MULTIVARIABLE CALCULUS

Sam C. Collingbourne
Department of Mathematics, Columbia University, New York

April 9, 2024

Overview and Motivation

Beware: These notes are subject to change! If you're reading ahead of lectures, it is very likely that there are mistakes. Use with care and please inform me of typos/mistakes.

A first course in calculus studies *univariate* calculus. Univariate means that one studies functions that give a rule from the real line $\mathbb{R}$ to itself, i.e. one has a function f that takes a point/variable $x \in \mathbb{R}$ to $f(x) \in \mathbb{R}$, whilst the calculus part means that one studies limits, derivatives and integrals of such functions. This course studies the generalisation of these concepts to vector-valued *multivariable* functions. In plain terms, this means that the functions one studies depend on more than one variable and can produce values which cannot be expressed as a single number but only as a ordered list of numbers. More precisely, a vector-valued multivariable function is a map/rule f which takes n inputs, $(x_1, ..., x_n)$, with $x_i \in \mathbb{R}$ for $i = 1, ..., n$ and outputs m real numbers $(y_1, ..., y_m)$. In this course, the multivariable functions studied will typically have $m, n \leq 3$. For example, one could define f by

$$(x, y) \mapsto f(x, y) = x^2 + y^2. \tag{1}$$

A natural question to ask is why one would want to study such things. Essentially this is motivated by the physical world. Multivariable calculus has applications everywhere in physics, mathematics, economics and engineering to name only a few. Below are some examples:

- **Physical positions require more than one number:** Putting string theories aside, we perceive 3 dimensions (plus time, so in reality 4). You can move forwards/backwards, left/right and up/down. Suppose you wanted to specify a place to meet someone in Manhattan (or on Earth in general), you need 2 bits of information: a street and an avenue or you need to tell them how far North and how far West (a longitude and a latitude). For example, $(40°48'2''N, 73°57'43''W)$ is Columbia University. Suppose further you wanted to meet them on the fifth floor of a building then you need a third number. You might even want to add a time giving a fourth number.
- **Many quantities change in space and time:** If you wanted to arrive at the above place by a certain time you need to know at what velocity you need to travel. Velocity is a directed rate of change of distance over time, i.e. you need a direction (which is more than one number) and a speed. Therefore, studying geometry in 2 and 3 dimensions and rates of change (i.e. derivatives) is important to understand physical problems.
- **Physical objects have 2 or 3-dimensional extent:** understanding areas/volumes of objects or how quantities leave a areas/volume (say energy of a water wave leaving a circular disk) requires integration in 2 and 3 dimensions.
- **Partial differential equations:** You may/may not have come across 'partial differential equations' which are equations which govern how a given quantity varies in space and time. They are fundamental in the study of many physical systems such as electrodynamics (Maxwell's equations), fluid mechanics (the Navier–Stokes equation), differential game theory (the Black–Scholes equation) and gravity (the Einstein equation). In aiming to study such equations one requires a good working knowledge of multivariable calculus.

Contents

Notation

The following is a list of notation that is used in lectures and these notes:

- $\{...\}$: denotes a set or a collection of elements, usually numbers, i.e. $\{1, 3, 7, \sqrt{2}\}$, $\{1, 2, 3, 4, ...\}$. One can have a set with a condition, this is written with a colon as follows:

$$\{\text{elements} : \text{condition}\}. \tag{2}$$

  For example

$$\{x \in \mathbb{R} : x > 0\} \tag{3}$$

  is the set of positive reals.

- $\{$ can denote a set of equations you usually want solve simultaneuously, i.e.

$$\begin{cases} ax + by + cz + d = 0 \\ ex + fy + gz + h = 0. \end{cases} \tag{4}$$

- $\mathbb{R}$: the set/collection of real numbers, i.e. (inprecisely) the set or collection of all numbers which have an infinite decimal expansion. Examples would be 1, 2, -7, $\pi = 3.1415...$, $\sqrt{2}$ and (uncountably) many more.

- $\in$: belongs to, i.e. $a \in \mathbb{R}$, a belongs to the reals.

- $A \subseteq B$: denotes that A is a 'subset' of B meaning that all elements of A are contained in B. For example, if $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$ then $A \subseteq B$. Note that A and B can be equal, i.e. in the above example $A = \{1, 2, 3, 4\}$ is a subset of B.

- (a, b) is the open interval from a to b, i.e. it is the set of all real numbers between a and b, excluding a and b.

- $(a, b]$ is a half-open interval from a to b, i.e. it is the set of all real numbers between a and b, excluding a but containing b.

- $[a, b)$ is a half-open interval from a to b, i.e. it is the set of all real numbers between a and b, including a but excluding b.

- $[a, b]$ is the closed interval from a to b, i.e. it is the set of all real numbers between a and b, including a and b.

- $A \times B$ is called the Cartesian product of A and B, it is the set of pairs (a, b) such that $a \in A$ and $b \in B$. In set notation:

$$\{(a, b) : a \in A, b \in B\} \tag{5}$$

- $\mathbb{R}^2$: denotes the Cartesian product $\mathbb{R} \times \mathbb{R}$, i.e. it is the set of ordered pairs of real numbers, (a, b), with $a, b \in \mathbb{R}$. Notationally, this is written,

$$\mathbb{R}^2 = \{(a, b) : a \in \mathbb{R}, b \in \mathbb{R}\}, \tag{6}$$

  where the colon : denotes 'such that' or equivalently that a condition follows.

- $\mathbb{R}^3$: denotes the Cartesian product $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$, i.e. the set of ordered triples of real numbers, (a, b, c), with $a, b, c \in \mathbb{R}$. Notationally, this is written as follows:

$$\mathbb{R}^3 = \{(a, b, c) : a, b, c \in \mathbb{R}\}, \tag{7}$$

- $\mathbb{R}^n$ for $n \geq 1$: denotes the n-fold Cartesian product $\mathbb{R} \times ... \times \mathbb{R}$, i.e. the set of ordered n-tuples of real numbers, $(x_1, ..., x_n)$, with $x_i \in \mathbb{R}$ for all $i = 1, ..., n$. Notationally, this is written as follows:

$$\mathbb{R}^n = \{(x_1, ..., x_n) : x_i \in \mathbb{R} \text{ for all } i = 1, ..., n\}. \tag{8}$$

- $\mathbb{S}^1_R$: denotes a circle of radius R. This has equation $x^2 + y^2 = R^2$. Notationally, this is written as follows:

$$\mathbb{S}^1_R = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = R^2\}. \tag{9}$$

- $\mathbb{S}^2_R$: denotes a sphere of radius R. This has equation $x^2 + y^2 + z^2 = R^2$. Notationally, this is written as follows:

$$\mathbb{S}^2_R = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = R^2\}. \tag{10}$$

- $f : A \to B$: denotes a function taking inputs from a set A (called its domain) and outputing elements of B (called its codomain).
- $\sum_i$: denotes a sum, i.e. $\sum_{i=1}^n a_i = a_1 + a_2 + ... + a_n$.
- $\pm, \mp$: denotes $+$ or $-$ compactly, i.e. ± 1 means plus or minus 1, $f_\pm(x) = \pm\sqrt{1 - x^2}$ means two functions defined simultaneously $f_+(x) = +\sqrt{1 - x^2}$, $f_-(x) = -\sqrt{1 - x^2}$. This notation can be used in more elaborate ways:

$$f_\pm(x) = a \mp b \pm cx^7 \tag{11}$$

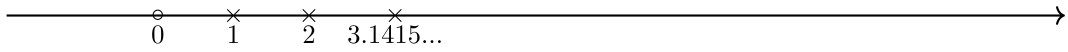
which translated to

$$f_+(x) = a - b + cx^7 \qquad f_-(x) = a + b - cx^7. \tag{12}$$

- $\implies$: implies, i.e. $A \implies B$ for example $x^2 = a \implies x = \pm a$.
- $\doteq$: definition via an equality, i.e. define a function $f(x) \doteq$ ..... For example $f_\pm(x) \doteq \pm\sqrt{1 - x^2}$ defines two functions f_+ f_- by the right-hand side with their respective signs.
- $\mapsto$: Arrow notation defines the rule of a function inline, without requiring a name to be given to the function. For example $(x, y) \mapsto x^2 - y^2$ should be read take (x, y) in the domain and map them to $x^2 - y^2$ in the codomain $\mathbb{R}$.
- γ: usually a line or a curve.

1 Coordinate Systems on the Plane and $3D$ Space

When you think of the real numbers $\mathbb{R}$ you probably, quite involintarily, think of an infinite straight line with each point representing an infinite decimal expansion:



This lecture is about how draw pairs or triples of numbers, which requires a coordinate system.

A **coordinate system** is a one-to-one assignment of a **coordinate**, which is an *ordered* list of n numbers (often called an n-tuple) to points on a surface or space. In plain terms a coordinate system is a way of labelling points on a surface or space with numbers. The notion of one-to-one just means that one assigns distinct elements to distinct elements: here one assigns a distinct coordinate to a distinct point on the surface or space. The reason for this is you want to uniquely label your points.

In this course, the number n will be 2 or 3 and the surface or space will be the 2-dimensional Euclidean plane, or 3-dimensional Euclidean space. These are often called just 'the plane' or '$3D$ space' respectively.

Remark 1.1. *The reader should be aware that there are notions of $2D$ or $3D$ spaces that are not Euclidean. For example, the surface of a ball, also called a sphere, is a two dimensional space (a given point on the surface of a sphere can be specified by two numbers) that is not Euclidean.*

1.1 Coordinate Systems on the Plane

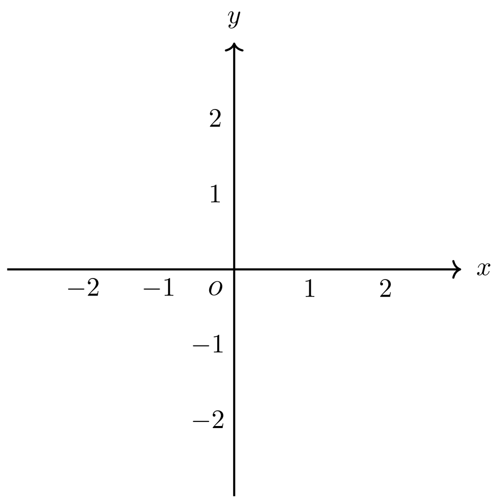
1.1.1 Cartesian Coordinates

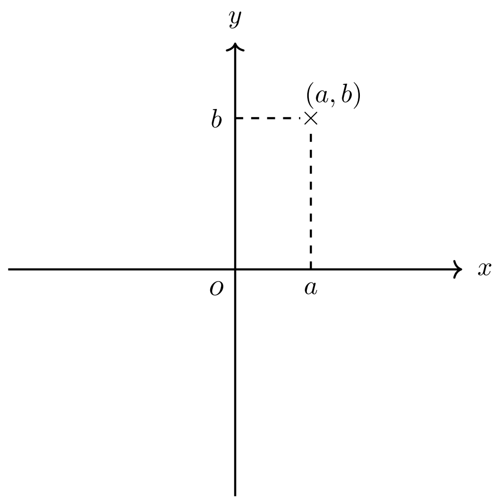
On the plane

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}\}, \tag{13}$$

one can set up **Cartesian coordinates** (due to French mathematician René Descartes). There is not much to do here: if $p \in \mathbb{R}^2$ then Cartesian coordinates simply its ordered pair (a, b).

Just as the real numbers $\mathbb{R}$ can be visualised as points on a straight infinite line, the plane can be visualised with points on a flat two-dimensional surface with infinite extent. To draw the Cartesian coordinate system, one picks an a point for $o = (0, 0)$ the origin and sets up two perpendicular axes or lines which intersect at $(0, 0)$: conventionally one horizontally, called the x-axis and one vertically, called the y-axis. One identifies each of these lines with the real numbers $\mathbb{R}$ such that the real numbers labelling the x-axis increase to the right and the real numbers labelling the y-axis increase upwards. This is easiest to visualise with a picture:





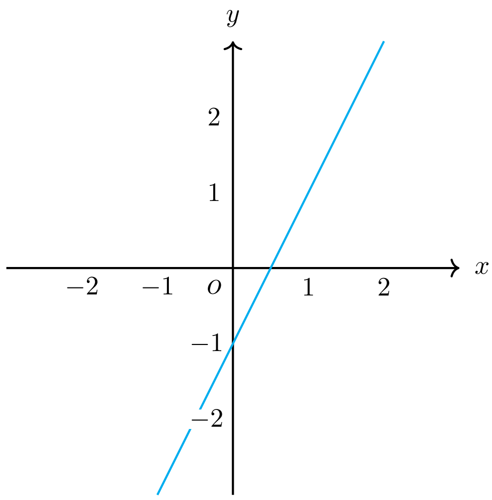
Given a point in the above diagram its Cartesian coordinates be determined by the following procedure (shown in the right hand figure of the above diagram):

1. Draw a line perpendicular to the x-axis through p and another line perpendicular to the y-axis through p.

2. a is given by the point at which line perpendicular to the x-axis meets the x-axis.

3. b is given by the point at which line perpendicular to the y-axis meets the y-axis.

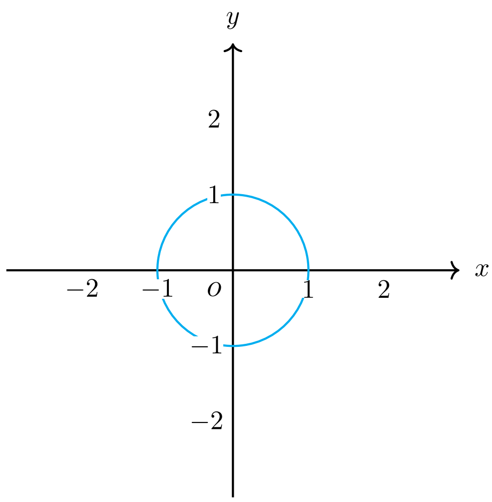
It is very easy to describe straight lines in Cartesian coordinates, for example the straight line determined by the equation $y = 2x - 1$:



Round shapes are slightly tricker. The unit circle is the subset of $\mathbb{R}^2$:

$$\mathbb{S}^1 \doteq \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}. \tag{14}$$

This is drawn below:



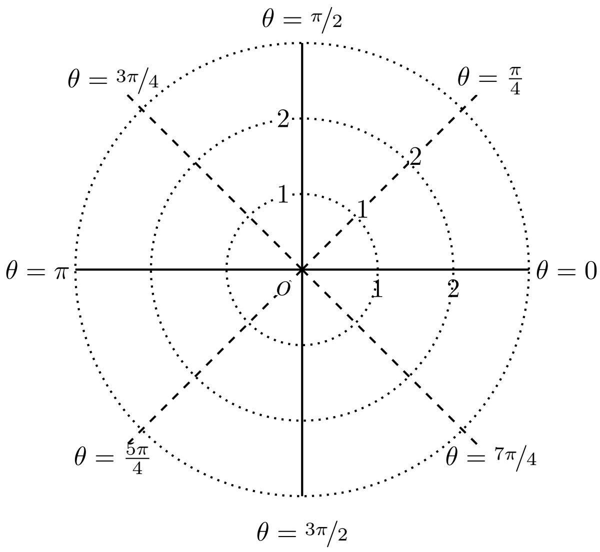
Polar coordinates on $\mathbb{R}^2$ give a cleaner description of round shapes.

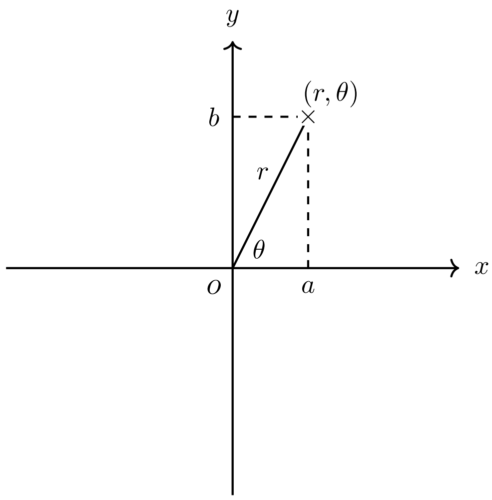
1.1.2 Polar Coordinates

Polar coordinates provide a one-to-one map from $(0, \infty) \times [0, 2\pi)$ tow $\mathbb{R}^2 \setminus \{p\}$ where p is usually chosen to be the origin o in Cartesian coordinates, i.e. $(0, 0)$.[1] Concretely, one takes the origin in $\mathbb{R}^2$ and the positive part of the x-axis, i.e. the set $\{(x, y) : x \in (0, \infty), y = 0\}$. This is often called the polar axis. Now to

[1]For those unfamiliar with set notation $\mathbb{R}^2 \setminus \{p\}$ this means $\mathbb{R}^2$ without the point p.

construct the polar coordinates of a point $q \in \mathbb{R}^2$ one take the distance r from the origin o to q and the anticlockwise angle between the polar axis and the line/ray from o to q. So the coordinate $r \in (0, \infty)$ and $\theta \in [0, 2\pi)$. Again, this is best visualised with a figure as follows:





Remark 1.2. *Whilst these comments may seem pedantic there are technical situations where they can be important. In general you will not have to worry about them but it is good to be aware.*

1. *Notice that polar coordinates here have been defined on $\mathbb{R}^2 \setminus (0,0)$. This is because the point $(0,0)$ can be represented in many different ways as $(0,\theta)$ for all $\theta \in [0, 2\pi)$. This is non-unique and therefore we don't have a one-to-one correspondence between a polar coordinate and the origin. One can either ignore this and 'represent o as $(0,\theta)$' or one could state that $(r,\theta) = (0,0)$ is the origin. Both of these solutions are mostly fine (especially when integrating) but have problems when you want to take limits at the origin such as the ones that arise in taking derivatives.*

2. *Related to point 1 is values of θ outside of $[0, 2\pi)$. Note that any other length 2π interval is fine; you can pick your favourite. The reason for the restriction is to have a unique θ coordinate for each point in $\mathbb{R}^2 \setminus \{(0,0)\}$.*

   Intuitively. one can image revolving around the origin any number of times and assigning a that θ value to represent the θ coordinate of a point. Say $r = 1$ then $\theta = \frac{\pi}{2}$ can be represented by $\theta = 5\pi/2$ or $\theta = 9\pi/2$ or $\theta = -3\pi/2$ etc. However, one runs into the same non-uniqueness issue as 1 if you allow this. So either one restricts θ to an interval of 2π length or periodically identifies θ, i.e. one declares θ is equivalent to $\theta + 2n\pi$ for $n \in \mathbb{Z}$ (the integers). One can represent this in notation as $\theta \sim \theta + 2n\pi$ for $n \in \mathbb{Z}$. In practise what this means is the following: You compute the angle to be $\tilde{\theta}$. Then one finds the n such that $\theta \doteq \tilde{\theta} + 2n\pi \in [0, 2\pi)$. This is the θ coordinate one assigns to that point.

Relations between Cartesian and Polar Coordinates

To find the Cartesian coordinates give polar coordinates one uses,

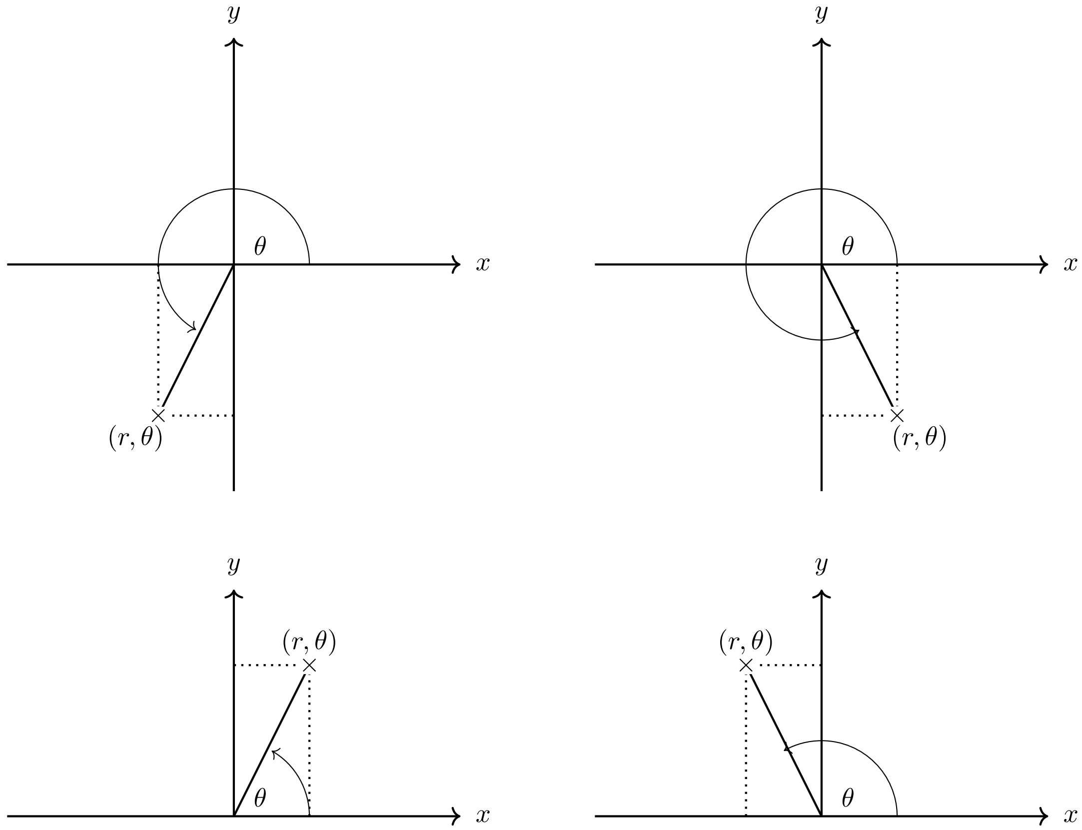
$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{15}$$

These can be inverted to find the polar coordinates from Cartesian coordinates. The resulting formulas are the following,

$$r = \sqrt{x^2 + y^2}, \qquad \theta = \begin{cases} \arctan\left(\frac{y}{x}\right) & x > 0, y \geq 0 \\ \arctan\left(\frac{y}{x}\right) + 2\pi & x > 0, y < 0 \\ \arctan\left(\frac{y}{x}\right) + \pi & x < 0 \\ \frac{\pi}{2} & x = 0, y > 0 \\ \frac{3\pi}{2} & x = 0, y < 0 \\ \text{undefined} & x = 0, y = 0. \end{cases} \tag{16}$$

The reason for the cases for θ is due to $\arctan : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$.

