r}|\rho_{n,k+q}(x_{q-1},x_q)|\geq 2^{-n}\left\{C\sum_{q=0}^{r-1}|x_q|+\lambda r^{1/2}|x_0|\right\}\right)\leq C\lambda^{-p}$$

for any $\lambda>0$.

Proof. We use $C_1,\cdots$ to denote constants which depend only on d and p. We write $\mathcal{F}_j$ for $\mathcal{F}_{(k+j)2^{-n}}$. Note first that x_q is $\mathcal{F}_q$ measurable and $\mathbb{E}(|\sigma_{n,k+q}(x_q)|^p|\mathcal{F}_q)\leq C_1 2^{-np/2}|x_q|^p$ by (8). Hence $\mathbb{E}|\sigma_{n,k+q}(x_q)|^p\leq C_1 2^{-np/2}\mathbb{E}|x_q|^p$. It follows that $\mathbb{E}|x_{q+1}|^p\leq(1+C_1^{1/p}2^{-n/2})^p\mathbb{E}|x_q|^p$ and so

$$\mathbb{E}|x_q|^p\leq(1+C_1^{1/p}2^{-n/2})^p|x_0|^p\leq C_2|x_0|^p \tag{16}$$

for $1\leq q\leq r$.

Now let $Y_q=|\rho_{n,k+q}(x_{q-1},x_q)|$, $Z_q=\mathbb{E}(Y_q|\mathcal{F}_q)$ and $X_q=Y_q-Z_q$. Then X_q is $\mathcal{F}_{q+1}$ measurable and $\mathbb{E}(X_q|\mathcal{F}_q)=0$ so by Burkholder's inequality

$$\begin{aligned}\mathbb{E}|\sum_{q=1}^{r}X_q|^p&\leq C_3\mathbb{E}(\sum X_q^2)^{p/2}\leq C_3r^{p/2-1}\mathbb{E}\sum|X_q|^p\leq C_4r^{p/2-1}\sum\mathbb{E}(Y_q^p)\\&\leq C_5r^{p/2-1}2^{-np/2}\sum\mathbb{E}|x_q-x_{q-1}|^p=C_5r^{p/2-1}2^{-np/2}\sum\mathbb{E}|\sigma_{n,k+q-1}(x_{q-1})|^p\\&\leq C_6r^{p/2-1}2^{-np}\sum_{q=1}^{r}\mathbb{E}|x_{q-1}|^p\end{aligned}$$

from which we deduce using (16) that

$$\mathbb{E}|\sum_{q=1}^{r}X_q|^p\leq C_7r^{p/2}2^{-np}|x_0|^p \tag{17}$$

Also let $V_q=\mathbb{E}(Z_q|\mathcal{F}_{q-1})$ and $W_q=Z_q-V_q$. Noting that $Z_q\leq C_8 2^{-n/2}\sigma_{n,q-1}(x_{q-1})$ we get in a similar way that

$$\mathbb{E}|\sum W_q|^p\leq C_9r^{p/2}2^{-np}|x_0|^p \tag{18}$$

We also have

$$|V_q| \le C_{10}2^{-N}|X_{q-1}| \tag{19}$$

Now $Y_q = X_q + W_q + V_q$. By (17) and (18) we have $\mathbb{P}(|\sum_{q=1}^{r}(X_q + W_q)| > 2^{-n}\lambda r^{1/2}|x_0|) \le C_{11}\lambda^p$ and the result then follows by (19). □

Lemma 3.6. *With probability 1 there exists $C > 0$ such that for any $n, r \in \mathbb{N}$ with $r \le 2^{n/4}$, any $k \in \{0, 1, \cdots, 2^n - r\}$ and any $y_0, \cdots, y_r \in Q$ we have*

$$\sum_{q=1}^{r} |\rho_{n,k+q}(y_{q-1}, y_q)| \le C\left(2^{-3n/4}|y_0| + 2^{-n/4}\sum_{q=0}^{r-1}|\gamma_q| + 2^{-2^{n/2}}\right) \tag{20}$$

where $\gamma_q = y_{q+1} - y_q - \sigma_{n,k+q}(y_q)$.