One can see why these formulas in (15) are true from using SOHCAHTOA plus trig.-identities on the triangles in the following pictures



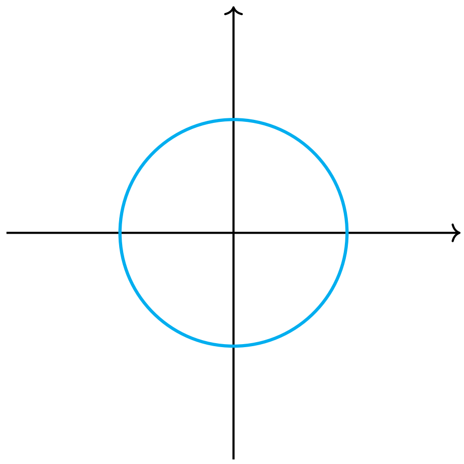
Polar Curves
In Cartesian coordinates one can consider functions that depend on both x and y, i.e. $F = F(x, y)$. For example,

$$(x, y) \mapsto x^2 + y^2 - 1. \tag{17}$$

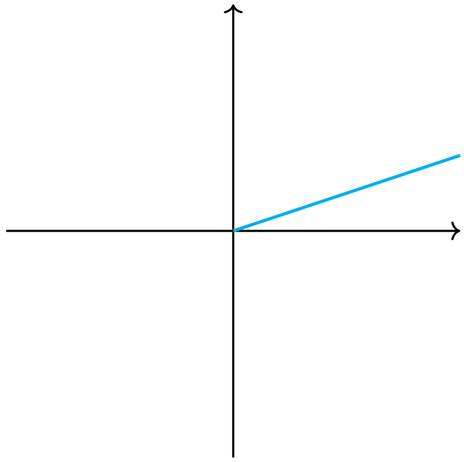
One can do the same in terms of polar coordinates. Then one writes $F(r, \theta)$. One can consider the **level set** of a such a function which is the set of points in $\mathbb{R}^2$ such that $F(r, \theta) = 0$. One can consider plots of such a set or the plot of a polar curve.

Example 1.1. *Here are some examples of plot of polar curves:*

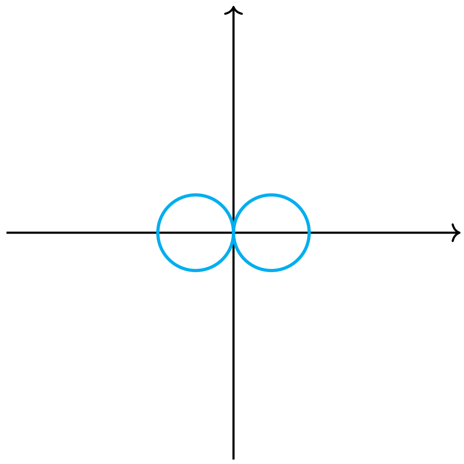
1. $r = c$ *where* $c \in (0, \infty)$*:*



2. $\theta = \theta_0$, *for* $\theta_0 \leq {}^{\pi}/_4$*:*



3. $r = a|\cos\theta|$,



1.2 Coordinate Systems on $3D$ Euclidean space

The Euclidean 3-space $\mathbb{R}^3$ is the set of all ordered *triples* of real numbers:

$$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) : x \in \mathbb{R}, y \in \mathbb{R}, z \in \mathbb{R}\}. \tag{18}$$

One visualises this as flat 3-dimensional space with infinite extent in all directions. It is often used as the model for the physical world, i.e. where all physical processes take place.

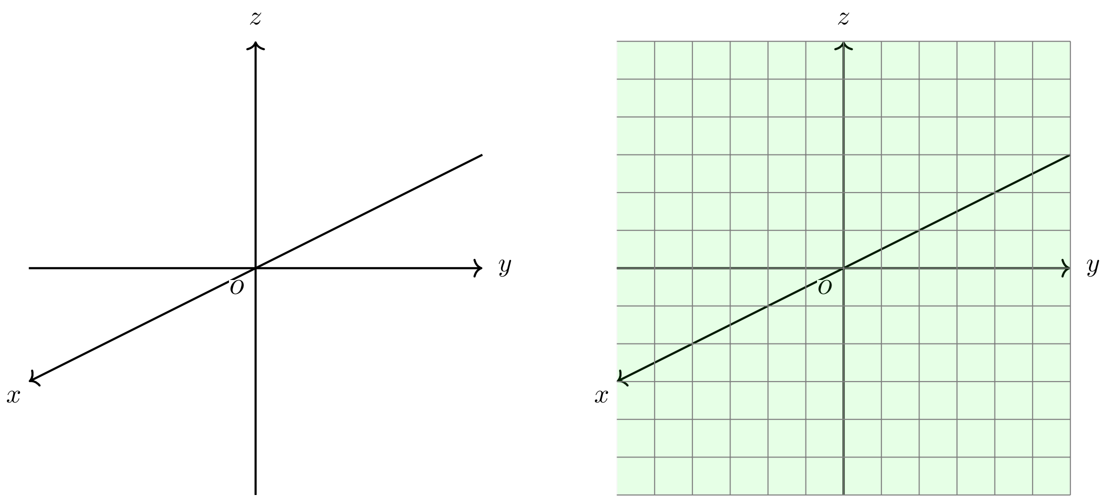
1.2.1 Cartesian Coordinates

If $p \in \mathbb{R}^3$ then **Cartesian coordinates** simply its ordered triple (a, b, c).

One can draw Cartesian coordinates for $\mathbb{R}^3$ in the following way:

1. Pick a point for origin $o = (0, 0, 0)$.
2. Draw three perpendicular axes intersecting at o according to the right-hand rule: label the axis associated to your thumb 'z', the axis associated to your index finger 'x' and the axis associated to your middle finger 'y'.
3. Identify each of these lines with the real numbers $\mathbb{R}$.

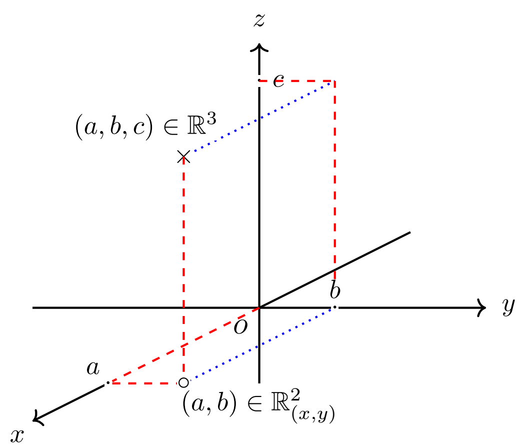
This construction is shown one the left hand side of the following diagram:



To each pair of axes one can associate a plane or $\mathbb{R}^2$. For example the yz-plane is depicted above on the right. One also has a xy-plane and a xz-plane. One then draws the point (a, b, c) in $\mathbb{R}^3$ as follows:

1. one goes a along the x axis.
2. one draws a line parallel to the y-axis in the xy-plane emanating from a and goes directed distance b along this line. This gives the point (a, b) in the xy-plane.
3. From (a, b) in the xy-plane one goes c along a line parallel to the z-axis through (a, b) in the xy-plane.

As usual this is best visualised in a diagram as follows:

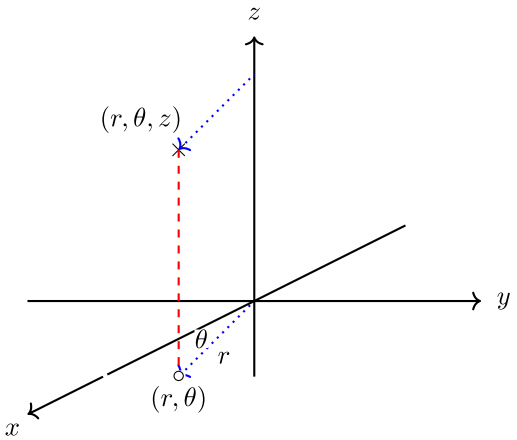


1.2.2 Cylindrical Coordinates

Another set of coordinates on $\mathbb{R}^3$ are cylindrical coordinates, which effectively extend polar coordinates on the xy-plane all of $\mathbb{R}^3$ by using the standard Cartesian z coordinate to complete the triple. More precisely, $p \in \mathbb{R}^3$ is assigned cylindrical coordinates as follows:

1. project p to the xy-plane and assign the usual polar coordinates (r, θ) to the projection of p.
2. z is then the directed distance of p from the xy-plane.
3. p is then given the cylindrical coordinates (r, θ, z).

This can be visualised as follows:



To convert between cylindrical and Cartesian coordinates one uses the relations (15) and (16) with $z = z$.

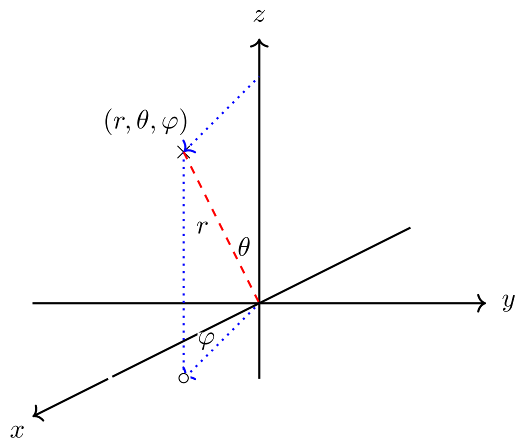
Remark 1.3. *Since polar coordinates on the plane do not cover the origin, cylindrical coordinates for* $\mathbb{R}^3$ *do not cover the* z*-axis. So one can think of cylindrical coordinates mapping* $\mathbb{R}^3 \setminus \{(0, 0, z) : z \in \mathbb{R}\}$ *to* $(0, \infty) \times [0, \pi) \times \mathbb{R}$.

1.2.3 Spherical Coordinates

Spherical coordinates on $\mathbb{R}^3$ are often very convenient for problems with symmetry about a point. They are a natural generalisation of polar coordinates on the plane. In particular, if one has a $p \in \mathbb{R}^3 \setminus \{(0, 0, z) : z \in \mathbb{R}\}$ (the usual remarks about the set $\{(0, 0, z) : z \in \mathbb{R}\}$ apply with spherical coordinates) one assigns spherical coordinates as follows:

1. Let r be the distance from the origin to p. Then one needs two angles (θ, φ) to give a unique representation of the point.
2. Let θ be the angle between the line segment from the origin o to p and the positive z-axis. Thus $\theta \in (0, \pi)$
3. Let φ be the angle in the xy-plane of the projection of p to the xy-plane as measured from the positive x-axis, i.e. the usual polar coordinate. Therefore, $\varphi \in [0, 2\pi)$.

The point $p \in \mathbb{R}^3$ then has polar coordinates (r, θ, φ). This can be drawn as follows:



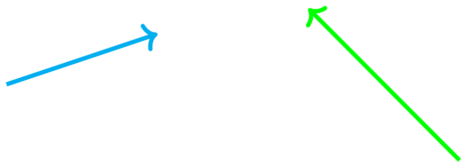
Remark 1.4. *It is very common for authors to make the swap the labelling of θ and φ. Stewart for example uses this convention.*

The relationship between Cartesian coordinates an spherical coordinates is

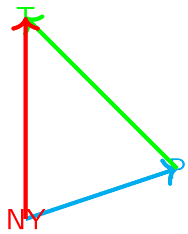
$$x = r \sin\theta \cos\varphi, \qquad y = r \sin\theta \sin\varphi, \qquad z = r \cos\theta. \tag{19}$$

2 Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$

One often associates the term vector to a quantity (often physical, i.e. velocity) that has both a magnitude and direction. This is often depicted with an arrow which has a length that depicts the magnitude of the vector and which has a orientation depicting the direction. For example:

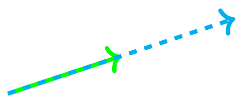


For example, this could represent 'displacement' (a distance one has to travel in a certain direction), i.e. New York to Providence, approximately $x$km ENE. Further, one could travel on to Toronto which you could represent with an arrow of length $y$km from Providence NW. You could then draw a third arrow which is your final displacement from NY to Toronto which would be the sum of the displacement from NY-Providence and the displacement from Providence-Toronto:



This requires a notion of **vector addition**.

Suppose you're driving along a straight section of motorway on your way to Providence. For simplicity, say it runs ENE. So you have some velocity (the directed rate of change of distance over time) that points ENE and has say a length of 100km/h. Suppose someone breaks in front of you and you have to slow down to 50km/h. The velocity that results has the same direction but half the magnitude, i.e. the arrow that one draws would be **scaled** by 50%:



These notions of **scalar multiplication** and **addition** are the key 'vector operations' that must satisfy certain properties. This lecture is about how to make these notions mathematically precise.

One can think about the content of this lecture in the following alternative way: at this stage you're happy to add single numbers $a + b$. How would you add ordered pairs $(a, b) + (c, d)$ of numbers? You're also probably happy to multiply numbers. What about multiplying a pair (a, b) by a number c? Very simply put this lecture is about how to define such operations and what properties these operations have.

2.1 The Algebraic Approach

In lecture 1, $\mathbb{R}^2$ and $\mathbb{R}^3$ were discussed. Recall that $\mathbb{R}^2$, which is the set of all ordered pairs (x, y), and $\mathbb{R}^3$, which is the set of all ordered triples (x, y, z), can be thought of as a plane or as 'ordinary space' respectively. To ease notation one often denotes an element of $\mathbb{R}^2$ (or $\mathbb{R}^3$) with a single boldface letter, i.e. $\mathbf{x} = (x, y)$. Other common notations are an single underlined letter, i.e. $\underline{x} = (x, y)$.[2] One can define notions of addition and 'scalar' multiplication on $\mathbb{R}^n$ as follows. These turn $\mathbb{R}^n$ from a **set** into a **vector space**.

[2] Often authors do not distinguish at all and simply write $x \in \mathbb{R}^2$.

Definition 2.1. *The* **addition** *of ordered pairs* $\mathbf{x}_1 = (x_1, y_1)$, $\mathbf{x}_2 = (x_2, y_2)$ *in* $\mathbb{R}^2$ *is defined by adding corresponding (Cartesian) coordinates, i.e.*

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2). \tag{20}$$

The resultant ordered pair $(x_1 + x_2, y_1 + y_2)$ *is denoted* $\mathbf{x}_1 + \mathbf{x}_2$.

The **scalar multiplication** *of a ordered pair* $\mathbf{x} = (x, y) \in \mathbb{R}^2$ *by a number (here called a scalar)* $\lambda \in \mathbb{R}$ *is defined by multiplying each (Cartesian) coordinate by* λ, *i.e.*

$$\lambda(x, y) = (\lambda x, \lambda y). \tag{21}$$

The resultant ordered pair $(\lambda x, \lambda y)$ *is denoted* $\lambda\mathbf{x}$.

Analogous statements hold for ordered triples in $\mathbb{R}^3$ *(in fact this extends to all* $\mathbb{R}^n$*).*

One can prove that the sets $\mathbb{R}^n$ with addition and scalar multiplication defined above satisfy the following **axioms of a vector space**[3]:

(i) **Commutativity** of addition: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.

(ii) **Associativity** of addition and scalar multiplication: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ and $(ab)\mathbf{v} = a(b\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and $a, b \in \mathbb{R}$.

(iii) The existence of an **additive identity**: there exists $\mathbf{0} \in \mathbb{R}^n$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^n$.

(iv) The existence of an **additive inverse**: for all $\mathbf{x} \in \mathbb{R}^n$ there exists $-\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$. For example for $\mathbb{R}^2$, $-\mathbf{x} = (-x, -y)$.

(v) The existence of a **multiplicative identity**: there exists $1 \in \mathbb{R}$ such that $1\mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^n$.

(vi) **Distributive** properties: $a(\mathbf{v} + \mathbf{u}) = a\mathbf{v} + a\mathbf{u}$ and $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.

(vii) **Compatibility** of scalar multiplication and multiplication in $\mathbb{R}$: $a(b\mathbf{v}) = (ab)\mathbf{v}$ for all $a, b \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}$.

So when we think of $\mathbb{R}^n$ in this manner (i.e. with the notions of addition and scalar multiplication defined as above), $\mathbb{R}^n$ is a vector space and any element of $\mathbf{x} \in \mathbb{R}^n$ is a **vector**. The ordered list $(x_1, x_2, ..., x_n)$ is known as the components of $\mathbf{x}$, i.e. x_1 is the first component of the vector $\mathbf{x}$. Additionally, one can define the **difference** of two vectors $\mathbf{u}, \mathbf{v}$ by

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}). \tag{22}$$

Remark 2.1. *Stewart uses the notation* $\langle x_1, x_2, ..., x_n \rangle$ *to write a vector in its components. This notation could lead to confusion in other classes (especially if you take a class where no boldface is used to distinguish vectors from numbers, i.e.* x *represents vectors rather than* $\mathbf{x}$*): if one is dealing with* $\mathbb{R}^2$ *then* $\langle x_1, x_2 \rangle$ *could mean the 'inner product' of two* <u>*vectors*</u>. *Moreover,* $\langle x_1, x_2, ..., x_n \rangle$ *is common notation for something known as the 'span' of the vectors* $x_1, x_2, ..., x_n$. *In this class you can use the notation* $\langle \rangle$ *notation if you like it or* $()$ *or even* $[]$ *but just be aware that the first is quite uncommon and used for other things (do not use* $\{\}$ *as this is typically set notation).*

One can define lots of additional structure on $\mathbb{R}^n$ (as will be covered in lectures 3 and 4). One useful notation is the length/magnitude/norm of a vector $\mathbf{v}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$:

$$||\mathbf{v}|| \doteq \begin{cases} \sqrt{v_1^2 + v_2^2} & \text{for } \mathbb{R}^2, \\ \sqrt{v_1^2 + v_2^2 + v_3^2} & \text{for } \mathbb{R}^3, \\ \sqrt{\sum_{i=1}^{n} v_i^2} & \text{for } \mathbb{R}^n. \end{cases} \tag{23}$$

[3]To define a vector space in a more abstract sense one needs a notion of vector addition and scalar multiplication such that these axioms are satisfied (see any course on Linear Algebra for more).

The vector space $\mathbb{R}^n$ with the norm defined in equation (23) gives $\mathbb{R}^n$ a **normed space structure**, which can be defined more abstractly. We call a vector $\mathbf{v}$ with norm 1 a unit vector. Any non-zero vector can be **normalised** to give a vector $\hat{\mathbf{v}} = (v_1, v_2, v_3)$ of length 1 by dividing by its norm:

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{||\mathbf{v}||}. \tag{24}$$

There are two vectors that are often distinguished in $\mathbb{R}^2$:

$$\mathbf{e_1} \doteq (1,0) \qquad \mathbf{e_2} \doteq (0,1). \tag{25}$$

These are the standard basis (unit) vectors for $\mathbb{R}^2$. Other common notation is $\mathbf{i}$ and $\mathbf{j}$. In $\mathbb{R}^3$ this generalises to

$$\mathbf{e_1} \doteq (1,0,0), \qquad \mathbf{e_2} \doteq (0,1,0), \qquad \mathbf{e_3} \doteq (0,0,1), \tag{26}$$

or $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. For $\mathbb{R}^n$ with $n \geq 3$ this generalises to

$$\mathbf{e_1} \doteq (1,0,...,0), \qquad \mathbf{e_2} \doteq (0,1,...,0), \quad \dots \quad \mathbf{e_n} \doteq (0,0,...,1). \tag{27}$$

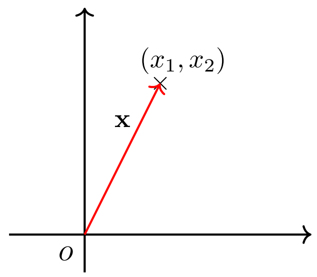
With the rules of vector addition laid out above one can express the vector $\mathbf{v} = (v_1, v_2, v_3)$ as

$$\mathbf{v} = v_1\mathbf{e_1} + v_2\mathbf{e_2} + v_3\mathbf{e_3} \tag{28}$$

in the basis $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$.

2.2 The Geometric Approach/Drawing Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$

As before one can sketch $\mathbb{R}^2$ on a piece of paper or blackboard. When we discussed Cartesian coordinates on the **set** $\mathbb{R}^2$ we represented an ordered pair $\mathbf{x} = (x_1, x_2)$ with a point on the plane. Instead when we think of $\mathbb{R}^2$ as a **vector space** we draw $\mathbf{x}$ as an arrow from the origin of $\mathbb{R}^2$ to the point (x_1, x_2) as shown below:



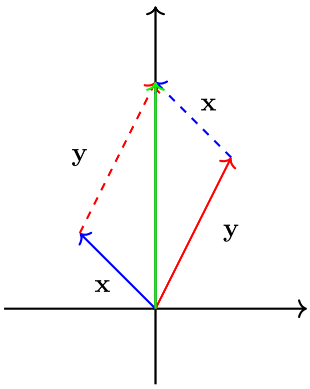
When one thinks of $\mathbf{x}$ as an arrow one calls it a **vector**.

Vector addition in this pictorial sense works as follows. Suppose you have two vectors $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ in $\mathbb{R}^2$. Vector addition gives you the vector $\mathbf{x}+\mathbf{y} = (x_1+y_1, x_2+y_2)$. So you draw an arrow from $\mathbf{0}$ to $\mathbf{x}+\mathbf{y}$. In practise what you can do is:

1. Draw the vector $\mathbf{x}$ (or $\mathbf{y}$) and move $\mathbf{y}$ parallel to itself to place the tail of $\mathbf{y}$ (or $\mathbf{x}$) at the tip of $\mathbf{x}$ (or $\mathbf{y}$).

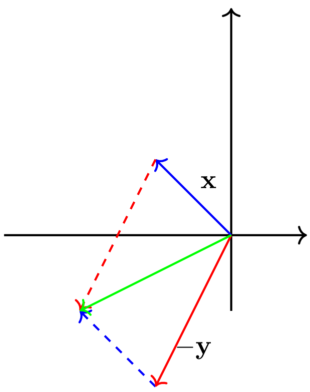
2. $\mathbf{x}+\mathbf{y}$ is then the vector that goes from the tail of $\mathbf{x}$ (or $\mathbf{y}$) to the tip of $\mathbf{y}$ (or $\mathbf{x}$).

Here $\mathbf{x}+\mathbf{y}$ (or $\mathbf{y}+\mathbf{x}$) is drawn in green:



Note that this figure illustrates the commutativity of vector addition.

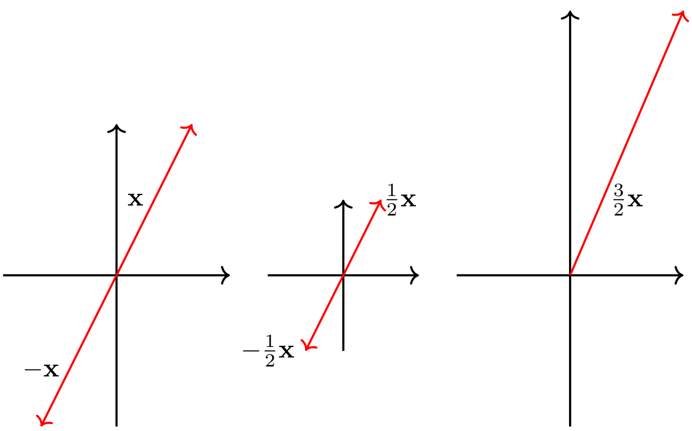
Using the definition of the difference of two vectors $\mathbf{x}$ and $\mathbf{y}$ one can draw $\mathbf{x} - \mathbf{y}$ using the addition prescription above:



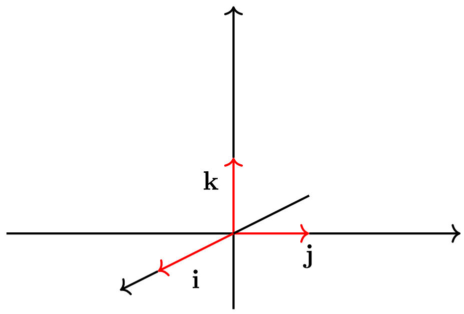
Scalar multiplication of a vector $\mathbf{v} \in \mathbb{R}^2$ by a scalar $a \in \mathbb{R}$ pictorially is carried out as follows:

1. Take the length of $\mathbf{v}$ and multiply it by $|a|$ call this b.
2. If $a > 0$ then $a\mathbf{v}$ is the vector of length b in the direction of $\mathbf{v}$.
3. If $a < 0$ then $a\mathbf{v}$ is the vector of length b in the opposite direction to $\mathbf{v}$.

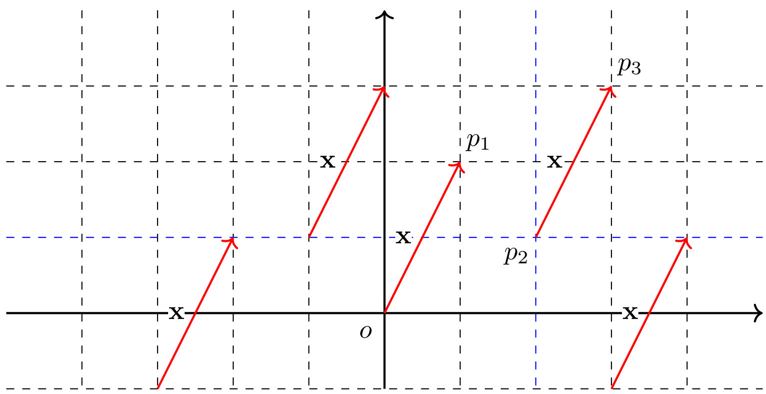
For example, pictorially one has:



All of the above is drawn in $\mathbb{R}^2$ but extends readily to $\mathbb{R}^3$. So that there is an example of drawing in $\mathbb{R}^3$, lets draw the unit basis vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$:



As defined in section 2.1, all vectors emanate from $(0,0)$. However, you may be wondering what about drawing arrows as depicted in the following diagram:



In this diagram the arrow representing the vector $\mathbf{x}$ has been translated without changing its length or direction. In this course we will say that these arrows (which we will again call vectors) are equivalent to (or representations of) the original vector (the one emanating from o).

Remark 2.2. *(Do not worry about this comment too much.) Strictly speaking, the arrows that are drawn from any point other than o belong to different copies of $\mathbb{R}^n$ (when viewed as a vector space). As defined in section 2.1, all vectors emanate from $(0,0)$. However, as represented in blue above one could realign the origin with $p_2 = (0,0)$ and draw $\mathbf{x}$. In technical terms, which you don't need to worry about, you are using something known as the 'affine structure' of $\mathbb{R}^n$. You may see this in further courses on linear algebra.*

One can often come across various terms associated to such vectors. For example, if a particle moves from p_2 to p_3 then it's **displacement vector** is $\mathbf{x}$. Sometimes one will denote this with $\overrightarrow{p_2p_3}$ instead of $\mathbf{x}$. The displacement vector $\overrightarrow{op_1}$ is often distinguished further and called the **position vector** of p_1. If one is given the coordinates of the points $p_3 = (3,3)$ and $p_2 = (2,1)$ then the displacement vector is computed by taking the difference of the position vectors of p_2 and p_3:

$$\mathbf{x} = \overrightarrow{p_2p_3} = (3,3) - (2,1) = (1,2). \tag{29}$$

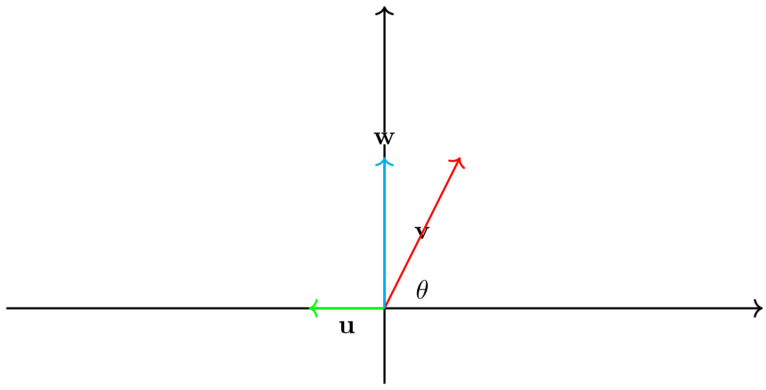
This tells us that to reach p_3 from p_2 one must travel across 1 and up 2. Note that our origin is at p_2 in this case.

2.3 Application: Crossing a River

Considering the following example of a computing velocity vector:

Example 2.1. *Suppose a woman launches a boat from the south shore of a straight river that flows directly west at 4km/h. She wants to land at the point directly across on the opposite shore. If the speed of the boat (relative to the water) is 8km/h, in what direction should she steer the boat in order to arrive at the desired landing point?*

Lets align the x-axis with the south shore of the river and the y-axis pointing across the river meeting the x-axis at the launching point. The boats velocity $\mathbf{v} = 8(\cos\theta, \sin\theta)$. The water velocity is $\mathbf{u} = (-4,0) = -4\mathbf{i}$. We want the resultant velocity $\mathbf{w}$ to be $\mathbf{w} = \omega\mathbf{j}$ for $\omega > 0$.



So,

$$\mathbf{w} = \mathbf{v} + \mathbf{u} = 4(2\cos\theta - 1, 2\sin\theta) = (0, \omega), \tag{30}$$

which has solution $\theta = \frac{2\pi}{3}$.

3 The Scalar Product

On the vector space $\mathbb{R}^n$, one can define even more structure. In particular, one can define a notion of multiplication of two vectors which called the **dot product** or **scalar product**. This is an example of an **inner product** on a vector space. Vector spaces with an inner product (known as inner product spaces) arise everywhere in physics. For example, the mathematical foundations of quantum mechanics are based on the theory of a particular type of inner product spaces called Hilbert spaces.

3.1 Definition and Properties

Definition 3.1 (Scalar Product)**.** *The* **scalar product** *or* **dot product** *of two vectors* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ *is defined by*

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} \doteq \sum_{i=1}^{n} u_i v_i. \tag{31}$$

Example 3.1. *Let's do an example: Suppose one has* $\mathbf{u} = (2, 5, -1)$ *and* $\mathbf{v} = (-3, 1, 0)$ *then one can compute using the formula* (31) *that*

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = 2 \times (-3) + 5 \times 1 + (-1) \times 0 \tag{32}$$
$$= -6 + 5 + 0 = -1. \tag{33}$$

One can see that the scalar product satisfies the following axioms of an inner product:

1. **Symmetry**: $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.
2. **Linearity** in the first argument: $\langle a\mathbf{v} + b\mathbf{w}, \mathbf{u} \rangle = a\langle \mathbf{v}, \mathbf{u} \rangle + b\langle \mathbf{w}, \mathbf{u} \rangle$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and $a, b \in \mathbb{R}$.
3. **Positive semi-definiteness**: $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ for all $\mathbf{u} \in \mathbb{R}^n$ with equality if and only if $\mathbf{u} = 0$.

Remark 3.1. *The terminology 'if and only if' is a logical statement about the equivalence of two statements:*

1. *Statement* A **if** *statement* B *means that* B *implies* A *written* $A \Longleftarrow B$*.*
2. *Statement* A **only if** *statement* B *means that* A *implies* B *written* $A \Longrightarrow B$*.*
3. *Statement* A **if and only if** *statement* B *mean* A *is equivalent to* B *written* $A \Longleftrightarrow B$*.*

Let's check the positive semi-definiteness property (you should check the rest of the above properties):

Proof. One can compute with formula (31) that

$$\langle \mathbf{u}, \mathbf{u} \rangle = \sum_{i=1}^{n} u_i u_i = \sum_{i=1}^{n} u_i^2. \tag{34}$$

So since $u_i^2 \geq 0$, $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$.

$\Longleftarrow$ If $\mathbf{u} = \mathbf{0}$ then $u_i = 0$ for all $i = 1, 2, ..., n$, then $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ from the right-hand side of (34).

$\Longrightarrow$ If $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ then, by the right-hand side of equation (34), $\sum_i u_i^2 = 0$ which implies $u_i^2 = 0$ for all $i = 1, 2, ..., n$. This implies $u_i = 0$ for all $i = 1, ..., n$. If a vectors components vanish then it is the zero vector. □

The scalar product (and indeed any inner product) satisfies the following additional properties:

1. **Linearity** in the second argument: $\langle \mathbf{u}, a\mathbf{v} + b\mathbf{w} \rangle = a\langle \mathbf{u}, \mathbf{v} \rangle + b\langle \mathbf{u}, \mathbf{w} \rangle$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and $a, b \in \mathbb{R}$.
2. $\langle \mathbf{0}, \mathbf{u} \rangle = \langle \mathbf{u}, \mathbf{0} \rangle = 0$.

Note that the right-hand side of equation (34) is $||\mathbf{u}||^2$ where $||\cdot||$ is the norm defined in equation (23). So,

$$||\mathbf{u}|| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}, \tag{35}$$

which is well-defined because $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$.

Remark 3.2. *This is a general property of an inner product: an inner product induces a norm on a vector space.*

This allows one to prove the following proposition:

Proposition 3.1 (Polarisation Identity)**.** *For* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$*,*

$$\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{2}\Big(||\mathbf{u}+\mathbf{v}||^2 - ||\mathbf{u}||^2 - |||\mathbf{v}||^2\Big). \tag{36}$$

Proof. Expand $||\mathbf{u}+\mathbf{v}||^2$ using equation (35) as

$$||\mathbf{u}+\mathbf{v}||^2 = \langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v} \rangle. \tag{37}$$

Using linearity in both arguments

$$||\mathbf{u}+\mathbf{v}||^2 = \langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle. \tag{38}$$

Using equation (35) and symmetry of the scalar product one has

$$||\mathbf{u}+\mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2 + 2\langle \mathbf{u}, \mathbf{v} \rangle. \tag{39}$$

Rearranging gives the result. □

Another useful notion associated to the scalar product is orthogonality of vectors:

Definition 3.2 (Orthogonal Vectors)**.** *Two vectors* $\mathbf{u}, \mathbf{v}$ *in* $\mathbb{R}^n$ *are said to be orthogonal if*

$$\langle \mathbf{u}, \mathbf{v} \rangle = 0. \tag{40}$$

3.2 Angles

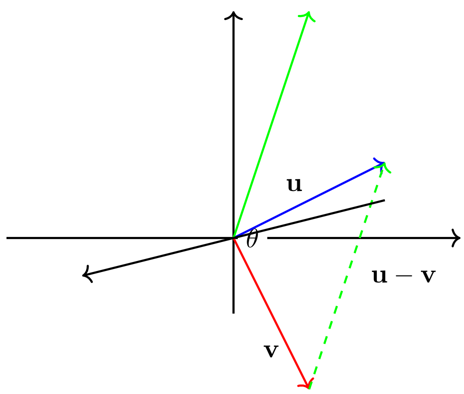
The angle between vectors can be related to their scalar product as follows:

Proposition 3.2. *Let* $\mathbf{u}, \mathbf{v}$ *be vectors in* $\mathbb{R}^n$ *then*

$$\langle \mathbf{u}, \mathbf{v} \rangle = ||\mathbf{u}||||\mathbf{v}|| \cos\theta, \tag{41}$$

where $\theta \in [0, \pi]$ *is the angle between* $\mathbf{u}, \mathbf{v}$*.*

Proof. Lets prove this in $\mathbb{R}^3$. As usual its helpful to draw a picture:



One now has a triangle with sides of length $||\mathbf{u}||$, $||\mathbf{v}||$ and $||\mathbf{u}-\mathbf{v}||$. The law of cosines gives

$$||\mathbf{u}-\mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2 - 2||\mathbf{u}||||\mathbf{v}||\cos\theta. \tag{42}$$

Replacing $\mathbf{v}$ with $-\mathbf{v}$ in the polarisation identity gives

$$\langle \mathbf{u}, \mathbf{v}\rangle = \frac{1}{2}\Big(||\mathbf{u}||^2 + |||\mathbf{v}||^2 - ||\mathbf{u}-\mathbf{v}||^2\Big). \tag{43}$$

Substituting $||\mathbf{u}-\mathbf{v}||^2$ from equation (42) gives the result. □

Remark 3.3. *The same proof works in $\mathbb{R}^2$. In fact this is the essential argument since any two vectors can be thought to lie in a plane in $\mathbb{R}^n$.*

Corollary 3.1. *If $\mathbf{u}, \mathbf{v} \neq 0$ then the angle between $\mathbf{u}, \mathbf{v} \neq 0$ is given by*

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{||\mathbf{u}||||\mathbf{v}||} = \langle \hat{\mathbf{u}}, \hat{\mathbf{v}}\rangle. \tag{44}$$

If $\langle \hat{\mathbf{u}}, \hat{\mathbf{v}}\rangle > 0$ then $\theta \in [0, \frac{\pi}{2})$ with $\theta = 0$ when $\mathbf{v} = c\mathbf{u}$ with $c > 0$. If $\langle \hat{\mathbf{u}}, \hat{\mathbf{v}}\rangle < 0$ then $\theta \in (\frac{\pi}{2}, \pi]$ with $\theta = \pi$ when $\mathbf{v} = c\mathbf{u}$ with $c < 0$. If $\langle \hat{\mathbf{u}}, \hat{\mathbf{v}}\rangle = 0$ then $\theta = \frac{\pi}{2}$ which means our terminology of orthogonality makes sense!

The following is are two **very** useful inequalities:

Proposition 3.3 (Cauchy-Schwarz Inequality)**.** *Let $\mathbf{u}, \mathbf{v}$ be vectors in $\mathbb{R}^n$ then*

$$|\langle \mathbf{u}, \mathbf{v}\rangle| \leq ||\mathbf{u}||||\mathbf{v}||. \tag{45}$$

Proof. By proposition 3.2 one has

$$|\langle \mathbf{u}, \mathbf{v}\rangle| = ||\mathbf{u}||||\mathbf{v}|||\cos\theta|. \tag{46}$$

Now $|\cos\theta| \leq 1$, so

$$|\langle \mathbf{u}, \mathbf{v}\rangle| \leq ||\mathbf{u}||||\mathbf{v}||. \tag{47}$$

□

Proposition 3.4 (Triangle Inequality)**.** *Let $\mathbf{u}, \mathbf{v}$ be vectors in $\mathbb{R}^n$ then*

$$||\mathbf{u}+\mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||. \tag{48}$$

Proof. See the problem sheet. □

3.3 Projections

In some cases you may wish to know how much some vector points along another vector. This leads to the definition of projection.

Definition 3.3 (Projection)**.** *Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Then the* **projection** *(or* **vector projection***) of $\mathbf{u}$ onto $\mathbf{v}$ is given by*

$$\mathrm{proj}_{\mathbf{v}}(\mathbf{u}) = \Pi_{\mathbf{v}}(\mathbf{u}) \doteq \langle \hat{\mathbf{v}}, \mathbf{u}\rangle \hat{\mathbf{v}}. \tag{49}$$

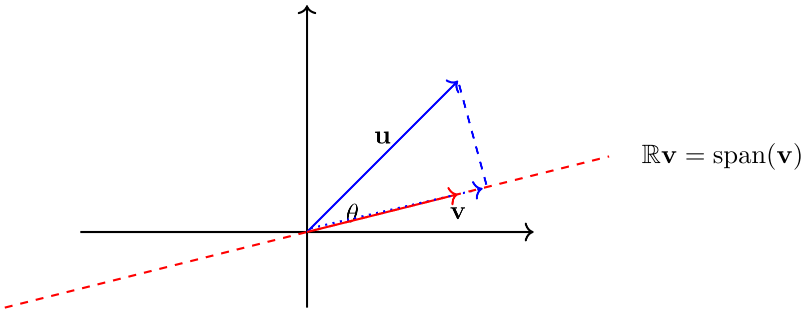
The **component** *(or* **scalar projection***) of $\mathbf{u}$ along $\mathbf{v}$ is given by*

$$\mathrm{comp}_{\mathbf{v}}(\mathbf{u}) \doteq \langle \hat{\mathbf{v}}, \mathbf{u}\rangle. \tag{50}$$

Notice that

$$\text{comp}_{\mathbf{v}}(\mathbf{u}) = ||\mathbf{u}|| \cos\theta \tag{51}$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$ as drawn in the following diagram (the dotted blue arrow is the projection of $\mathbf{u}$ onto $\mathbf{v}$):



Therefore, projection is doing what we want, i.e. it tells you how much some vector points along another vector.

4 The Cross Product

The cross product is a very special operation defined **only** for vectors in $\mathbb{R}^3$. Its usefulness is in the fact that if one is given two vectors $\mathbf{u}$, $\mathbf{v}$, the cross product of these vectors is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

Definition 4.1 (Cross Product)**.** *Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ then the* **cross product** *or* **vector product** *of $\mathbf{u}$ and $\mathbf{v}$ is*

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1). \tag{52}$$

You may be wondering how does one remember this formula. There are a few options. The first is with **determinants**.[4]

Definition 4.2 (Determinant of 2×2 Matrix)**.** *The determinant of a 2×2 matrix,*

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{53}$$

is

$$\det(\mathbf{A}) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \doteq ad - bc. \tag{54}$$

Remark 4.1. *The determinant operation is effectively to multiply across diagonals and then subtract.*

One can extend this definition to 3×3 matrices with the following:

Definition 4.3 (Determinant of 3×3 Matrix)**.** *The determinant of a 3×3 matrix,*

$$\mathbf{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}, \tag{55}$$

is

$$\det(\mathbf{A}) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \doteq a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}. \tag{56}$$

The matrices

$$\mathbf{M}_{1,1} \doteq \begin{pmatrix} e & f \\ h & i \end{pmatrix}, \qquad \mathbf{M}_{1,2} \doteq \begin{pmatrix} d & f \\ g & i \end{pmatrix}, \qquad \mathbf{M}_{1,3} \doteq \begin{pmatrix} d & e \\ g & h \end{pmatrix}, \tag{57}$$

are called the minors and result from removing the first row and the j^{th} ($j = 1, 2, 3$ respectively as the notation suggests) column of the matrix $\mathbf{A}$.