Proof. Let $\delta_n = 2^{-2^{n/2}}$. By Lemma 3.1, with probability 1 there exists $C > 0$ such that, for any $n, k \ge 0$ and any $x, y \in Q$, we have

$$\rho_{nk}(x, y) \le C2^{-n/4}|x - y| + \delta_n \tag{21}$$

As before, let $Q_s = \{x \in \mathbb{R}^d : \|x\| \le 2^{-s}\}$. Then, for integers s with $0 \le s < 2^{n/2}$, let Ω_{ns} be a set of not more than $(2^n d^{1/2})^d$ points of Q_s such that every $x \in Q_s$ is within distance 2^{-s-n} of a point of Ω_{ns} and let $\Omega_n = \cup_{0\le s<2^{-n/2}}\Omega_{ns}$. Let $p = 8(4 + d)$. Then by Lemma 3.5 there is $C_1 > 0$ such that the probability that

$$\sum_{q=1}^{r} |\rho_{n,k+q}(x_{q-1}, x_q)| \ge 2^{-n}\left(C_1\sum_{q=0}^{r-1}|x_q| + \lambda 2^{n/8}r^{1/2}|x_0|\right)$$

for some n, r, k as in the statement and some $x_0 \in \Omega_n$, is bounded above by $C_1\sum_{n=0}^{\infty}\lambda^{-p}2^{n(3+d)}2^{-pn/8}$ which approaches 0 as $\lambda \to \infty$. Hence with probability 1 there exists $C > 0$ such that

$$\sum_{q=1}^{r} |\rho_{n,k+q}(x_{q-1}, x_q)| < C2^{-n}\left(\sum_{q=0}^{r-1}|x_q| + 2^{n/8}r^{1/2}|x_0|\right) \tag{22}$$

for all n, k, r as above and $x_0 \in \Omega_n$.

We now suppose, as we may with probability 1, that (21) and (22) hold (with the same C). We fix $n, k, r, y_0 \cdots y_r, \gamma_0 \cdots \gamma_r$ as in the statement of the lemma. Take the smallest s such that $y_0 \in Q_s$, noting that then $2^{-s-1} \le |y_0| \le d^{1/2}2^{-s}$. Then we find $x_0 \in \Omega_{ns}$ with $|x_0 - y_0| < 2^{-s-n} \le 2^{1-n}|y_0|$ and define $x_1 \cdots x_r$ by the recurrence relation $x_{q+1} = x_q + \sigma_{n,k+q}(x_q)$. Then by (22)

$$\sum_{q=1}^{r} |\rho_{n,k+q}(x_{q-1}, x_q)| < C2^{-n}\left(\sum_{q=0}^{r-1}|x_q| + 2^{n/4}|x_0|\right)$$

Using (21) we have $|x_{q+1}| = |x_q + \sigma_{n,k+q}(x_q)| \leq (1 + C2^{-n/4})|x_q| + \delta_n$ so $|x_q| \leq C_1(|x_0| + r\delta_n)$ and

$$\sum_{q=1}^{r} |\rho_{n,k+q}(x_{q-1}, x_q)| < C_2 2^{-3n/4}(|x_0| + 2^{n/4}\delta_n) \tag{23}$$

Now let $u_q = x_q - y_q$. Then $|u_{q+1} - u_q| \leq |\rho_{n,k+q}(x_q, y_q)| + |\gamma_q|$ so

$$|u_{q+1}| \leq |u_q|(1 + C2^{-n/4}) + |\gamma_q| + \delta_n$$

and since $|u_0| \leq 2^{1-n}|y_0|$ we deduce that $|u_q| \leq C_3(2^{-n}|y_0| + r\delta_n + \sum_{q=0}^{r-1}|\gamma_q|)$ and so

$$|\rho_{n,k+q}(x_q, y_q)| \leq C_4 2^{-n/4}\left(2^{-n}|y_0| + r\delta_n + \sum_{q=0}^{r-1}|\gamma_q|\right) \tag{24}$$

and we have the same bound for $|\rho_{n,k+q}(x_{q-1}, y_{q-1})|$. Now

$$\rho_{n,k+q}(y_{q-1}, y_q) = \rho_{n,k+q}(x_{q-1}, x_q) + \rho_{n,k+q}(y_{q-1}, x_{q-1}) + \rho_{n,k+q}(x_q, y_q)$$

and then using (23), (24) and the fact that $|x_0 - y_0| \leq 2^{1-n}|y_0|$ we deduce that

$$\sum_{q=1}^{r} |\rho_{n,k+q}(y_{q-1}, y_q| \leq C_5\left(2^{-3n/4}|y_0| + 2^{-n/4}\sum_{q=0}^{r-1}|\gamma_j| + 2^{-n/2}\delta_n\right)$$

from which the result follows. □

We now proceed to complete the proof of the theorem. From now on we take $g = f$ in the definition of σ_{nk} and ρ_{nk}. We consider a Brownian path W satisfying the conclusions of Lemmas 3.1, 3.2, 3.6 and 3.4 for some $C > 0$. We shall show that for such a Brownian path the only solution u of (4) in Φ is $u = 0$. This will follow from the following:

Lemma 3.7. *Suppose W satisfies the conclusions of Lemmas 3.1, 3.2, 3.6 and 3.4 for some $C > 0$. Then there are positive constants K and m_0 such that, for all integers $m > m_0$, if u is a solution of (4) in Φ and for some $j \in \{0, 1, \cdots, 2^m - 1$ and some β with $2^{-2^{3m/4}} \leq \beta \leq 2^{-2^{2m/3}}$ we have $|u(j2^{-m})| \leq \beta$, then*

$$|u((j+1)2^{-m})| \leq \beta\{1 + K2^{-m}\log(1/\beta)\}$$

Proof. We use $C_1, C_2, \cdots$ for positive constants which depend only on the constant C and the dimension d. Fix m, j and β as in the statement, and suppose $|u(j2^{-m})| \leq \beta$. Let N be the integer part of $4\log_2(1/\beta)$. Suppose $u \in \Phi$ satisfies (4), and let u_n be the step function which takes the constant value $u(k2^{-n})$ on the interval I_{nk}, for $k = 0, 1, \cdots, 2^n - 1$.

Let α be the smallest nonnegative number such that

$$\sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1} |u((k+1)2^{-n}) - u(k2^{-n})| \leq \alpha 2^{-m}(n^{1/2}2^{n/2} + N) \tag{25}$$

for all n with $m \leq n \leq N$.

For $n \geq m$ let

$$\psi_n = \sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1} |u(k2^{-n})| \tag{26}$$

Then by (25)

$$\psi_n \leq 2\psi_{n-1} + \alpha 2^{-m}(n^{1/2}2^{n/2} + N)$$

for $n > m$,and since $\psi_m = \beta$ it follows that

$$\psi_n \leq 2^{n-m}\beta + \sum_{l=m+1}^{n} \alpha 2^{n-l-m}(l^{1/2}2^{l/2} + N) \leq C_1 2^{n-m}(\beta + \alpha 2^{-m}N) \tag{27}$$

for all n with $m \leq n \leq N$, where we have used the fact that $m^{1/2}2^{m/2}$ is bounded by const.N.

Now fix $n \geq m$. Then for $k = j2^{n-m}, \cdots, (j+1)2^{n-m} - 1$ we have, using (15)

$$u((k+1)2^{-n}) - u(k2^{-n}) = \int_{I_{kn}} \{f(W(t) + u(t)) - f(W(t))\}dt$$

$$= \int_{I_{kn}} \{f(W(t) + u_n(t)) - f(W(t))\}dt + \sum_{l=n}^{\infty} \int_{I_{kn}} \{f(W(t) + u_{l+1}(t)) - f(W(t) + u_l(t))\}dt$$

which we can write as

(28)