Remark 4.2. *One can generalise further to $n \times n$ matrices but there is no need in this course.*

The following is a method to remember/derive the formula for the cross product:

1. Write $\mathbf{u}$ and $\mathbf{v}$ in terms of the standard basis vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, i.e.

$$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}, \tag{58}$$
$$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}. \tag{59}$$

2. Construct the following matrix:

$$\mathbf{C} = \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix} \tag{60}$$

   and treat $\mathbf{i}, \mathbf{j}, \mathbf{k}$ as elements of a matrix, not vectors.

[4] Determinants crop up in linear algebra. For example a famous result is that if a square matrix has non-zero determinant then it is invertible.

3. Then $\mathbf{u}\times\mathbf{v} = \det(\mathbf{C})$. In otherwords, we can now rewrite the formula for the cross product in terms of the determinant of the 3×3 matrix $\mathbf{C}$:

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}. \tag{61}$$

Explicitly,

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \tag{62}$$

$$= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - v_1u_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}, \tag{63}$$

which is precisely formula (52) when written in the standard basis.

The following alternative is **non-examinable** but may be of interest to some. One can write the components of the cross product with a succinct formula:

$$(\mathbf{u}\times\mathbf{v})_i = \sum_{j=1}^{3}\sum_{k=1}^{3}\varepsilon_{ijk}u_jv_k \tag{64}$$

where ε_{ijk} is known as the (3-dimensional) Levi-Civita symbol which is defined as

$$\varepsilon_{ijk} = \begin{cases} (-1)^p & \text{if} \quad i\neq j\neq k \\ 0 & \text{otherwise} \end{cases} \tag{65}$$

where p is the number of pairwise interchanges of indices necessary to unscramble i, j, k to $1, 2, 3$. So,

$$\varepsilon_{123} = 1, \qquad \varepsilon_{231} = 1, \qquad \varepsilon_{312} = 1 \tag{66}$$

$$\varepsilon_{213} = -1, \qquad \varepsilon_{132} = -1, \qquad \varepsilon_{321} = -1, \tag{67}$$

whilst all other selection of indices vanish ($\varepsilon_{11j} = 0$ etc). Lets compute $(\mathbf{u}\times\mathbf{v})_1$. So,

$$(\mathbf{u}\times\mathbf{v})_1 = \sum_{j=1}^{3}\sum_{k=1}^{3}\varepsilon_{1jk}u_jv_k. \tag{68}$$

Now the only non-zero options for ε_{1jk} are $\varepsilon_{123} = 1$ and $\varepsilon_{132} = -1$. Therefore, the double sum in (68) only has two terms:

$$(\mathbf{u}\times\mathbf{v})_1 = \varepsilon_{123}u_2v_3 + \varepsilon_{132}u_3v_2 \tag{69}$$

$$= u_2v_3 - u_3v_2, \tag{70}$$

which if you compare to formula (52) is the correct result for $(\mathbf{u}\times\mathbf{v})_1$.

4.1 Properties

Proposition 4.1. *Let $\mathbf{u}, \mathbf{v}\in\mathbb{R}^3$ then the cross product $\mathbf{u}\times\mathbf{v}$ of $\mathbf{u}$ and $\mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.*

Proof. Lets compute directly the scalar product

$$\langle\mathbf{u}\times\mathbf{v}, \mathbf{u}\rangle = (u_2v_3 - u_3v_2)u_1 + (u_3v_1 - u_1v_3)u_2 + (u_1v_2 - u_2v_1)u_3. \tag{71}$$

Upon expanding the first and the fourth term cancel. Likewise, the second and the fifth term cancel and the third and the last term cancel. Computing $\langle\mathbf{u}\times\mathbf{v}, \mathbf{v}\rangle$ is completely analogous. □

Proof (**Non-examinable** *).* The formula (64) gives a slick proof that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal to $\mathbf{u} \times \mathbf{v}$. The scalar product of $\mathbf{v}$ and $\mathbf{u} \times \mathbf{v}$ in components is

$$\langle \mathbf{v}, \mathbf{u} \times \mathbf{v} \rangle = \sum_i v_i (\mathbf{u} \times \mathbf{v})_i = \sum_{i,j,k=1}^{3} \varepsilon_{ijk} u_j v_i v_k. \tag{72}$$

Observe that ε_{ijk} is totally antisymmetric. In particular, $\varepsilon_{ijk} = -\varepsilon_{kji}$. Therefore,

$$\langle \mathbf{v}, \mathbf{u} \times \mathbf{v} \rangle = \frac{1}{2} \sum_{i,j,k=1}^{3} \varepsilon_{ijk} u_j v_i v_k - \frac{1}{2} \sum_{i,j,k=1}^{3} \varepsilon_{kji} u_j v_i v_k. \tag{73}$$

You can now relabel the indices $i \leftrightarrow k$ since these both occur on $\mathbf{v}$ and they are summed over to find:

$$\langle \mathbf{v}, \mathbf{u} \times \mathbf{v} \rangle = \frac{1}{2} \sum_{i,j,k=1}^{3} \varepsilon_{ijk} u_j v_i v_k - \frac{1}{2} \sum_{i,j,k=1}^{3} \varepsilon_{ijk} u_j v_k v_i = 0. \tag{74}$$

□

Example 4.1. *Lets do some examples for some practise:*

(a) *Compute the cross product of the standard basis vectors:*

$$\mathbf{i} \times \mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \mathbf{j}. \tag{75}$$

(b) *The cross product of any vector with itself vanishes:*

$$\mathbf{u} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \end{vmatrix} = \mathbf{i}(u_2 u_3 - u_3 u_2) - \mathbf{j}(u_1 u_3 - u_3 u_1) + \mathbf{k}(u_1 u_2 - u_1 u_2) = \mathbf{0}. \tag{76}$$

(c) *Let* $\mathbf{u} = (1, -1, 5)$ *and* $\mathbf{v} = (2, 1, -2)$*. The cross product of* $\mathbf{u}$ *and* $\mathbf{v}$ *is*

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 5 \\ 2 & 1 & -2 \end{vmatrix} = (2-5)\mathbf{i} - (-2-10)\mathbf{j} + (1+2)\mathbf{k} = -3\mathbf{i} + 12\mathbf{j} + 3\mathbf{k}. \tag{77}$$

Proposition 4.2 (Properties of $\times$)**.** *Let* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ *and* $\mathbf{a} \in \mathbb{R}$*. The cross product satisfies the following properties:*

(a) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$.

(b) *Anticommutative:* $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$.

(c) *Distributive over vector addition:* $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$.

(d) *Compatible with scalar multiplication:* $(a\mathbf{u}) \times \mathbf{v} = a(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times (a\mathbf{v})$.

(e) *The* **scalar triple product** $\langle \mathbf{u}, (\mathbf{v} \times \mathbf{w}) \rangle$ *satisfies* $\langle \mathbf{u}, (\mathbf{v} \times \mathbf{w}) \rangle = \langle (\mathbf{u} \times \mathbf{v}), \mathbf{w} \rangle$.

(f) *The* **vector triple product** $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ *satisfies* $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \langle \mathbf{u}, \mathbf{w} \rangle \mathbf{v} - \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{w}$.

Proof. Property a) is proved above in example 4.1. Most are left to you to check yourself. Here is a proof of property f). After computing $\mathbf{v} \times \mathbf{w}$ one has

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (\mathbf{i}(v_2w_3 - v_3w_2) - \mathbf{j}(v_1w_3 - v_3w_1) + \mathbf{k}(v_1w_2 - w_1v_2)) \tag{78}$$

By using the distributive property, the anticommutivity property in conjunction with the distributive property and property a) one has

$$\begin{aligned} \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = & -\mathbf{i} \times \mathbf{j}u_1(v_1w_3 - v_3w_1) + \mathbf{i} \times \mathbf{k}u_1(v_1w_2 - w_1v_2) \\ & + \mathbf{j} \times \mathbf{i}u_2(v_2w_3 - v_3w_2) + \mathbf{j} \times \mathbf{k}u_2(v_1w_2 - w_1v_2)) \\ & + \mathbf{k} \times \mathbf{i}u_3(v_2w_3 - v_3w_2) - \mathbf{k} \times \mathbf{j}u_3(v_1w_3 - v_3w_1). \end{aligned} \tag{79}$$

Using the formulas from example 4.1 part a) and the anticommutative property one has

$$\begin{aligned} \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = & -\mathbf{k}u_1(v_1w_3 - v_3w_1) - \mathbf{j}u_1(v_1w_2 - w_1v_2) \\ & - \mathbf{k}u_2(v_2w_3 - v_3w_2) + \mathbf{i}u_2(v_1w_2 - w_1v_2) \\ & + \mathbf{j}u_3(v_2w_3 - v_3w_2) + \mathbf{i}u_3(v_1w_3 - v_3w_1). \end{aligned} \tag{80}$$

Suggestively collecting terms gives:

$$\begin{aligned} \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = & \, u_1v_1(-\mathbf{k}w_3 - w_2\mathbf{j}) + u_1w_1(v_3\mathbf{k} + v_2\mathbf{j}) \\ & + u_2v_2(-\mathbf{k}w_3 - \mathbf{i}w_1) + u_2w_2(v_3\mathbf{k} + v_1\mathbf{i}) \\ & + u_3w_3(v_2\mathbf{j} + v_1\mathbf{i}) + u_3v_3(-w_2\mathbf{j} - w_1\mathbf{i}). \end{aligned} \tag{81}$$

This can be rewritten as

$$\begin{aligned} \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = & \, u_1v_1(-\mathbf{w} + \mathbf{i}w_1) + u_1w_1(\mathbf{v} - v_1\mathbf{i}) \\ & + u_2v_2(-\mathbf{w} + \mathbf{j}w_2) + u_2w_2(\mathbf{v} - \mathbf{j}w_2) \\ & + u_3w_3(\mathbf{v} - \mathbf{k}v_3) + u_3v_3(-\mathbf{w} + \mathbf{k}w_3) \end{aligned} \tag{82}$$

$$= (u_1w_1 + u_2w_2 + u_3w_3)\mathbf{v} - (u_1v_1 + u_2v_2 + u_3v_3)\mathbf{w}. \tag{83}$$

□

Remark 4.3. *Note that the associative property from ordinary multiplication does* **not** *hold for the cross product, i.e.*

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} \neq \mathbf{u} \times (\mathbf{v} \times \mathbf{w}). \tag{84}$$

For example, $\mathbf{i} \times \mathbf{i} = \mathbf{0}$ *so* $(\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = 0$ *but* $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ *so* $\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = -\mathbf{j}$*. Note that proposition 4.2 and example 4.1 have been used.*

Remark 4.4. *You can check (and you should check) that computing the scalar triple product* $\langle \mathbf{u}, (\mathbf{v} \times \mathbf{w}) \rangle$ *is equivalent to computing the determinant of the following matrix:*

$$\mathbf{M} \doteq \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}. \tag{85}$$

4.2 Geometric Properties of Cross Product

Proposition 4.3. *Let* θ *denote the angle between* $\mathbf{u}$ *and* $\mathbf{v}$*. The norm/length of the vector* $\mathbf{u} \times \mathbf{v}$ *is then given by*

$$||\mathbf{u} \times \mathbf{v}|| = ||\mathbf{u}||||\mathbf{v}|| \sin \theta. \tag{86}$$

Proof. One can show by a direct computation that

$$||\mathbf{u} \times \mathbf{v}||^2 = ||\mathbf{u}||^2 ||\mathbf{v}||^2 - |\langle \mathbf{u}, \mathbf{v} \rangle|^2. \tag{87}$$

From proposition 3.2 one has

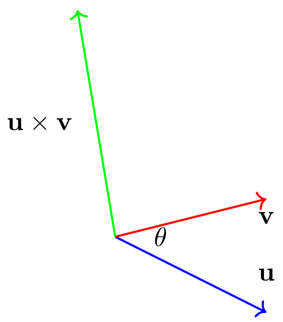
$$||\mathbf{u} \times \mathbf{v}||^2 = ||\mathbf{u}||^2 ||\mathbf{v}||^2 (1 - \cos^2\theta) = ||\mathbf{u}||^2 ||\mathbf{v}||^2 \sin^2\theta. \tag{88}$$

The angle θ is the angle between the two vectors, so $\theta \in [0, \pi]$ and therefore, $\sin\theta \geq 0$. Therefore, its square root is well defined on the reals. Additionally, the norm of a vector is a positive quantity. Therefore, we take the positive branch of the square root to complete the proof. □

Corollary 4.1. *Let* $\mathbf{u}$, $\mathbf{v}$ *be two non-zero vectors.* $\mathbf{u}$, $\mathbf{v}$ *are parallel if and only if* $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

Proof. The vectors $\mathbf{u}$, $\mathbf{v}$ are parallel if and only if $\theta = 0$ or $\theta = \pi$. In either case $\sin\theta = 0$. Therefore, by proposition 4.3, $\mathbf{u}$, $\mathbf{v}$ are parallel if and only if $||\mathbf{u} \times \mathbf{v}|| = 0$. From the properties of the norm $||\mathbf{u} \times \mathbf{v}|| = 0$ if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$. □

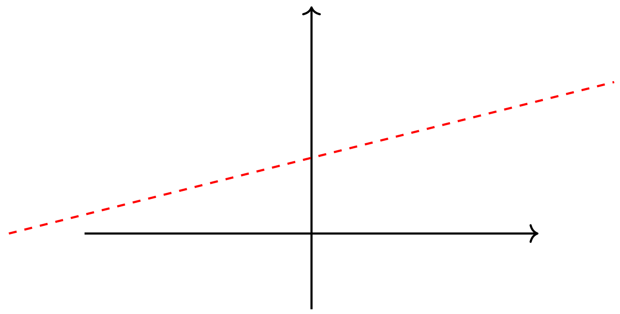
How should you visualise of cross product geometrically? Suppose you have two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$. Then $\mathbf{u} \times \mathbf{v}$ is a vector that points perpendicular to the plane through (spanned by) $\mathbf{u}$ and $\mathbf{v}$. It's direction is determined by the *right-hand rule*. Curl the fingers of your right hand in the direction of the smallest angle from $\mathbf{u}$ to $\mathbf{v}$. The vector $\mathbf{u} \times \mathbf{v}$ is then in the direction of your thumb. This drawn below:



5 Equations of Lines and Planes

5.1 Review: Lines in $\mathbb{R}^2$

Suppose one has a straight line, denoted γ, in the plane $\mathbb{R}^2$ as drawn in red below:



How do you go about describing this mathematically? First of all it's a subset of $\mathbb{R}^2$, i.e. it's a collection of points in the plane. You can determine it uniquely with two points (x_0, y_0) and (x_1, y_1), i.e. the straight line passing through these points is unique. From this you can compute the lines gradient

$$m = \frac{y_1 - y_0}{x_1 - x_0} \tag{89}$$

which is how much you go up for how much you go across. Then equation of a line is

$$y - y_0 = m(x - x_0), \tag{90}$$

which is simply a relation between the x and y coordinates of the points on the line. So γ is the subset:

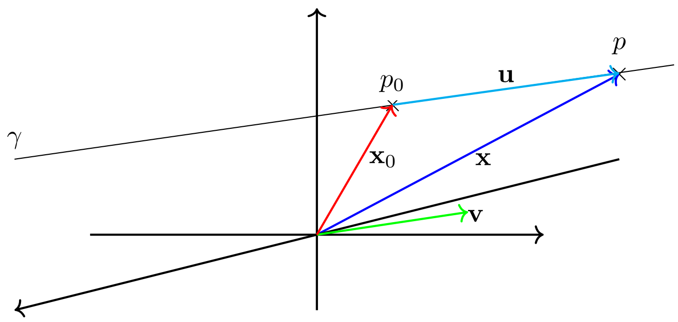
$$\gamma = \Big\{(x, y) \in \mathbb{R}^2 : y - y_0 = m(x - x_0)\Big\}. \tag{91}$$

There is an alternative, one could also describe the line with a point and a direction (which would be a vector in $\mathbb{R}^2$). This is what we are going to do to study lines in $3D$ space $\mathbb{R}^3$ but the same works in $\mathbb{R}^2$ and indeed $\mathbb{R}^n$.

5.2 Lines in $\mathbb{R}^3$

A line γ in $\mathbb{R}^3$ is specified by a point $p_0 \in \mathbb{R}^3$ that γ passes through and a direction for γ. This direction can be described by any vector $\mathbf{v}$ parallel to the line.

The setup is the following: Let $\mathbf{x}_0$ be the position vector of some **known** point p_0 on γ and $\mathbf{x}$ be the position vector of another **arbitrary** point p on γ.[5] Let $\mathbf{u}$ be the vector that points from p_0 to p, i.e. $\mathbf{u} = \overrightarrow{p_0p}$. Finally, let $\mathbf{v}$ be any vector that is parallel to $\mathbf{u}$. This is drawn below:



So from the discussion in lecture 2 one has that the position vector $\mathbf{x}$ is given by

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{u}. \tag{92}$$

[5] I will often drop the distinction between the position vector and the point, I will simply say let $\mathbf{x}_0$ be a point.

Now, since $\mathbf{u}$ is parallel to $\mathbf{v}$, one can express $\mathbf{u} = \lambda\mathbf{v}$ for $\lambda \in \mathbb{R}$. Hence, be written as equation (92) can be written as

$$\mathbf{x} = \mathbf{x}_0 + \lambda\mathbf{v}, \tag{93}$$

which is the **vector equation** of the line. Notice that $\mathbf{x}$ has dependence on the **parameter** λ; for each λ one has a the position vector of a point on γ. So the tip of the position vector traces out γ as λ varies. One can highlight these ideas by writing

$$\mathbf{x}(\lambda) = \mathbf{x}_0 + \lambda\mathbf{v}. \tag{94}$$

Additionally, one can write the above equation in components

$$(x, y, z) = (x_0 + \lambda v_1, y_0 + \lambda v_2, z_0 + \lambda v_3). \tag{95}$$

This gives the set of **parametric equations** for the line γ:

$$x = x_0 + \lambda v_1, \qquad y = y_0 + \lambda v_2, \qquad z = z_0 + \lambda v_3. \tag{96}$$

Note that $\mathbf{v} \neq 0$ otherwise one has a point in $\mathbb{R}^3$. So at least one of v_1, v_2, v_3 is non-vanishing. Suppose it is v_1 then one can solve for λ,

$$\lambda = \frac{x - x_0}{v_1} \tag{97}$$

and replace λ in (96) to give

$$y - y_0 = \frac{x - x_0}{v_1} v_2, \qquad z - z_0 = \frac{x - x_0}{v_1} v_3. \tag{98}$$

Remark 5.1. *One should compare to equation* (90)*, the equation for a line in* $\mathbb{R}^2$*. Note that if one was dealing with lines in the plane then one would simply have*

$$y - y_0 = \frac{v_2}{v_1}(x - x_0) \tag{99}$$

and no z*-equation. So we have precisely equation* (90) *with the gradient* $m = \frac{v_2}{v_1}$*.*

If $v_2 = 0 = v_3$ then the line one is describing is parallel to the x-axis. Suppose additionally, $v_2 \neq 0$ and $v_3 = 0$ then one can

$$\frac{y - y_0}{v_2} = \frac{x - x_0}{v_1}, \qquad z = z_0, \tag{100}$$

which describes a line in the plane $z = z_0$. Finally if $v_2 \neq 0$ and $v_3 \neq 0$ then one can see

$$\frac{y - y_0}{v_2} = \frac{x - x_0}{v_1} = \frac{z - z_0}{v_3}, \tag{101}$$

which is sometimes called the **symmetric equation** for γ.

In practise if you are asked to determine the equation of a straight line in $\mathbb{R}^3$ from two points p_0 at (x_0, y_0, z_0) and p_1 at (x_1, y_1, z_1) you would compute the following:

1. The position vectors of p_0 and p_1 are $\mathbf{x}_0 = (x_0, y_0, z_0)$ and $\mathbf{x}_1 = (x_1, y_1, z_1)$ respectively.