$$u((k+1)2^{-n}) - u(k2^{-n}) = \sigma_{nk}(u(k2^{-n})) + \sum_{l=n}^{\infty} \sum_{r=k2^{l-n}}^{(k+1)2^{l-n}-1} \rho_{l+1,2r+1}(u(2^{-l-1}(2r+1)), u(2^{-l}r))$$

from which we deduce

$$\sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1} |u((k+1)2^{-n}) - u(k2^{-n})| \leq \sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1} |\sigma_{nk}(u(k2^{-n}))| + \sum_{l=n}^{\infty} \Omega_l \tag{29}$$

where $\Omega_l = \sum_{r=j2^{l-m}}^{(j+1)2^{l-m}-1} |\rho_{l+1,2r+1}(u(2^{-l-1}(2r+1)), u(2^{-l}r))|$.

We now proceed to estimate the two sums on the right of (29), starting with the easier σ_{nk} term. Using Lemma 3.2 and the fact that $N < 2^m$, we have $|\sigma_{nk}(x)| \leq C_2 n^{1/2}2^{-n/2}(2^{-N} + |x|)$ and so

$$\begin{aligned} \sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1} |\sigma_{nk}(u(k2^{-n}))| &\leq C_2 \sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1} n^{1/2}2^{-n/2}(2^{-N} + |u(k2^{-n})|) \\ &\leq C_3 n^{1/2}2^{n/2-m}(\beta + 2^{-m}N\alpha + 2^{-N}) \end{aligned} \tag{30}$$

using (27).

Next we bound $\sum \Omega_l$, which we do in two stages. We first obtain a relatively crude bound by applying (11) to each term, and then obtain an improved by applying the crude bound together with Lemma (3.6). To start with the crude bound, from (11) we have $|\rho_{nk}(x,y)| \leq C_3 2^{-n/2} N^{1/2}(2^{-N} + |x-y|)$ and using this together with (25) gives

$$\Omega_l \leq C_4 2^{-l/2} N^{1/2} \{2^{-N} 2^{l-m} + \alpha 2^{-m}(l^{1/2} 2^{l/2} + N)\} \tag{31}$$

and so

$$\sum_{l=m}^{N} \Omega_l \leq C_5 (N^{1/2} 2^{-m-N/2} + \alpha 2^{-m} N^2) \tag{32}$$

For $l > N$ we use $|u(t) - u(t')| \leq |t - t'|$ and (11) to obtain

$$\sum_{l=N+1}^{\infty} \Omega_l \leq \sum_{l=N+1}^{\infty} C_6 2^{l-m} l^{1/2} 2^{-3l/2} \leq C_7 N^{1/2} 2^{-m-N/2} \tag{33}$$

and combining this with (32) we obtain

$$\sum_{l=m}^{\infty} \Omega_l \leq C_8 (N^{1/2} 2^{-m-N/2} + \alpha 2^{-m} N^2) \tag{34}$$

The second stage is to improve the estimate (34) by applying Lemma 3.6 to obtain a better estimate for Ω_n for larger n; we use (34) to bound the γ term in Lemma 3.6.

Let $N^{1/6} \leq n \leq N$. We define $\gamma_{nk} = u((k+1)2^{-n}) - u(k2^{-n}) - \sigma_{nk}(u(k2^{-n}))$, noting that (28) implies that

$$\sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1} |\gamma_{nk}| \leq \sum_{l=n}^{\infty} \Omega_l \leq C_8 (N^{1/2} 2^{-m-N/2} + \alpha 2^{-m} N^2) \tag{35}$$

Also we define

$$\Lambda_n = \sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1} |\rho_{n,k+1}(2^{-n}k, 2^{-n}(k+1))$$

so that $\Omega_n \leq \Lambda_{n+1}$. Let $r = \lfloor 2^{n/4} \rfloor$. In order to apply Lemma 3.6 to estimate Λ_n, we will split the sum into r-sized pieces. First we find $i \in \{0, 1, \cdots, r-1\}$ such that, writing $s = \lfloor r^{-1}(2^{n-m} - i) \rfloor$, we have $\sum_{t=0}^{s} |u(j2^{-m} + (i+tr)2^{-n})| \leq r^{-1}\psi_n$. Now we fix for the moment $t \in \{0, 1, \cdots, s\}$ and apply Lemma 3.6 with $y_q = u((k+q)2^{-n})$ where $k = j2^{n-m} + i + tr$. We obtain

$$\sum_{q=1}^{r} |\rho_{n,k+q}(y_{q-1}, y_q)| \leq C_9 \left(2^{-3n/4} |u(k2^{-n})| + 2^{-n/4} \sum_{q=0}^{r-1} |\gamma_{n,k+q}| + 2^{-2^{n/2}} \right)$$

Summing over t then gives

$$\sum_{k=j2^{n-m}+i}^{(j+1)2^{n-m}-1} |\rho_{n,k+1}(2^{-n}k, 2^{-n}(k+1))| \leq C_9 2^{-3n/4} \sum_{t=0}^{s} |u(j2^{-m} + (i+tr)2^{-n})| + C_9 \left(+2^{-n/4} \sum_{k=j2^{n-m}+i}^{(j+1)2^{n-m}-1} |\gamma_{n,k}| + 2^{n-2^{n/2}} \right)$$

Also

$$\sum_{k=j2^{n-m}}^{j2^{n-m}+i-1} |\rho_{n,k+1}(2^{-n}, 2^{-n}(k+1))| \leq C_9 \left(2^{-3n/4}|u(j2^{-m})| + 2^{-n/4} \sum_{k=j2^{n-m}}^{j2^{n-m}+i-1} |\gamma_{n,k}| + r2^{-2^{n/2}} \right)$$

From the last two inequalities, using (27), (35) and $|u(j2^{-m})| \leq \beta$, we find that

$$\Lambda_n \leq C_{10}\{2^{-m}(\beta + \alpha 2^{-m}N) + 2^{-m-n/4}(N^{1/2}2^{-N/2} + \alpha N^2) + 2^{n-2^{n/2}}\}$$

Since $n \geq N^{1/6}$ the first term dominates so $\Lambda_n \leq C_{11}2^{-m}(\beta + \alpha 2^{-m}N)$, and the same bound holds for $\Omega_n \leq \Lambda_{n+1}$. We deduce that

$$\sum_{N^{1/6} \leq l \leq N} \Omega_l \leq C_{12}N2^{-m}(\beta + \alpha N 2^{-m})$$

Using the original bound (31) for $l < N^{1/6}$ we have

$$\sum_{m \leq l < N^{1/6}} \Omega_l \leq C_{13}N^{1/2}\{2^{-N+N^{1/4}/2-m} + \alpha 2^{-m}(N^{1/4} + 2^{-m/2}N)\}$$

Combining these two estimates with (33) we get our improved bound.

$$\sum_{l=m}^{\infty} \Omega_l \leq C_{14}\{N2^{-m}(\beta + \alpha N 2^{-m}) + \alpha(2^{-m}N^{3/4} + 2^{-3m/2}N^{3/2})\}$$

To conclude the proof we use this bound along with (30) in (29) and obtain

$$\sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1} |u((k+1)2^{-n}) - u(k2^{-n})| \leq C_{15}(n^{1/2}2^{n/2-m} + N2^{-m}) \times \{\beta + \alpha(N2^{-m} + N^{-1/4} + 2^{-m/2}N^{1/2})\}$$

for all n with $m \leq n \leq N$. Comparing this with (25) we see by the minimality of α that

$$\alpha \leq C_{15}\{\beta + \alpha(N2^{-m} + N^{-1/4} + 2^{-m/2}N^{1/2})\}$$

Then if m is large enough to ensure $C_{15}(N2^{-m} + N^{-1/4} + 2^{-m/2}N^{1/2}) < 1/2$ it follows that $\alpha \leq 2C_{15}\beta$. Then applying (25) with $n = m$ gives $|u((j+1)2^{-m})| \leq \beta + 2C_{15}\beta(m^{1/2}2^{m/2} + N)2^{-m} \leq \beta(1 + C_{16}N2^{-m})$ from which the required result follows. $\square$