2. The vector $\mathbf{u} = \mathbf{x}_1 - \mathbf{x}_0$ is parallel to the line.

3. Therefore,

$$\mathbf{x}(\lambda) = \mathbf{x}_0 + \lambda\mathbf{u} \tag{102}$$

is the equation of the line or alternatively, using $\mathbf{u} = \mathbf{x}_1 - \mathbf{x}_0$,

$$\mathbf{x}(\lambda) = (1 - \lambda)\mathbf{x}_0 + \lambda\mathbf{x}_1. \tag{103}$$

Notice that $\mathbf{x}(1) = (x_1, y_1, z_1)$ and $\mathbf{x}(0) = (x_0, y_0, z_0)$. So if one limits the values that λ can take one has a description of a **line segment**, i.e. the line segment from p_0 to p_1 is given by

$$\mathbf{x}(\lambda) = (1-\lambda)\mathbf{x}_0 + \lambda\mathbf{x}_1, \qquad 0 \leq \lambda \leq 1. \tag{104}$$

Lets do an example:

Example 5.1. *Let γ_1 be the line passing through the point $(1, -2, 4)$ with parallel vector $\mathbf{v} = (1, 3, -1)$. Let γ_2 be the line passing through the points $(2, 4, 1)$ and $(0, 3, -3)$.*

1. *Determine their vector equations and parametric equations.*
2. *Determine where they intersect the xy-plane.*
3. *Show that they do not intersect and are not parallel.*

1. Denote the position vector $(1, -2, 4)$ as $\mathbf{x}_{0,1}$. So denoting the position vector of an arbitrary point p along γ_1 as $\mathbf{r}_1$, the vector equation for γ_1 is

$$\mathbf{r}_1(\lambda_1) = \mathbf{x}_{0,1} + \lambda_1\mathbf{v}, \tag{105}$$

which can be written as

$$\mathbf{r}_1(\lambda_1) = (1 + \lambda_1, -2 + 3\lambda_1, 4 - \lambda_1). \tag{106}$$

So its parametric equations can be read off as

$$x = 1 + \lambda_1, \qquad y = -2 + 3\lambda_1, \qquad z = 4 - \lambda_1. \tag{107}$$

Now one has the position vectors $\mathbf{x}_{1,2} = (2, 4, 1)$ and $\mathbf{x}_{0,2} = (0, 3, -3)$. So denoting the position vector of an arbitrary point p along γ_2 as $\mathbf{r}_2$ and using equation (103) *one has*

$$\mathbf{r}_2(\lambda_2) = (1 - \lambda_2)\mathbf{x}_{0,2} + \lambda_2\mathbf{x}_{1,2} = (2\lambda_2, 3 + \lambda_2, 4\lambda_2 - 3). \tag{108}$$

So the parametric equations for γ_2 are

$$x = 2\lambda_2, \qquad y = 3 + \lambda_2, \qquad z = 4\lambda_2 - 3. \tag{109}$$

2. To determine where γ_1 intersects the xy-plane one checks what value of λ_1 gives $z = 0$. From above, this means $\lambda_1 = 4$. Therefore, γ_1 intersects the xy-plane at $(5, 10, 0)$. For γ_2 one has $\lambda_2 = \frac{3}{4}$ and $(\frac{3}{2}, \frac{15}{4}, 0)$.

3. To be parallel $\mathbf{v}$ and $\mathbf{u} = \mathbf{x}_{1,2} - \mathbf{x}_{0,2} = (2, 1, 4)$ would have to be proportional: one should be able to solve

$$(1, 3, -1) + a(2, 1, 4) = 0 \tag{110}$$

for a, which gives a contradiction.

If they were to intersect one should be able to find λ_1 and λ_2 such that

$$\mathbf{r}_1(\lambda_1) = \mathbf{r}_2(\lambda_2) \Longleftrightarrow 2\lambda_2 = 1 + \lambda_1, \qquad 3 + \lambda_2 = -2 + 3\lambda_1, \qquad 4\lambda_2 - 3 = 4 - \lambda_1. \tag{111}$$

Solving the first gives

$$\lambda_1 = 2\lambda_2 - 1 \tag{112}$$

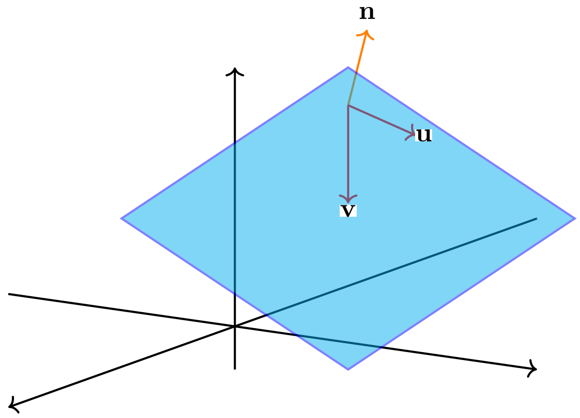
which can be used to eliminate λ_2 in the second and third equations to give:

$$\lambda_2 = \frac{8}{5}, \qquad \lambda_2 = \frac{4}{3} \tag{113}$$

which is inconsistent and therefore, the lines do not intersect.

5.3 Planes

Suppose you have a plane in $\mathbb{R}^3$. How do you describe such and object mathematically? You want to specify a point which is in the plane and how the plane sits in $3D$ space relative to that point. Two non-parallel vectors which lie in the plane would suffice to specify the 'direction' of the plane:



Alternatively, one could work out the planes **normal vector**, $\mathbf{n}$, which is a vector orthogonal to the plane (drawn in orange above). Therefore, any vector $\mathbf{v}$ lying in the plane satisfies

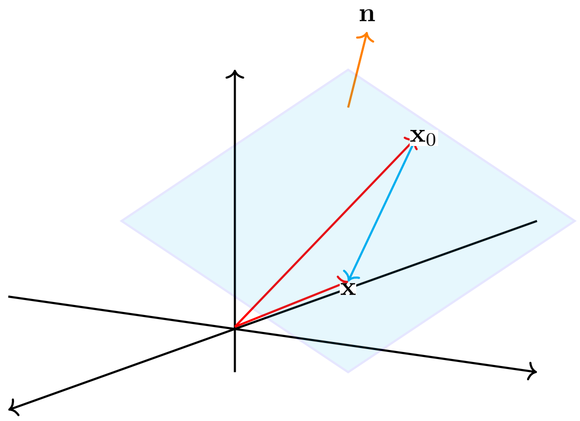
$$\langle \mathbf{n}, \mathbf{v} \rangle = 0. \tag{114}$$

In other words, what this equation is encoding is the $2D$ space of vectors orthogonal to the normal vector, which is precisely the plane you want to describe.

Given any two points p_0 and p in the plane with position vectors $\mathbf{x}_0 = (x_0, y_0, z_0)$ and $\mathbf{x} = (x, y, z)$ respectively. Then $\mathbf{v} = \mathbf{x} - \mathbf{x}_0$ lies in the plane. So,

$$\langle \mathbf{n}, \mathbf{x} - \mathbf{x}_0 \rangle = 0, \tag{115}$$

which is often called the **equation of the plane**. This is drawn below



You can write equation (115) using the definition of the scalar product. Suppose $\mathbf{n} = (n_1, n_2, n_3)$, $\mathbf{x} = (x, y, z)$ and $\mathbf{x}_0 = (x_0, y_0, z_0)$. Then, equation (115) becomes

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0. \tag{116}$$

Remark 5.2. *Stewart likes to call* (115) *the* **vector** *equation of a plane and* (116) *the* **scalar** *equation of the plane. This seems a bit nonsensical. First,* (115) *is just compact notation for* (116)*. Secondly, the object one computes in equation* (115) *is the scalar product, which is a number (in particular* 0*) not a vector!*

Remark 5.3. *The general form of equation* (116) *is*

$$ax + by + cz + d = 0 \tag{117}$$

where we've written $(a, b, c) = (n_1, n_2, n_3)$ *and* $d = -(n_1x_0 + n_2y_0 + n_3z_0)$. *One can view this as the general equation for a plane provided one of* a, b, c *is non-vanishing: Suppose* $a \neq 0$ *(the same argument works with* b *or* c *non-vanishing). Then,*

$$a\left(x + \frac{d}{a}\right) + b(y - 0) + c(z - 0) = 0 \tag{118}$$

which can be rewritten as

$$\langle \mathbf{n}, \mathbf{x} - \mathbf{x}_0 \rangle = 0 \tag{119}$$

with $\mathbf{n} = (a, b, c)$ *and* $\mathbf{x}_0 = (-d/a, 0, 0)$.

Remark 5.4. *It is very common for the normal vector to be 'normalised' to be a unit vector,* $\hat{\mathbf{n}}$.

The **process for finding a normal**:

1. Construct two vectors lying in the plane. Call them $\mathbf{u}$ and $\mathbf{v}$.
2. Recall that the cross product has the following useful property: $\langle \mathbf{u}, \mathbf{u} \times \mathbf{v} \rangle = 0 = \langle \mathbf{v}, \mathbf{u} \times \mathbf{v} \rangle$. So, $\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$ and, therefore, the plane.
3. Any two vectors orthogonal to the plane must be parallel, i.e. the normal is proportional to $\mathbf{u} \times \mathbf{v}$. Picking the constant of proportionality to be 1 is perfectly valid. Therefore,

$$\mathbf{n} = \mathbf{u} \times \mathbf{v}. \tag{120}$$

Two planes are **parallel** if their normal vectors are parallel. If the two planes are not parallel then they intersect in a straight line and the **angle between the planes** is defined as the acute angle between their normal. In particular, if one has two planes with normals $\mathbf{n}_1$ and $\mathbf{n}_2$ respectively then the angle between the planes is

$$\theta = \arccos\big(|\langle \hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2 \rangle|\big) \tag{121}$$

Finally, one can consider the distance of a point $p \in \mathbb{R}^3$ to a plane. Let p_0 be some point in the plane and let $\mathbf{x}_0$ be its position vector. Additionally, let $\mathbf{x}$ be the position vector of p. The displacement vector from p_0 to p is then $\mathbf{v} = \mathbf{x} - \mathbf{x}_0$. Then the distance from p to the plane is the absolute value of the component of $\mathbf{v}$ onto $\mathbf{n}$:

$$d = |\mathrm{comp}_{\mathbf{n}} \mathbf{v}| = |\langle \hat{\mathbf{n}}, \mathbf{v} \rangle|. \tag{122}$$

Example 5.2. *Let's do a prototypical practise problem to illustrate all these ideas:*

1. *Suppose* $\mathbf{u} = (-1, 1, 1)$, $\mathbf{v} = (7, -4, -5)$ *lie in the plane which passes through* $(1, 0, 0)$. *Find its normal and therefore it's equation.*
2. *Find the equation of the plane which passes through the points* $(-1, 1, 1)$, $(1, -2, 2)$, $(4, -3, 0)$.
3. *Find the angle between these planes and equation of the line of intersection of the planes.*

To do 1 we need to find a normal. For this we simply need to compute the cross product of $\mathbf{u}$ *and* $\mathbf{v}$ *to find a vector orthogonal to* $\mathbf{u}$ *and* $\mathbf{v}$:

$$\mathbf{n}_1 = \mathbf{u} \times \mathbf{v} = \left| \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 1 \\ 7 & -4 & -5 \end{pmatrix} \right| = -\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}. \tag{123}$$

Therefore, denoting $\mathbf{x}_0 = (1, 0, 0)$ *one has*

$$\langle \mathbf{n}_1, \mathbf{x} - \mathbf{x}_0 \rangle = 0, \tag{124}$$

which can be expanded as

$$-(x-1) + 2y - 3z = 0 \Longleftrightarrow x - 2y + 3z = 1. \tag{125}$$

For 2 one can do the following. From the position vectors of the points $(-1, 1, 1)$, $(1, -2, 2)$, $(4, -3, 0)$ *one can construct two vectors in the plane:*

$$\mathbf{u} = (5, -4, -1), \qquad \mathbf{v} = (2, -3, 1). \tag{126}$$

Their cross product gives,

$$\mathbf{n}_2 = (-7, -7, -7). \tag{127}$$

Therefore, if $\mathbf{x}_0 = (-1, 1, 1)$ *the equation of the plane is:*

$$-7(x+1) - 7(y-1) - 7(z-1) = 0 \Longleftrightarrow x + y + z = 1. \tag{128}$$

The angle between the planes is

$$\cos\theta = \frac{|-1+2-3|}{\sqrt{42}} = \frac{2}{\sqrt{42}} \implies \theta = \arccos(2/\sqrt{42}) \tag{129}$$

To find the line of intersection you need a point on the line an a vector along that line. Set $z = 0$ *in both plane equations gives:*

$$x + y = 1, \qquad x - 2y = 1 \implies y = 0, \quad x = 1. \tag{130}$$

So the point $(1, 0, 0)$ *lies on the line. Now note that the vector along the line must lie in both planes, i.e. it's perpendicular to both normals. Therefore, one can determine it by computing the cross product of the normals:*

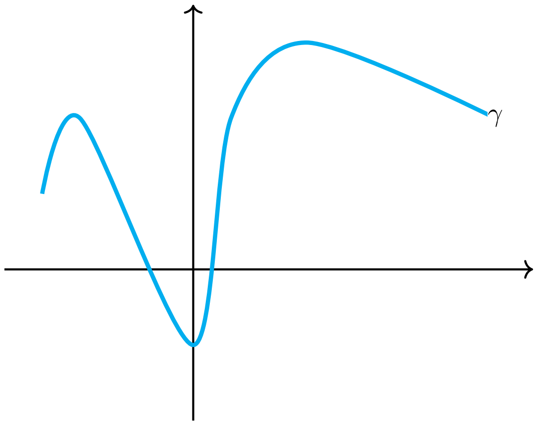
$$\mathbf{w} = \mathbf{n}_1 \times \mathbf{n}_2 = (5, -2, -3). \tag{131}$$

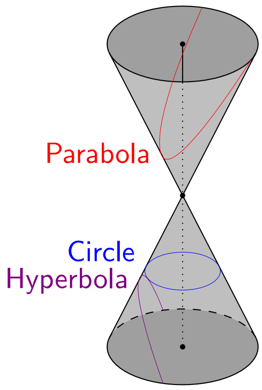
Therefore,

$$\mathbf{x} = (1, 0, 0) + t\mathbf{w}. \tag{132}$$

6 Curves, Conic Sections, Generalised Cylinders and Quardric Surfaces

In last lecture we studied straight lines in $\mathbb{R}^3$. This lecture is about how to write down a mathematical description of lines that are not straight in $\mathbb{R}^2$, i.e. ones that curve in the plane. The first half will cover how to write curves in terms of a parameter and the second half will look at conics. The term conic refers to a particular collection of curves that result from intersecting a plane with a cone: the options are an ellipse (including a circle), a parabola and a hyperbola as drawn below.





6.1 Curves in terms of a Parameter in $\mathbb{R}^2$

In this section we will study how to write down an equation or set of equations for lines that are not straight, i.e. ones that curve, like γ above. Depending on the curve one can sometimes express y as a function of x, i.e. $y = f(x)$. A key restriction when doing this is that a function cannot produce multiple outputs for a given input.[6] What we've drawn above, on the left, is the **graph** of some function f:

$$\mathrm{Graph}(f) = \{(x, f(x)) : x \in I\}, \tag{133}$$

where I is some interval in $\mathbb{R}$. Therefore, the curve γ above would be precisely this set: the graph of $f(x)$.

In many instances curves cannot be represented as the graph of a single function. For example, even something as simple as a unit circle,

$$\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}, \tag{134}$$

cannot be represented as the graph of a single function. In particular, one requires two functions[7]:

$$y = f_\pm(x) = \pm\sqrt{1 - x^2}. \tag{135}$$

One can also abandon trying to write

$$y = f(x)$$

for the curve γ. Instead one can treat x and y independently and allow them to depend on a **parameter**, λ. This involves writing

$$x = f(\lambda), \qquad y = g(\lambda). \tag{136}$$

The equations in (136) are then known as **parametric equations** of the curve γ. Then γ would be the set

$$\gamma = \{(x, y) = (f(\lambda), g(\lambda))\}. \tag{137}$$

The parameter λ may take values in all of the real numbers $\mathbb{R}$ or just some interval I of the real numbers.

[6] This is in the *definition* of a function: a function f is a rule from a set X to a set Y which prescribes to each input $x \in X$ **precisely one** output $y = f(x) \in Y$

[7] You may wonder when a curve can be expressed as more than one function *locally*. This is covered by the implicit function theorem, which morally says that most systems of equations (here the curve $x^2 + y^2 = 1$) can be written *locally* as a graph of a function.

Remark 6.1. *In the previous section, on lines in $\mathbb{R}^3$, we wrote down the parametric equations of lines.*

Returning to the circle example. A circle of radius R centred at (a,b) can be represented with the parametric equations

$$x = a + R\sin(\lambda), \qquad y = b + R\cos(\lambda), \qquad \lambda \in [0, 2\pi). \tag{138}$$

One can sanity check this,

$$(x-a)^2 + (y-b)^2 = R^2\sin^2\lambda + R^2\cos^2\lambda = R^2, \tag{139}$$

which is the equation of a circle of radius R centred at (a,b).

Remark 6.2. *Sometimes the parameter has physical meaning, sometimes it does not. For example, if one was modelling the curve traced out by a particle on a plane then the parameter one could use is time, i.e. $\lambda = t$.*

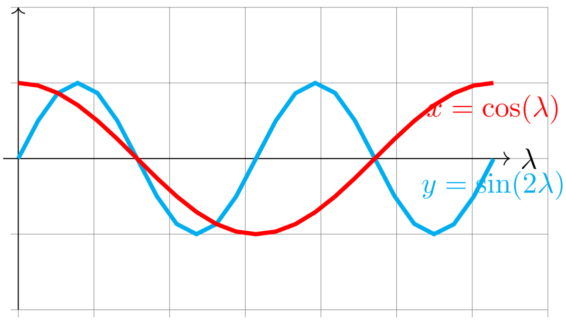
Remark 6.3. *In the circle example the sanity check computation is an example of eliminating the parameter, λ. Be warned that it is not always possible to eliminate the parameter.*

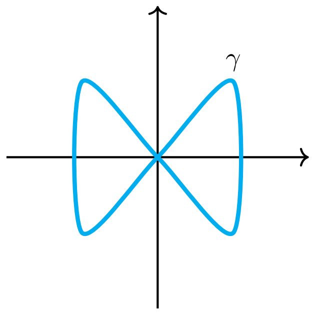
Let's do an example:

Example 6.1. *Consider the curve γ defined by*

$$\gamma = \{(x,y) : x = \cos(\lambda), y = \sin(2\lambda))\}. \tag{140}$$

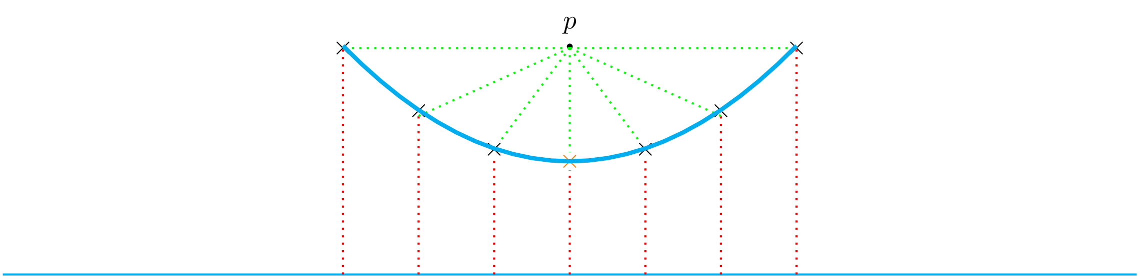
It's useful to plot x and y as functions of λ to help you visualise the curve. This done on the left with γ on the right:





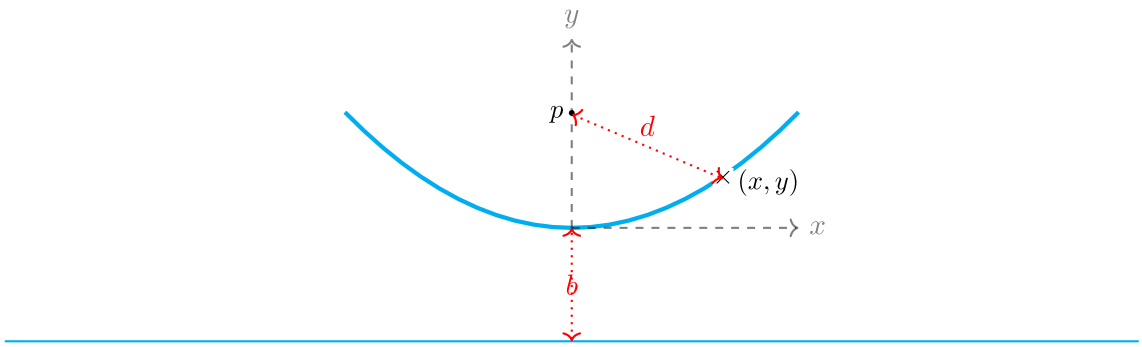
6.2 The Parabola

Suppose you take a line in $\mathbb{R}^2$ and a point p that does not lie on the line:



The curve that lies equidistant from the point and the line is a parabola. The line here is given a special name: **the directrix**. The point p is called the **focus**. The point denoted with a orange cross in the above diagram is called the **vertex**.

Let's work out a simple formula for the parabola by placing the vertex at the origin of our Cartesian coordinates on $\mathbb{R}^2$:



From pythagoras, one has

$$d^2 = (b - y)^2 + x^2. \tag{141}$$

Also, by the definition of the parabola, $|b + y| = d$. Therefore, equating and rearranging gives

$$y = \frac{1}{4b} x^2. \tag{142}$$

So, this is the equation of a parabola with focus at $(0, b)$ and directrix at $y = -b$.

Remark 6.4. *The sign of b controls whether the parabola points up or down, i.e. $b > 0$ then $y \geq 0$, so the parabola points up, $b < 0$ then $y \leq 0$, so the parabola points down.*

As is evident from its drawing the parabola is symmetric with repect to $x \mapsto -x$.