To complete the proof of Theorem 1.1, using the notation of Lemma 3.7 let $m > m_0$ and $\beta_0 = 2^{-2^{3m/4}}$, and define β_j for $j = 1, 2, \cdots, 2^m$ by the recurrence relation $\beta_{j+1} = \beta_j(1 + K2^{-m}\log(1/\beta_j))$. Writing $\gamma_j = \log(1/\beta_j)$ we then have

$$\gamma_{j+1} = \gamma_j - \log(1 + K2^{-m}\gamma_j) \geq \gamma_j(1 - K2^{-m})$$

so the sequence (γ_j) is decreasing and

$$\gamma_j \geq \gamma_0(1 - K2^{-m})^j \geq \gamma_0 e^{-K-1} = 2^{3m/4}e^{-K-1} \geq 2^{2m/3}$$

for all $j = 1, 2, \cdots, 2^m$, provided m is large enough. Then for each j, β_j is in the range specified in Lemma 3.7, and it follows from that lemma by induction on j that $|u(j2^{-m})| \leq \beta_j$ for each j. Hence $|u(j2^{-m})| \leq 2^{-2^{2m/3}}$ for each j. This holds for all large enough m, and hence u vanishes at all dyadic points in [0,1], and, as u is continuous, $u = 0$ on [0,1]. This completes the proof of the theorem.

4. An Application

We give an application of Theorem 1.1 to convergence of Euler approximations to (1) with variable step size.

In this section we assume f is continuous and consider (1) on a bounded interval $[0, T]$. Given a partition $\mathcal{P} = \{0 = t_0 < t_1 < \cdots < t_N = T\}$ of $[0, T]$ we consider the Euler approximation to (1) given by:

$$x_{n+1} = x_n + W(t_{n+1}) - W(t_n) + (t_{n+1} - t_n)f(t_n, x_n)$$

for $n = 0, \cdots, N-1$, with $x_0 = 0$. For such a partition $\mathcal{P}$ we let $\delta(\mathcal{P}) = \max_{n=1}^{N}(t_n - t_{n-1})$. Then we have the following:

Corollary 4.1. *For almost every Brownian path W, for any sequence*

$$\mathcal{P}_k = \{t_0^{(k)}, \cdots, t_{N_k}^{(k)}\}$$

of partitions with $\delta(\mathcal{P}_k) \to 0$, we have

$$\max_{n=1}^{N_k} |x_n^{(k)} - x(t_n^{(k)})| \to 0$$

as $k \to \infty$, where $x(t)$ is the unique solution of (1) and $\{x_n^{(k)}\}$ is the Euler approximation using the partition $\mathcal{P}_k$.

Proof. Suppose W is a path for which the conclusion of Theorem 1.1 holds, and suppose there is a sequence of partitions with $\delta(\mathcal{P}_k) \to 0$ such that $\max_{n=1}^{N_k} |x_n^{(k)} - x(t_n^{(k)})| \geq \delta > 0$. Then if we let $u_n^{(k)} = x_n^{(k)} - W(t_n^{(k)})$ we have $|u_{n+1}^{(k)} - u_n^{(k)}| \leq \|f\|_\infty (t_{n+1}^{(k)} - t_n^{(k)})$ so by Ascoli-Arzela, after passing to a subsequence we have a continuous u on $[0, T]$ such that $\max_{n=1}^{N_k} |u_n^{(k)} - u(t_n^{(k)})| \to 0$. Then writing $y(t) = u(t) + W(t)$ we see that $y \neq x$ and, using the continuity of f, that y satisfies (1), contradicting the conclusion of the theorem. Corollary 4.1 is proved. $\square$

The point of Corollary 4.1 is that the partitions can be chosen arbitrarily, no 'non-anticipating' condition is required. For general SDE's with non-additive noise and sufficiently smooth coefficients Euler approximations will converge to the solution provided the partition points t_n are stopping times, but this condition is rather restrictive for numerical practice, and an example is given in section 4.1 of [1] of a natural variable step-size Euler scheme for a simple SDE which converges to the wrong limit. [1] also contains related results and discussion.

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