In general, a upright parabola with vertex at (a, c) and focus at $(a, c + b)$ has formula

$$y = \frac{1}{4b}(x - a)^2 + c. \tag{143}$$

6.3 The Ellipse

Take a circle and stretch it, you have an ellipse. The standard equation for an ellipse of height $2b > 0$ and width $2a > 0$ centred at (x_0, y_0) is

$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} = 1. \tag{144}$$

Note that for $a = b$ this is the equation of a circle of radius a^2 centred at (x_0, y_0). For simplicity, let's assume that $(x_0, y_0) = (0, 0)$ and that $a \geq b$. Then there are two distinguished points called **foci** $f_\pm$ at $(\pm c, 0)$ with

$$c = \sqrt{a^2 - b^2}. \tag{145}$$

Let p be some point on the ellipse at (x, y), with $x > 0$, $y > 0$. Then the distance from the foci f_+ at $(c, 0)$ to p is

$$d(p, f_+) = \sqrt{(c - x)^2 + y^2}. \tag{146}$$

Similarly, the distance from the foci f_- at $(-c, 0)$ to p is

$$d(p, f_-) = \sqrt{(x + c)^2 + y^2}. \tag{147}$$

Let's compute $d = d(p, f_+) + d(p, f_-)$. Squaring gives

$$\frac{1}{2} d^2 = x^2 + c^2 + y^2 + \sqrt{(c - x)^2 + y^2}\sqrt{(c + x)^2 + y^2}. \tag{148}$$

Solving for the square-root terms and squaring gives:

$$((c-x)^2+y^2)((c+x)^2+y^2)-(\frac{1}{2}d^2-x^2-c^2-y^2)^2=0, \tag{149}$$

which, using $c^2=a^2-b^2$ and the equation (144) reduces to the polynomial:

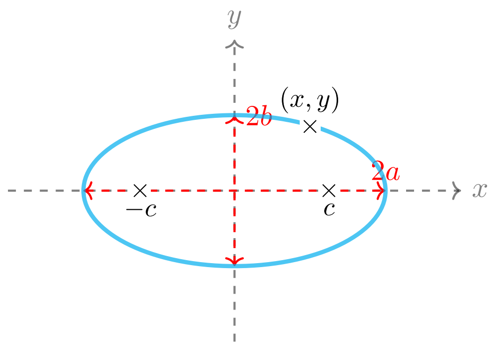
$$\frac{(4a^2-d^2)(4b^2x^2+a^2(d^2-4x^2))}{4a^2}=0. \tag{150}$$

This holds for all $x\in(-a,a)$. Hence, the only way this is possible is if

$$d=2a. \tag{151}$$

This means that the defining feature of an ellipse is that the sum of the distances from each foci to a given point has to be constant (here equal to $2a$).

The line through the foci is called the **(semi)-major axis** and the line perpendicular to this is called the **(semi)-minor axis**. The ellipse and the above discussion is drawn below:



Remark 6.5. *The parametric equations defining the ellipse centred* $(x_0,y_0)=(0,0)$ *with* $a\geq b$ *are*

$$(x,y)=(a\cos(\lambda),b\sin(\lambda)), \qquad \lambda\in[0,2\pi). \tag{152}$$

6.4 The Hyperbola

Recall that defining feature of an ellipse is that the sum of the distances from each foci to a given point has to be constant. The hyperbolas defining feature is that the difference of the distances from each foci to a given point has to be constant, i.e if one returns for the above computation

$$d(p,f_+)-d(p,f_-)=\pm a. \tag{153}$$

If one lets $c=\sqrt{a^2+b^2}$ then the equation for a hyperbola with foci at $(\pm c,0)$ is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1. \tag{154}$$

When $y=0$ then $x=\pm a$. These are the **vertices** of the hyperbola. Also, observe

$$x^2=a^2+\frac{a^2}{b^2}y^2\geq a^2 \implies x\geq a \text{ or } x\leq -a. \tag{155}$$

These are two **branches** of the hyperbola. Finally, note that there is no y-intercept since $y^2=-b^2<0$.

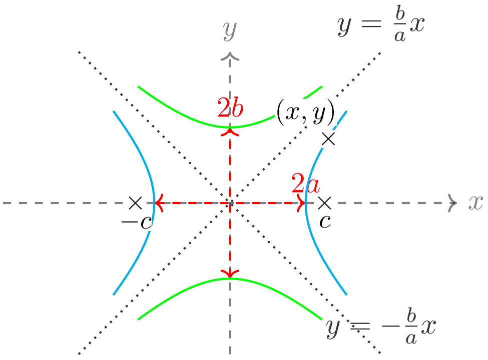
The hyperbola in equation (154) has two asymptotes:

$$y=\pm\frac{b}{a}x. \tag{156}$$

One can reverse the roles of x and y by sending x/a to y/b and visa versa to obtain the equation

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1, \tag{157}$$

which has the same asymptotes and is known as the **conjugate hyperbola**. This is drawn below:



Note that the parametric form of the equation for the hyperbola is

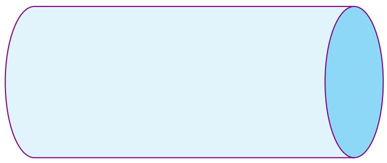
$$x = \pm a\cosh(\lambda), \qquad y = b\sinh(\lambda). \tag{158}$$

6.5 Generalised Cylinders

Up to a rotation and a translation a **cylinder** of radius R is the set of points in $\mathbb{R}^3$:

$$\{(x, y, z) : x^2 + y^2 = R^2\}. \tag{159}$$

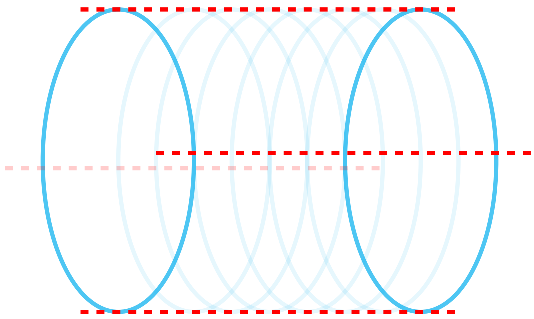
This is drawn below as:



One can think of this is the Cartesian product of the circle times a line:

$$\mathbb{S}^1_R \times \mathbb{R}, \tag{160}$$

where $\mathbb{S}^1_R$ denotes the circle of radius R. In other words, its the surface that results from taking the circle and take all lines that pass through the circle and are parallel to a given line:



One can use this type of construction to consider more general objects called **generalised cylinders** (Stewart simply calls these cylinders but we will make the distinction). More precisely, a generalised cylinder is defined as a surface consisting of all the points on all parallel lines which are orthogonal to a fixed curve in a plane. In particular, given a curve γ in the plane it is the set:

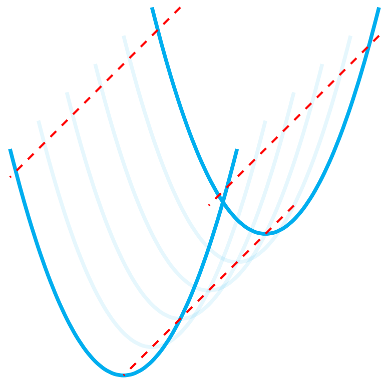
$$\{(x, y, z) : (x, y) \in \gamma, z \in \mathbb{R}\}. \tag{161}$$

A **cylindric section** is the intersection of the cylinder with a plane. If we orientate the generalised cylinder such that the lines of the cylinder are parallel to one of the axes (say the z axis). Then the **cylindric section** that results from intersecting the generalised cylinder with the coordinate planes (or any parallel plane to the a coordinate plane) is called the **cross-section** or trace.

Example 6.2. *Consider the parabola* $\mathbb{P} = \{(x, y) \in \mathbb{R}^2 : y = x^2\}$*. Now take its Cartesian product with* $\mathbb{R}$ *to give*

$$\mathbb{P} \times \mathbb{R} = \{(x, y, z) \in \mathbb{R}^3 : y = x^2\}. \tag{162}$$

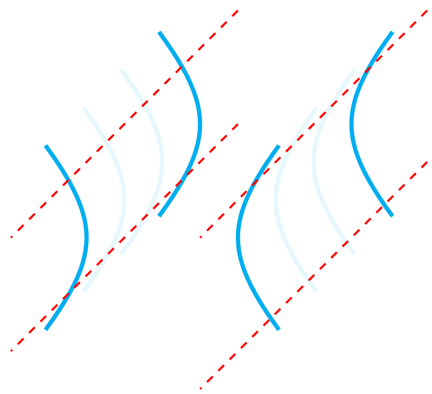
This is a parabolic cylinder, which we can draw as



Example 6.3. *Consider the hyperbola* $\mathbb{H} = \{(x, y) \in \mathbb{R}^2 : x^2 - y^2 = 1\}$*. Now take its Cartesian product with* $\mathbb{R}$ *to give*

$$\mathbb{H} \times \mathbb{R} = \{(x, y, z) \in \mathbb{R}^3 : x^2 - y^2 = 1\}. \tag{163}$$

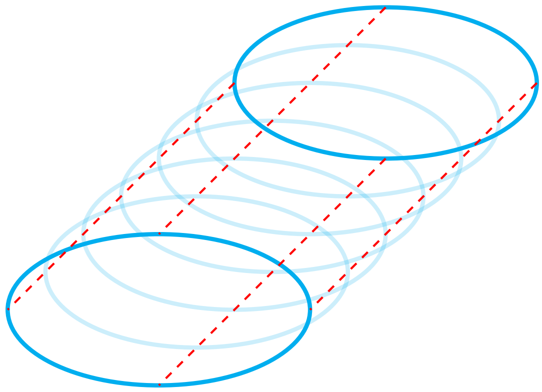
This is a hyperbolic cylinder, which we can draw as



Example 6.4. *Consider the ellipse* $\mathbb{E} = \{(x, y) \in \mathbb{R}^2 : \frac{x^2}{2} + y^2 = 1\}$*. Now take its Cartesian product with* $\mathbb{R}$ *to give*

$$\mathbb{E} \times \mathbb{R} = \{(x, y, z) \in \mathbb{R}^3 : \frac{x^2}{2} + y^2 = 1\}. \tag{164}$$

This is an elliptic cylinder, which we can draw as



6.6 Quadric Surfaces

A **quadric surface**, $\mathcal{Q}$, is a set of points in $\mathbb{R}^3$ which satisfy an equation of the following form:

$$P(x,y,z) \doteq Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0, \tag{165}$$

where $A, ..., J$ are constants, i.e. $\mathcal{Q}$ is the zero set of a quadratic equation in 3 variables.[8] By a rotation or translation one can always bring the equation of the quadric into one of the two following 'standard forms':

$$Ax^2 + By^2 + Cz^2 + D = 0, \qquad Ax^2 + By^2 + Cz = 0, \tag{166}$$

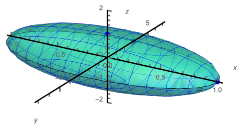
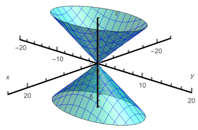
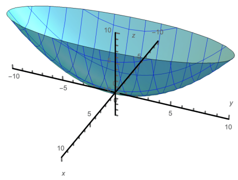
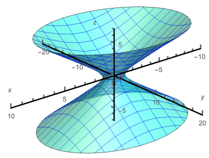
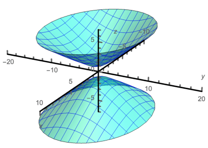
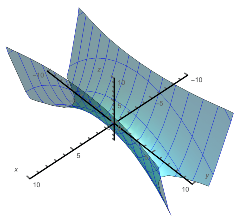
where A, B, C, D are not necessarily those above. Compare these equations to the quadratic/ellipse/hyperbola equations:

$$y = ax^2, \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{167}$$

One can consider these quadrics a generalisation of the conic sections.

In this course we are going to content ourselves with looking at the standard quadrics. These are listed below:

[8] In $\mathbb{R}^n$, $\mathcal{Q}$ is the zero set of a quadratic equation in n variables.

Ellipsoid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	
Cone	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$	
Elliptic Paraboloid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$	
Hyperboloid of 1 Sheet	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$	
Hyperboloid of 2 Sheets	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$	
Hyperbolic Paraboloid	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$	

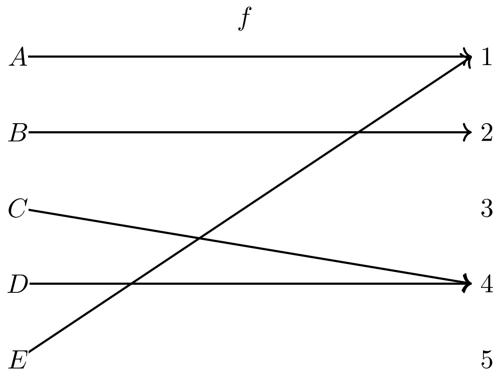
We will come back to a constructive way of plotting these when we consider multivariable functions.

7 Vector-Valued Functions and Curves in Space

A function (or sometimes called a map) is a rule/assignment of each element of one set X to exactly one element of another set Y. This is denoted $f : X \to Y$. The set X is called the functions domain and Y is called the codomain. Often you will see these denoted $X = \mathrm{dom}(f)$ and $Y = \mathrm{codom}(f)$. The term range can be slightly ambiguous since some people use range to refer to the codomain of a function whilst others use it to mean the image, denoted $\mathrm{im}(f)$, of a function which is the set of all possible values in the codomain reached by f:

$$\mathrm{im}(f) \doteq \{f(x) \in Y : x \in \mathrm{dom}(f)\}. \tag{168}$$

Note that the image may not be the whole codomain as demonstrated by this picture:



Here the $\mathrm{dom}(f) = \{A, B, C, D, E\}$, the $\mathrm{im}(f) = \{1, 2, 4\}$ and $\mathrm{codom}(f) = \{1, 2, 3, 4, 5\}$. A functions domain can be specified as part of the definition of a function.

Thus far, you have probably studied functions that map some subset of the real line to another subset of the real line, i.e. $\mathrm{dom}(f) \subseteq \mathbb{R}$ and $\mathrm{im}(f) \subseteq \mathbb{R}$. Such functions typically have a **natural domain** or **domain of definition**,

$$\{x \in \mathbb{R} : f(x) \in \mathbb{R}\}, \tag{169}$$

which is the set of x on which the function produces **real** (in particular, not infinite) values, i.e. where the function is well-defined. In this course, unless otherwise stated, a functions domain will be its natural domain. Here are some examples of functions with various domains and the corresponding images:

$$f(x) = x^2 \qquad \mathrm{dom}(f) = \mathbb{R} \qquad \mathrm{im}(f) = [0, \infty) \tag{170}$$

$$f(x) = x^2 \qquad \mathrm{dom}(f) = [2, 3] \qquad \mathrm{im}(f) = [4, 9] \tag{171}$$

$$f(x) = \frac{1}{x} \qquad \mathrm{dom}(f) = (-\infty, 0) \cup (0, \infty) \qquad \mathrm{im}(f) = (-\infty, 0) \cup (0, \infty) \tag{172}$$

$$f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases} \qquad \mathrm{dom}(f) = \mathbb{R} \qquad \mathrm{im}(f) = \mathbb{R}. \tag{173}$$

In this lecture we will study **vector-valued functions**. These are functions whose image is a subset of $\mathbb{R}^n$, i.e. the function outputs a vector with real valued components. For now we will consider functions that have a domain which is a subset of $\mathbb{R}$.[9] These functions take as an input some real number $x \in \mathrm{dom}(f)$ and output a vector:

$$\mathbf{f}(x) = (f_1(x), f_2(x), f_3(x)) = f_1(x)\mathbf{i} + f_2(x)\mathbf{j} + f_3(x)\mathbf{k}. \tag{174}$$

Note that f_1, f_2, f_3 are called the component functions of $\mathbf{f}$.

Example 7.1. *Consider the vector-valued function*

$$\mathbf{f}(x) = \Big(\frac{1}{x}, x^2, \sin(x)\Big). \tag{175}$$

Then its domain is $\mathbb{R} \setminus \{0\}$ *and its image is* $\mathbb{R} \setminus \{0\} \times \mathbb{R} \times [-1, 1]$.

[9] Later we will consider functions whose domain is a subset of $\mathbb{R}^m$, i.e multivariable functions.

7.1 Review of Limits for Real-Valued Functions

Recall the definition of limits for functions of single variables

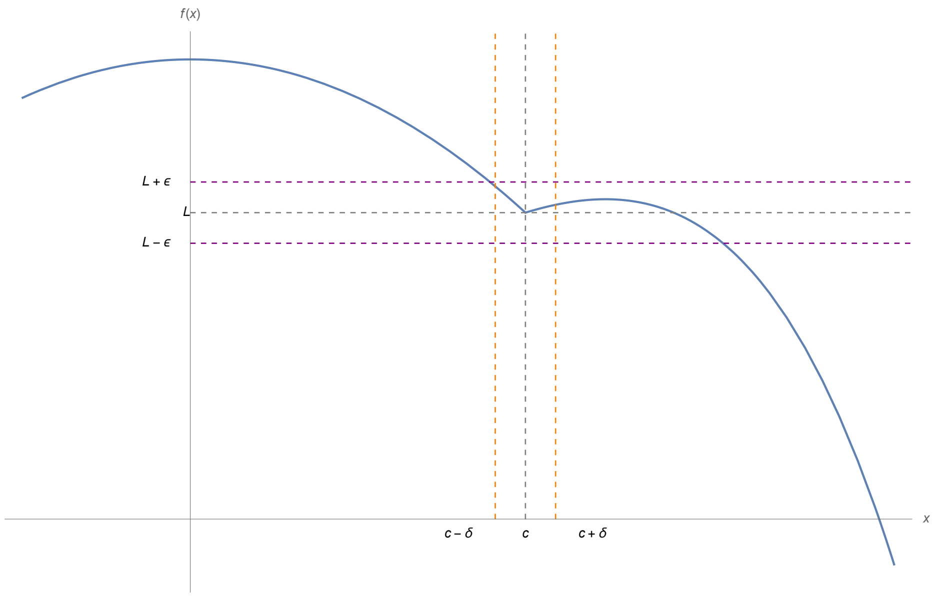
Definition 7.1 (Limits of Scalar Functions)**.** *Let f be a function defined on some open interval $I = (a, b)$, except possibly at some point $c \in I$. Then one says that the limit of f as x tends to c is $L \in \mathbb{R}$ if for all $\epsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - c| < \delta$ and $x \in I$ implies $|f(x) - L| < \epsilon$. In this case one writes*

$$\lim_{x \to c} f(x) = L. \tag{176}$$

What this definition is saying is that L is the limit of f at c if $f(x)$ gets closer and closer to L as x is gets closer and closer to c. The $\epsilon - \delta$ terminology can often come across as confusing. You can think of it in the following way:

- You give me an error, let's call it ϵ.
- I can give you a distance, let's call it δ, away from the point c such that all of the functions values $f(x)$ are within error ϵ away from L when x is within distance δ of the point c.

This is illustrated on the following plot:



Example 7.2. *Let $f(x) = |x|$, let's show that $\lim_{x\to 0} f(x) = 0$.*

For any $\epsilon > 0$ we must show we can find a $\delta > 0$ such that $|f(x) - 0| < \epsilon$ when $|x - 0| < \delta$. So,

$$|f(x) - 0| = |f(x)| = ||x|| = |x| < \delta. \tag{177}$$

So picking $\delta \leq \epsilon$ we have

$$|f(x) - 0| < \epsilon. \tag{178}$$

Example 7.3. *Let $f(x) = x^2 - x + 1$, let's show that $\lim_{x\to 0} f(x) = 1$.*

For any $\epsilon > 0$ we must show we can find a $\delta > 0$ such that $|f(x) - 1| < \epsilon$ when $|x - 0| < \delta$. So,

$$|f(x) - 1| = |x^2 - x| \leq |x^2| + |x| \tag{179}$$

by the triangle inequality. So,

$$|f(x) - 1| \leq |x|^2 + |x| < \delta^2 + \delta \tag{180}$$

So picking $\delta \leq \frac{1}{2}\min(\sqrt{\epsilon}, \epsilon)$ we have

$$|f(x) - 1| < \epsilon. \tag{181}$$

Recall also the definition of one-sided limits:

Definition 7.2 (One-Sided Limits). *Let f be a function defined on some open interval $I = (a, b)$, except possibly at some point $c \in I$. Then one says that the limit from below of f as x tends to c is $L^- \in \mathbb{R}$ if for all $\epsilon > 0$, there exists a $\delta > 0$ such that if $c - \delta < x < c$ then $|f(x) - L^-| < \epsilon$. In this case one writes*

$$\lim_{x \to c^-} f(x) = L^-. \tag{182}$$

Similarly, one says that the limit from above of f as x tends to c is $L^+ \in \mathbb{R}$ if for all $\epsilon > 0$, there exists a $\delta > 0$ such that if $c < x < c + \delta$ then $|f(x) - L^+| < \epsilon$. In this case one writes

$$\lim_{x \to c^+} f(x) = L^+. \tag{183}$$

Recall the useful fact that the limit of definition 7.3 exists if and only if both one sided limits exist <u>and</u> are equal, i.e.

$$\lim_{x \to c} f(x) = L \Longleftrightarrow \lim_{x \to c^-} f(x) = L^- = L^+ = \lim_{x \to c^+} f(x). \tag{184}$$

This is often a good way to show the **limit does not exist**.

Example 7.4. *Define the* $\mathrm{sign} : \mathbb{R} \to \mathbb{R}$ *function*

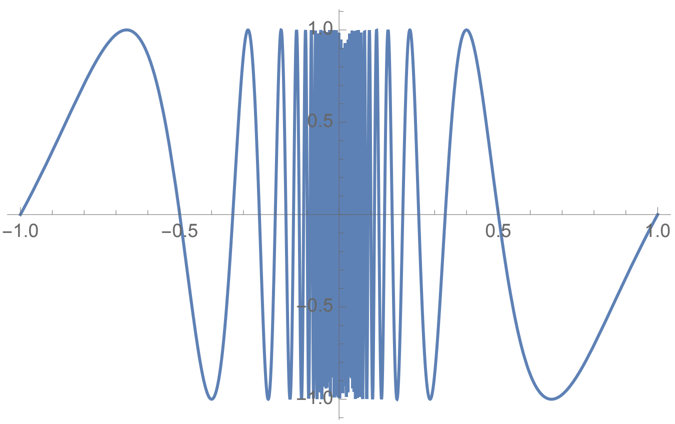
$$\mathrm{sign}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}. \tag{185}$$

Then $\lim_{x \to 0^+} \mathrm{sign}(x) = 1$, $\lim_{x \to 0^-} \mathrm{sign}(x) = -1$. *So,* $\lim_{x \to 0} \mathrm{sign}(x)$ *does not exist.*

What are other ways limits fail to exist? One can have a function that increasingly rapidly oscillates the closer you get to c and therefore it does not approach a fixed number. Such a function would be

$$f(x) = \sin\left(\frac{\pi}{x}\right), \tag{186}$$

which oscillates between 1 and -1 increasingly rapidly as $x \to 0$. This is plotted below:



To see directly that the limit does not exist consider the sequences $x_n = \frac{1}{2n}$ and $x'_n = \frac{1}{2n + \frac{1}{2}}$. Then

$$f(x_n) = 0 \quad \forall n \qquad f(x'_n) = 1 \quad \forall n. \tag{187}$$

Therefore, $\lim_{n \to \infty} f(x_n) = 0$ and $\lim_{n \to \infty} f(x'_n) = 1$ but $x_n \to 0$ and $x'_n \to 0$, i.e. one has a contradiction to $\epsilon = 1/2$ since $|f(x'_n)| = 1 > 1/2$ but n can always be chosen to satisfy $|x'_n| < \delta$ for any $\delta > 0$.

Additionally, a limits fail can to exist if the function diverges, i.e. the function values grow arbitrarily large. For example the function

$$f(x) = \frac{1}{x}. \tag{188}$$

One can introduce the notion of an infinite limit in this case (note we do not think of the limit as existing in this case since $\pm\infty$ is not a number, it is a symbol to denote that a function becomes unbounded):

Definition 7.3 (Infinite Limit of Single Variable Function)**.** *Let f be a function defined on some open interval $I = (a, b)$, except possibly at some point $c \in I$. Then one says that the limit of f as x tends to c is ∞ $(-\infty)$ if for all $M > 0$, there exists a $\delta > 0$ such that if $0 < |x - c| < \delta$ then $f(x) > M$ $(f(x) < -M$ resp.). In this case one writes*

$$\lim_{x\to c} f(x) = \infty \qquad \left(\lim_{x\to c} f(x) = -\infty \quad \textit{resp.}\right) \tag{189}$$

Remark 7.1. *One can generalise the definition of infinite limit to one-sided limits.*

Recall some properties of limits:

Proposition 7.1 (Limit Properties)**.** *Suppose f and g are defined on $I = (a, b)$, except possibly at some point $c \in I$. Further suppose,*

$$\lim_{x\to c} f(x), \qquad \lim_{x\to c} g(x) \tag{190}$$

exist and $k \in \mathbb{R}$. Then

1. $\lim_{x\to c}(f(x) + g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$.
2. $\lim_{x\to c}(f(x) - g(x)) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$.
3. $\lim_{x\to c}(kf(x)) = k \lim_{x\to c} f(x)$.
4. $\lim_{x\to c}(f(x)g(x)) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$.
5. $\lim_{x\to c}({}^{f(x)}/_{g(x)}) = {}^{\lim_{x\to c} f(x)}/_{\lim_{x\to c} g(x)}$ *if* $\lim_{x\to c} g(x) \neq 0$.

Proof. You can find a proof of these statements can be found in Stewart. ☐

Finally let's note some helpful theorems and tips for finding limits. The first is l'Hôpital's rule which is applicable in the '$0/0$' or the '∞/∞' situation:

Theorem 7.1 (L'Hôpital's Rule)**.** *Let I an open interval and $a \in I$. Suppose f and g are differentiable and $g'(x) \neq 0$ on I except possibly at a. Further suppose,*

$$\lim_{x\to a} f(x) = 0 \quad \textit{and} \quad \lim_{x\to a} g(x) = 0 \tag{191}$$

or

$$\lim_{x\to a} f(x) = \pm\infty \quad \textit{and} \quad \lim_{x\to a} g(x) = \pm\infty. \tag{192}$$

Then,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} \tag{193}$$

if the limit on the right-hand side exists.

Example 7.5. *Evaluate the limit*

$$\lim_{x\to 0}\frac{\ln(x+1)}{x}. \tag{194}$$

Using l'Hôpital's rule one has

$$\lim_{x\to 0}\frac{\ln(x+1)}{x}=\lim_{x\to 0}\frac{1}{x+1}=1. \tag{195}$$

Another useful theorem is the 'Squeeze Theorem':

Theorem 7.2 (Squeeze Theorem)**.** *Let I an open interval and $a\in I$. Suppose f, g and h are functions defined on I except possibly at a. Suppose, $f(x)\leq g(x)\leq h(x)$ when $x\neq a$ and*

$$\lim_{x\to a}f(x)=L=\lim_{x\to a}h(x). \tag{196}$$

Then,

$$\lim_{x\to a}g(x)=L. \tag{197}$$

Example 7.6. *Show that*

$$\lim_{t\to 0}t^2\sin\left(\frac{1}{t}\right)=0. \tag{198}$$

Above we noted that $\lim\sin\left(\frac{1}{t}\right)$ does not exist. So one can't just use the product of limits rule. However, for all $t\neq 0$, $\sin(\frac{1}{t})$ is bounded above by 1 and below by -1. So, for all $t\neq 0$,

$$-t^2\leq t^2\sin\left(\frac{1}{t}\right)\leq t^2. \tag{199}$$

Applying the squeeze theorem gives the result.

7.2 Review of Continuity for Real-Valued Functions

Lastly here is a brief review of continuity:

Definition 7.4 (Continuity)**.** *Let f be a function defined on some an open interval $I=(a,b)$ of the real line $\mathbb{R}$. Then one says that f* **is continuous at** *$c\in I$ if*

$$\lim_{x\to c}f(x)=f(c). \tag{200}$$

The function $f:I\to\mathbb{R}$ is continuous if it is continuous at every $c\in I$.

Note that this definition assumes that $c\in\mathrm{dom}(f)$, i.e. $f(c)$ exists. Additionally it assumes that the limit on the left-hand side exists and is equal to $f(c)$.[10]

Example 7.7. *Take the function $f(x)=x$, this has that $\lim_{x\to a}x=a$ and $f(a)=a$. Therefore, it is continuous at $a\in\mathbb{R}$. In fact $f:\mathbb{R}\to\mathbb{R}$ is continuous.*

On the other hand take:

$$f(x)=\begin{cases}0 & x\in(-\infty,0]\\ 1 & x=(0,\infty).\end{cases} \tag{201}$$

Then $f(0)=0$ by definition but $\lim_{x\to 0^+}f(x)=1$ and $\lim_{x\to 0^-}f(x)=0$. So f is not continuous at 0. However, it is continuous everywhere else.

[10] One can unpack this in $\epsilon-\delta$ form, as follows: Let $I=(a,b)\subseteq\mathbb{R}$ and $c\in I$. Let $f:I\to\mathbb{R}$. Then we say that f is continuous at c if, for all $\epsilon>0$, there exists a $\delta>0$ such that for all $y\in I$ with $|y-c|<\delta$, we have $|f(y)-f(c)|<\epsilon$.

Finally let's note that the following proposition:

Proposition 7.2. *The following functions are continuous on all of* $\mathbb{R}$*:* e^t, $\sin$, $\cos$ *and polynomial functions. The function* $\ln(x)$ *is continuous on for* $x > 0$ *and a rational function*

$$R(x) = \frac{P(x)}{Q(x)} \tag{202}$$

is continuous on the set of values for which $Q(x) \neq 0$*.*

Example 7.8. *Where is*

$$f(x) \doteq \frac{t^2 \ln(t)}{t-1} \tag{203}$$

continuous?

It's continuous where its defined: $\ln$ *restricts this to* $t > 0$ *and* $\frac{1}{t-1}$ *restricts this to* $t \neq 1$*. So, its continuous on* $(0, 1) \cup (1, \infty)$*.*

7.3 Vector-Valued Functions

One can give two definitions of limits for vector valued functions. Let's start with the $\epsilon - \delta$ definition:

Definition 7.5 (Limit of a Vector-Valued Function)**.** *Let* $\mathbf{f}$ *be a function defined on some open interval* $I = (a, b)$*, except possibly at some point* $c \in I$*. Then one says that the limit of* $\mathbf{f}$ *as* x *tends to* c *is* $\mathbf{L} \in \mathbb{R}^n$ *if for all* $\epsilon > 0$*, there exists a* $\delta > 0$ *such that if* $0 < |x - c| < \delta$ *and* $x \in I$ *implies* $||\mathbf{f}(x) - \mathbf{L}|| < \epsilon$*. In this case one writes*

$$\lim_{x \to c} \mathbf{f}(x) = \mathbf{L}. \tag{204}$$

Alternatively,

Definition 7.6 (Limit of a Vector-Valued Function)**.** *Let* $\mathbf{f}$ *be a function defined on some open interval* $I = (a, b)$*, except possibly at some point* $c \in I$*. Then the limit of* $\mathbf{f}$ *as* x *tends to* c *is defined as*

$$\lim_{x \to c} \mathbf{f}(x) = \Big(\lim_{x \to c} f_1(x), \lim_{x \to c} f_2(x), \lim_{x \to c} f_3(x) \Big), \tag{205}$$

provided the limits of the component functions exist.

Both of these are natural extensions of the definition of limits for scalar valued functions and are equivalent, which you should try to prove.

Continuity generalises naturally:

Definition 7.7 (Continuity)**.** *Let* $\mathbf{f}$ *be a vector-valued function defined on an interval* $I = (a, b)$ *of the real line* $\mathbb{R}$*. Then one says that* $\mathbf{f}$ **is continuous at** $c \in I$ *if*

$$\lim_{x \to c} \mathbf{f}(x) = \mathbf{f}(c). \tag{206}$$

The function $\mathbf{f} : I \to \mathbb{R}$ *is continuous if it is continuous at every* $c \in I$*.*

7.4 Application: Curves in Space

Suppose $\mathbf{r}$ is a continuous vector-valued function with domain $I = (a, b)$ and let $t \in I$. The set of points (x, y, z) where

$$x = r_1(t), \qquad y = r_2(t), \qquad z = r_3(t) \tag{207}$$

is a curve in $\mathbb{R}^3$. The equations in (207) are known as the parametric equations and t is called the parameter. You can think of of this curve being traced out by the tip of a vector from the origin to $(r_1(t), r_2(t), r_3(t))$.

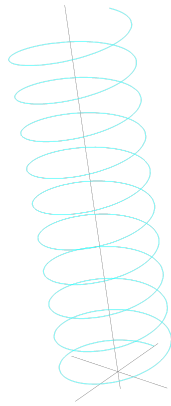
Example 7.9. *Let's sketch the curve*

$$\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + t\mathbf{k}. \tag{208}$$

In the xy-plane the motion is circular since

$$x^2 + y^2 = \cos^2(t) + \sin^2(t) = 1. \tag{209}$$

However, the z component means that the one moves upwards with circular motion, i.e. one has a helix:



7.5 Differentiation

Recall that the idea behind differentiability is the 'rate of change' of a function with respect to its variable(s). For functions of a single variable one has the following definition:

Definition 7.8 (Derivative)**.** *A function f of a single variable x is differentiable at $a \in \mathbb{R}$ if*

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \tag{210}$$

exists. In this case, we define the derivative of f at a, denoted $f'(a)$ or $df/dx(a)$, as

$$f'(a) \doteq \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \quad \text{or} \quad \frac{df}{dx}(a) \doteq \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \tag{211}$$

One can extend this naturally to vector-valued functions:

Definition 7.9 (Derivative of a Vector-Valued Function)**.** *A vector-valued function $\mathbf{f}$ of a single variable x is differentiable at $a \in \mathbb{R}$ if*

$$\lim_{h\to 0} \frac{\mathbf{f}(a+h) - \mathbf{f}(a)}{h}, \tag{212}$$

exists. In this case, we define the derivative of $\mathbf{f}$ at a, denoted $\mathbf{f}'(a)$ or $\frac{d\mathbf{f}}{dx}(a)$, as

$$\mathbf{f}'(a) \doteq \lim_{h\to 0} \frac{\mathbf{f}(a+h) - \mathbf{f}(a)}{h} \quad \text{or} \quad \frac{d\mathbf{f}}{dx}(a) \doteq \lim_{h\to 0} \frac{\mathbf{f}(a+h) - \mathbf{f}(a)}{h}. \tag{213}$$

Recall that the limit of a vector-valued function can be defined through the limits of its components. Therefore, a vector-valued function is differentiable if and only if its components are differentiable, i.e. if $\mathbf{f} = (f_1, f_2, f_3)$ and f_1, f_2, f_3 are differentiable then

$$\frac{d\mathbf{f}}{dx}(a) = \left(\lim_{h\to 0} \frac{f_1(a+h) - f_1(a)}{h}, \lim_{h\to 0} \frac{f_2(a+h) - f_2(a)}{h}, \lim_{h\to 0} \frac{f_3(a+h) - f_3(a)}{h}\right), \tag{214}$$

which is the same as

$$\frac{d\mathbf{f}}{dx}(a) = \Big(\frac{df_1}{dx}(a), \frac{df_2}{dx}(a), \frac{df_3}{dx}(a)\Big) = \frac{df_1}{dx}(a)\mathbf{i} + \frac{df_2}{dx}(a)\mathbf{j} + \frac{df_3}{dx}(a)\mathbf{k}. \tag{215}$$

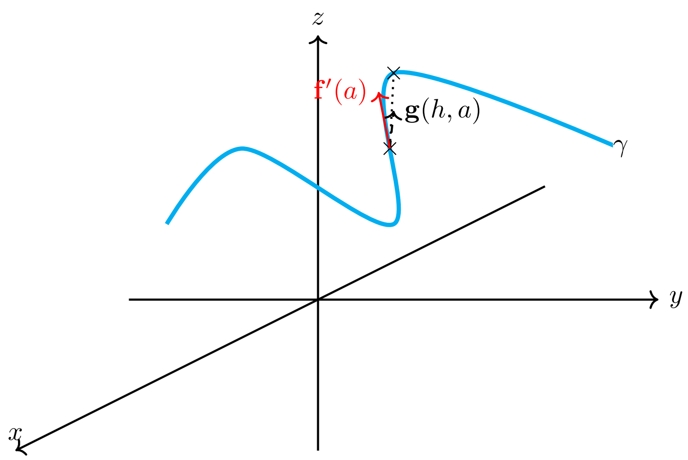
How do we visualise this this definition? Suppose we consider the curve γ defined by a vector-valued function $\mathbf{f}(t)$:

$$\gamma = \{\mathbf{x} = (x, y, z) \in \mathbb{R}^3 : \mathbf{x} = \mathbf{f}(t), t \in (a, b)\}. \tag{216}$$

Let p be the point with position vector $\mathbf{f}(a)$ and q be the point with position vector $\mathbf{f}(a+h)$. Then,

$$\mathbf{g}(h, a) \doteq \frac{1}{h}\big(\mathbf{f}(a+h) - \mathbf{f}(a)\big), \tag{217}$$

is a vector that points in the direction of the displacement vector $\overrightarrow{pq}$. As $h \to 0$, $\mathbf{g}(h, a)$ tends to a vector that is tangent to the curve defined by $\mathbf{f}$ at p. Therefore, $\mathbf{f}'(t)$ is often called the **tangent vector** to the curve defined by $\mathbf{f}$. As usual this is best draw as below:



Remark 7.2. *Just as for scalar functions the n^{th}-derivative of a vector-valued function (if it exists) is the derivative of the $(n-1)^{\text{th}}$ derivative (if it exists), i.e. the second derivative $\mathbf{f}'' = (\mathbf{f}')'$.*

Example 7.10. *Let $\mathbf{r}(t) = (7t^3, (1 - \frac{1}{2}t)e^t, \cosh(t))$.*

1. *Find the derivative of $\mathbf{r}(t)$.*
2. *Find the unit tangent vector at the point with at $t = 0$.*

One can differentiate each component in turn

$$\mathbf{r}'(t) = \Big(21t^2, \frac{1}{2}(1-t)e^t, \sinh(t)\Big). \tag{218}$$

The tangent vector at the point with at $t = 0$ is $\mathbf{r}'(0) = (0, \frac{1}{2}, 0)$. The unit is $\widehat{\mathbf{r}'(0)} = (0, 1, 0)$.

Proposition 7.3. *Let c be a scalar and let h be a real valued function. Suppose the vector-valued functions $\mathbf{f}$ and $\mathbf{g}$ are differentiable in some interval (a, b). Then one has*

1. $\frac{d}{dx}(\mathbf{f}(x) + \mathbf{g}(x)) = \frac{d}{dx}\mathbf{f}(x) + \frac{d}{dx}\mathbf{g}(x)$.
2. $\frac{d}{dx}(c\mathbf{f}(x)) = c\frac{d}{dx}\mathbf{f}(x)$.
3. $\frac{d}{dx}(h(x)\mathbf{f}(x)) = \mathbf{f}(x)\frac{dh}{dx}(x) + h(x)\frac{d}{dx}\mathbf{f}(x)$.
4. $\frac{d}{dx}\langle \mathbf{f}(x), \mathbf{g}(x)\rangle = \langle \frac{d}{dx}\mathbf{f}(x), \mathbf{g}(x)\rangle + \langle \mathbf{f}(x), \frac{d}{dx}\mathbf{g}(x)\rangle$.
5. $\frac{d}{dx}(\mathbf{f}(x) \times \mathbf{g}(x)) = \frac{d}{dx}\mathbf{f}(x) \times \mathbf{g}(x) + \mathbf{f}(x) \times \frac{d}{dx}\mathbf{g}(x)$ *for $\mathbf{f}$ and $\mathbf{g}$ with image in $\mathbb{R}^3$.*

6. *the chain rule:* $\frac{d}{dx}\mathbf{f}(h(x)) = \frac{d\mathbf{f}}{dh}(h(x))\frac{dh}{dx}(x)$.

Proof. The proof of all these statements follows from writing the vector-valued function in terms of components and using the above properties for real-valued functions. □

Proposition 7.4. *Suppose $\mathbf{r}(t)$ has constant norm for all t, then $\frac{d}{dt}\mathbf{r}(t)$ is orthogonal to $\mathbf{r}(t)$ for all t.*

Proof. If $||\mathbf{r}(t)|| = k$ for $k \in \mathbb{R}$ then

$$\langle \mathbf{r}(t), \mathbf{r}(t)\rangle = ||\mathbf{r}(t)||^2 = k^2. \tag{219}$$

Therefore, using proposition 7.3 one has

$$\langle \mathbf{r}'(t), \mathbf{r}(t)\rangle + \langle \mathbf{r}(t), \mathbf{r}'(t)\rangle = 0. \tag{220}$$

The symmetry of the scalar product then gives

$$2\langle \mathbf{r}'(t), \mathbf{r}(t)\rangle = 0, \tag{221}$$

or in other words $\mathbf{r}$ and $\mathbf{r}'$ are orthogonal. □

7.6 Integration

Definition 7.10. *Let $\mathbf{f}(t) = (f_1(t), f_2(t), f_3(t))$ where $f_1(t)$, $f_2(t)$ and $f_3(t)$ are continuous real valued functions on the interval $[a, b]$. Then we define the integral of $\mathbf{f}$ as the vector*

$$\int_a^b \mathbf{f}(x)dx = \Big(\int_a^b f_1(x)dx\Big)\mathbf{i} + \Big(\int_a^b f_2(x)dx\Big)\mathbf{j} + \Big(\int_a^b f_3(x)dx\Big)\mathbf{k}. \tag{222}$$

We now extend the fundamental theorem of calculus to vector-valued functions:

Theorem 7.3. *Let $\mathbf{f} : [a, b] \to \mathbb{R}^n$ be continuous. For $x \in [a, b]$ define the* **antiderivative**

$$\mathbf{F}(x) \doteq \int_a^x \mathbf{f}(y)dy. \tag{223}$$

Then $\mathbf{F}$ is continuous on $[a, b]$ and differentiable with $\mathbf{F}'(x) = \mathbf{f}(x)$ for every $x \in (a, b)$.

Additionally, suppose that $\tilde{\mathbf{F}} : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b) with $\tilde{\mathbf{F}}' = f$ on (a, b) then

$$\tilde{\mathbf{F}}(b) - \tilde{\mathbf{F}}(a) = \int_a^b \mathbf{f}(y)dy. \tag{224}$$

The fundamental theorem of calculus is a very useful theorem in mathematics. For example:

- It tells us that integration and differentiation are essentially inverse operations. If you can find the appropriate $\tilde{F}$ such that $\tilde{F}' = f$ then you can compute the functions integral. This is how we integrate by hand!
- In particular, suppose a function is defined through an integral

  $$g(x) = \int_a^x f(y)dy \tag{225}$$

  for $x \geq a$ and $f(x)$ continuous. The fundamental theorem of calculus tells us this is a well-defined object and we can simply compute from the fundamental theorem of calculus that

  $$g'(x) = f(x). \tag{226}$$

  One can take this further and suppose

  $$g(x) = \int_a^{h(x)} f(y)dy \tag{227}$$

  for $h(x)$ differentiable then by the chain rule and the fundamental theorem of calculus we have:

  $$g'(x) = f(h(x))h'(x). \tag{228}$$

- It allows us to compute areas under curves.
- When trying to solve ordinary differential equation (ODE)

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(t) \tag{229}$$

for a function $\mathbf{x}(t)$ with a given $\mathbf{f}$. The fundamental theorem of calculus tells us

$$\mathbf{x}(t) = \mathbf{x}(0) + \int_0^t \mathbf{f}(s)ds \tag{230}$$

is a solution to this ODE.

Remark 7.3. *Often we will use the notation $\int \mathbf{f}(x)dx$ for an indefinite integral.*

Example 7.11. *Let $\mathbf{r}(t) = (7t^3, (1 - \frac{1}{2}t)e^t, \cosh(t))$.*

1. *Find the indefinite integral of $\mathbf{r}(t)$.*
2. *Find the definite integral from 0 to 1.*

The indefinite integral is computed component-wise but one now must add a vector of constants $\mathbf{c} = (c_1, c_2, c_3)$:

$$\mathbf{F}(x) = \frac{7}{4}x^4\mathbf{i} + \Big(\frac{3}{2} - \frac{1}{2}t\Big)e^t\mathbf{j} + \sinh(t)\mathbf{k} + \mathbf{c}, \tag{231}$$

where we've used integration by parts on $(1 - {}^1\!/\!{}_2t)e^t$. The definite integral is

$$\mathbf{F}(1) - \mathbf{F}(0) = \frac{7}{4}\mathbf{i} + \Big(e - \frac{3}{2}\Big)\mathbf{j} + \sinh(1)\mathbf{k}. \tag{232}$$

7.7 Application: Motion in $\mathbb{R}^3$

Suppose $\mathbf{r}$ is a continuous vector-valued function with domain $I = (a, b)$ and let $t \in I$. Suppose $\mathbf{r}$ models the motion of a particle in space, i.e. its position vector. If $\mathbf{r}(t)$ is differentiable, the particles **velocity** at t, $\mathbf{v}(t)$, is the first derivative of $\mathbf{r}$, i.e.

$$\mathbf{v}(t) = \mathbf{r}'(t). \tag{233}$$

The particles **speed** is the norm of its velocity,

$$s(t) = ||\mathbf{v}(t)||. \tag{234}$$

If $\mathbf{r}(t)$ is twice differentiable, the particles acceleration at t, $\mathbf{a}(t)$, is the second derivative of $\mathbf{r}$, i.e.

$$\mathbf{a}(t) = \mathbf{v}'(t). \tag{235}$$

Let's do some examples:

Example 7.12. *The helix:*

$$\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + t\mathbf{k}. \tag{236}$$

Let's compute $\mathbf{v}(t)$ and $\mathbf{a}(t)$:

$$\mathbf{v}(t) = \mathbf{r}'(t) = -\sin(t)\mathbf{i} + \cos(t)\mathbf{j} + \mathbf{k}, \tag{237}$$
$$\mathbf{a}(t) = \mathbf{r}''(t) = -\cos(t)\mathbf{i} - \sin(t)\mathbf{j}. \tag{238}$$

Example 7.13. *Suppose a particle has acceleration vector*

$$\mathbf{a}(t) = (3t, -2t, 1), \tag{239}$$

with initial velocity $\mathbf{v}(0) = \mathbf{i} + \mathbf{j} - \mathbf{k}$ and position $\mathbf{r}(0) = (0, 1, 0)$. Find its velocity as a function of t.

8 Multivariable Functions I: Introduction and Limits

8.1 Introduction to Multivariable Functions

In lecture 7, we discussed domains, codomains and images of functions. We want to consider 'multivariable functions', i.e. functions that take more than one input.

Definition 8.1. *Let D be a subset of $\mathbb{R}^n$ then a (real-valued) multivariable function f is a rule that assigns to each ordered n-tuple $(x_1, ..., x_n)$ to a real number denoted $f(x_1, ..., x_n)$. In other words, the domain of f, $\mathrm{dom}(f) = D$, is a subset of $\mathbb{R}^n$ and the image of $\mathrm{im}(f)$ is a subset of $\mathbb{R}$. In notation,*

$$f : D \to \mathbb{R}. \tag{240}$$

For us we are going to restrict $n \leq 3$, i.e. functions of two variables $f(x, y)$ and functions of three variables $f(x, y, z)$. For the moment, we are not going to allow for *vector-valued* multivariable functions, i.e. functions with domain in $\mathbb{R}^n$ *and* codomain $\mathbb{R}^m$ for $m > 1$. All functions will map to a subset of the real numbers ($m = 1$). Let's do some examples:

Example 8.1. *Define $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ by*

$$f(x, y) = x^2 + y^2. \tag{241}$$

One could pick $D = [1, 2] \times [1, 10]$. However, its domain of definition is $\mathbb{R}^2$.

Example 8.2. *Define $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ by*

$$f(x, y) = \frac{1}{x^2 + y^2}. \tag{242}$$

This function is well-defined as long as its denominator does not vanish, i.e. as long as $x \neq y \neq 0$. So its domain of definition is $\mathbb{R}^2 \setminus \{(0, 0)\}$.

Example 8.3. *Define $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ by*

$$f(x, y) = \frac{1}{x^2 - y^2}. \tag{243}$$

This function is well-defined as long as its denominator does not vanish, i.e. as long as $x^2 - y^2 \neq 0$. One can factorise to find the set of ill-definition:

$$(x - y)(x + y) = 0 \iff x = y, \quad x = -y. \tag{244}$$

So its domain of definition is

$$\{(x, y, z) \in \mathbb{R}^3 : x \neq y, x \neq -y\}. \tag{245}$$

Example 8.4. *Define $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ by*

$$f(x, y) = \log(x - y). \tag{246}$$

The natural logarithm is well-defined for it's argument in $(0, \infty)$. Therefore, for $f(x, y)$ to be well-defined $x - y > 0$. So, its domain of definition is

$$\{(x, y, z) \in \mathbb{R}^3 : x - y > 0\}. \tag{247}$$

Example 8.5. *Define $f : D \subseteq \mathbb{R}^3 \to \mathbb{R}$ by*

$$f(x, y, z) = \frac{\sqrt{y}}{z} e^{-x^2}. \tag{248}$$

The exponential function and $-x^2$ are well-defined on all of $\mathbb{R}$. The only issues appearing is when $z = 0$ and when $y < 0$ (due to the square-root). So its domain of definition is the set

$$\{(x, y, z) \in \mathbb{R}^3 : y \geq 0, z \neq 0\} \tag{249}$$

(i.e. $\mathbb{R}^3$ without a the $z = 0$ plane and restricted to $y \geq 0$).

There are two groups of functions that are very common:

Definition 8.2 (Polynomial/Rational Functions). *A polynomial function of n-variables $(x_1, ..., x_n)$ is a (finite) sum of terms of the form*

$$cx_1^{m_1}...x_n^{m_n}, \tag{250}$$

for $m_1, ..., m_n \in \mathbb{N}_0$ (the natural numbers including zero, $\mathbb{N}_0 = \{0, 1, 2, 3, 4, ...\}$).

A rational function is a ratio of two polynomials.

Example 8.6. *Two examples of polynomials of two variables are*

$$P(x, y) = x^5 - y^2 + 7, \qquad Q(x, y) = x^7y^3 + y^5 + xy - 3y - 1. \tag{251}$$

An example of a rational function would be the ratio of P and Q,

$$R(x, y) = \frac{P(x, y)}{Q(x, y)} = \frac{x^5 - y^2 + 7}{x^7y^3 + y^5 + xy - 3y - 1}. \tag{252}$$

8.2 Drawing Multivariable Functions: Graphs

To visualise and draw functions we need to extend the idea of graphs:

Definition 8.3. *Let $f : \mathrm{dom}(f) \subseteq \mathbb{R}^n \to \mathbb{R}$ and denote $\mathbf{x} = (x_1, ..., x_n)$ then the graph of f, $\mathrm{Graph}(f)$, is the set*

$$\{(x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} = f(\mathbf{x}), \mathbf{x} \in \mathrm{dom}(f)\} = \{(\mathbf{x}, f(\mathbf{x})) : \mathbf{x} \in \mathrm{dom}(f)\}. \tag{253}$$

Note that this is a subset of $\mathbb{R}^{n+1}$, which in general is not within our capability to visualise. However, if $n \leq 2$ this is possible, i.e. functions of two variables are still within our capability to visualise. These sets can be drawn as surfaces in $\mathbb{R}^3$. Again, lets do some examples

Example 8.7. *Sketch the graph of the function $f(x, y) = 1 - 7x - y$.*

The graph of f is

$$\mathrm{Graph}(f) = \{(x, y, f(x, y)) : (x, y) \in \mathrm{dom}(f)\}. \tag{254}$$

The domain of definition for f is $(x, y) \in \mathbb{R}^2$. The graph is determined by $z = 1 - 7x - y$, which is the equation of a plane. Let's find three points on this plane:

$$z = 0 = y \implies x = \frac{1}{7} \tag{255}$$

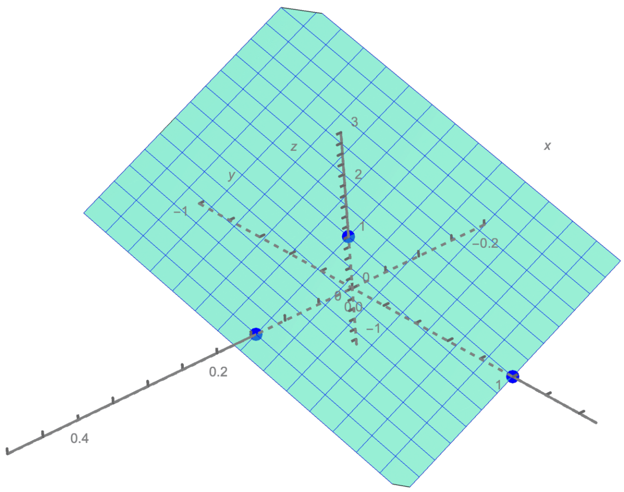
$$z = 0 = x \implies y = 1 \tag{256}$$

$$x = 0 = y \implies z = 1 \tag{257}$$

Therefore, we have three points in $\mathbb{R}^3$ in the plane:

$$\mathbf{x}_1 = (1/7, 0, 0), \qquad \mathbf{x}_2 = (0, 1, 0), \qquad \mathbf{x}_3 = (0, 0, 1). \tag{258}$$

The plot of the graph is then:



Let's look at some of the standard quadrics from lecture 6. Note that sometimes one cannot write them as the graph of a single function but two functions will suffice if necessary.

Example 8.8. *Sketch the graph of the functions* $f_\pm(x,y) = \pm\sqrt{1 - x^2 - \frac{y^2}{10}}$.

The graph of $f_\pm$ *is*

$$\text{Graph}(f_\pm) = \{(x, y, f_\pm(x,y)) : (x,y) \in \text{dom}(f_\pm)\}. \tag{259}$$

The domain of definition for $f_\pm$ *is* $\{(x,y) \in \mathbb{R}^2 : 1 - x^2 - \frac{y^2}{10} \geq 0\}$, *which is the interior (including the boundary) of the ellipse:*

$$x^2 + \frac{y^2}{10} = 1. \tag{260}$$

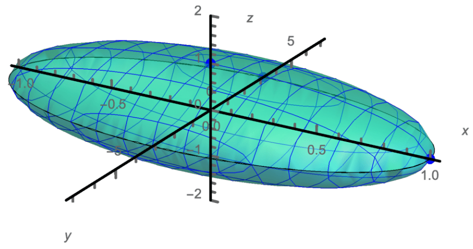
The graph of $f_\pm$ *is determined by* $z = \pm\sqrt{1 - x^2 - \frac{y^2}{10}}$. *Let's find some points on this surface:*

$$x = 1 \implies y = 0, z = 0 \tag{261}$$
$$y = \sqrt{10} \implies x = 0, z = 0 \tag{262}$$
$$z = 0 \implies x^2 + \frac{y^2}{10} = 1 \tag{263}$$
$$x = 0 = y \implies z = \pm 1. \tag{264}$$

Take $z = f_\pm = k_\pm = const.$ *with* $0 < k_+ < 1$, $-1 < k_- < 0$ *then,*

$$\frac{x^2}{1 - k_\pm^2} + \frac{y^2}{10(1 - k_\pm^2)} = 1, \tag{265}$$

which is the equation of an ellipse. So, for k_+ *increasing from* 0 *to* 1 *we have a smaller and smaller ellipse at each* $z = k_+$. *Similarly, for* k_- *decreasing from* 0 *to* -1 *we have a smaller and smaller ellipse at each* $z = k_-$. *The plot of the combined graphs of* $f_\pm$ *is then:*



This is an ellipsoid.

Example 8.9. *Sketch the graph of the functions* $f(x,y) = x^2 + \frac{y^2}{10}$.

The graph of f *is*

$$\mathrm{Graph}(f) = \{(x, y, f(x, y)) : (x, y) \in \mathrm{dom}(f)\}. \tag{266}$$

The domain of definition for f *is* $\mathbb{R}^2$. *The graph of* f *is determined by* $z = x^2 + \frac{y^2}{10}$. *Note that* $z \geq 0$ *and at* $z = 0$ $x = 0 = y$. *Take* $z = f = k = const.$ *with* $0 < k$, *then,*

$$\frac{x^2}{k} + \frac{y^2}{10k} = 1, \tag{267}$$

which is the equation of an ellipse. So, for k *increasing from* 0 *we have a larger and larger ellipse at each* $z = k$.

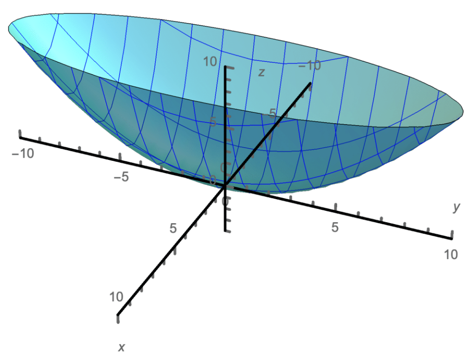
If we take $y = k$ *then*

$$z = \frac{k^2}{10} + x^2 \tag{268}$$

which is the equation of a parabola with variable x. *Similarly, if* $x = k$ *then*

$$z = k^2 + \frac{y^2}{10} \tag{269}$$

which is also a parabola in y. *The plot of the graphs of* f *is then:*



This is an elliptic paraboloid.

Remark 8.1. *The process employed above of considering $z = k$, $y = k$, $x = k$ means we are looking at intersections of the surface with planes parallel to the coordinate planes xy, zx and yz. The curves that result from such intersections are called* **cross-sections***.*

Example 8.10. *Sketch the graph of the function $f(x, y) = x^2 - \frac{y^2}{10}$.*

The graph of f is

$$\mathrm{Graph}(f) = \{(x, y, f(x, y)) : (x, y) \in \mathrm{dom}(f)\}. \tag{270}$$

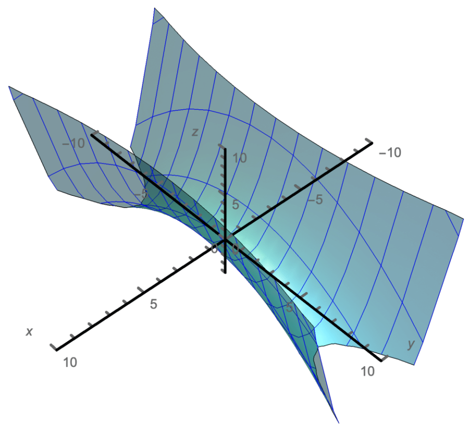
The domain of definition for f is $\mathbb{R}^2$. The graph of f is determined by $z = x^2 - \frac{y^2}{10}$. Take $z = f = k = const.$ with $0 < k$, then,

$$\frac{x^2}{k} - \frac{y^2}{10k} = 1, \tag{271}$$

which is the equation of a hyperbola. Take $z = f = k = const.$ with $0 > k$, then,

$$\frac{y^2}{10|k|} - \frac{x^2}{|k|} = 1, \tag{272}$$

which is also the equation of the conjugate hyperbola. So, for k increasing from 0 we have hyperbola at each $z = k$ and for k decreasing from 0 we have the conjugate hyperbola at each $z = k$. If one looks at $x = k$ and $y = k$ one will find parabola. The plot of the graphs of f is then:



This is an hyperbolic paraboloid.

Example 8.11. *Sketch the graph of the functions $f_{\pm}(x, y) = \pm\sqrt{-1 + x^2 + \frac{y^2}{2}}$.*

The graph of f_+ is the set

$$\{(x, y, z) : z = \sqrt{\frac{y^2}{2} + x^2 - 1}\}. \tag{273}$$

The graph of f_- is the set

$$\{(x, y, z) : z = -\sqrt{\frac{y^2}{2} + x^2 - 1}\}. \tag{274}$$

Let's look at cross-sections $f_{\pm}$:

1. *If* $z = k$ *then for* f_+ *we have*

$$f_+(x, y) = \sqrt{\frac{y^2}{2} + x^2 - 1} = k. \tag{275}$$

For there to be a solution $k \geq 0$. *If* $k = 0$, *then*

$$\frac{y^2}{2} + x^2 = 1 \tag{276}$$

which is an ellipse. If $k > 0$ *then*

$$\frac{y^2}{2(k^2+1)} + \frac{x^2}{k^2+1} = 1 \tag{277}$$

which is an ellipse with increasing size. So with increasing z *we have ellipses with increasing size from* $z = 0$.

If we now look look at f_- *we have*

$$f_-(x, y) = -\sqrt{\frac{y^2}{2} + x^2 - 1} = k. \tag{278}$$

for solution $k \leq 0$ *with* $\frac{y^2}{2} + x^2 = 1$ *when* $k = 0$. *Now as* k *decreases we have ellipses with equation:*

$$\frac{y^2}{2(k^2+1)} + \frac{x^2}{k^2+1} = 1. \tag{279}$$

Therefore, down the z *axis we have ellipses of increasing size.*

2. $x = k$:

$$z^2 = \frac{y^2}{2} + k^2 - 1 \tag{280}$$

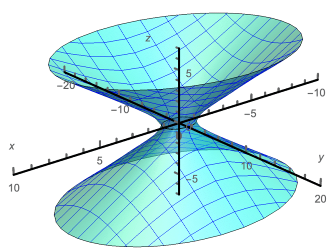
which can be written as

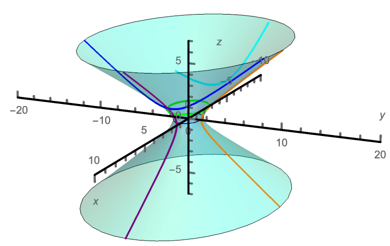
$$z^2 - \frac{y^2}{2} = k^2 - 1. \tag{281}$$

If $k = \pm 1$ *then* $z = \pm\frac{\sqrt{|y|}}{2}$ *so we have two lines in the* $z = \pm 1$ *planes. If* $k^2 - 1 \neq 0$ *then we can write*

$$\frac{z^2}{|k^2-1|} - \frac{y^2}{2|k^2-1|} = \text{sign}(k^2-1) = \begin{cases} 1 & k^2 - 1 > 0 \\ -1 & k^2 - 1 < 0 \end{cases} \tag{282}$$

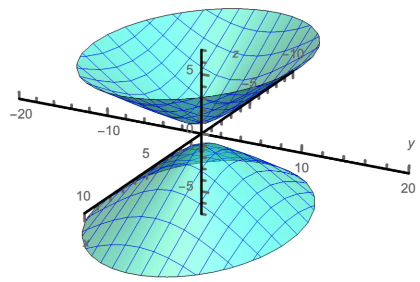
3. $y = k$ *is completely analogous to* $x = k$.





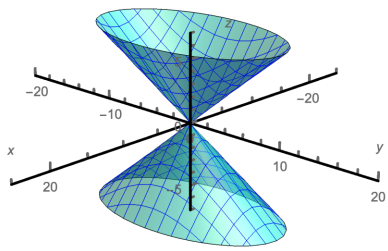
This is an hyperbola of one sheet.

Example 8.12. *Sketch the graph of the functions* $f_{\pm}(x,y) = \pm\sqrt{1 + x^2 + \frac{y^2}{2}}$.



This is an hyperbola of two sheets.

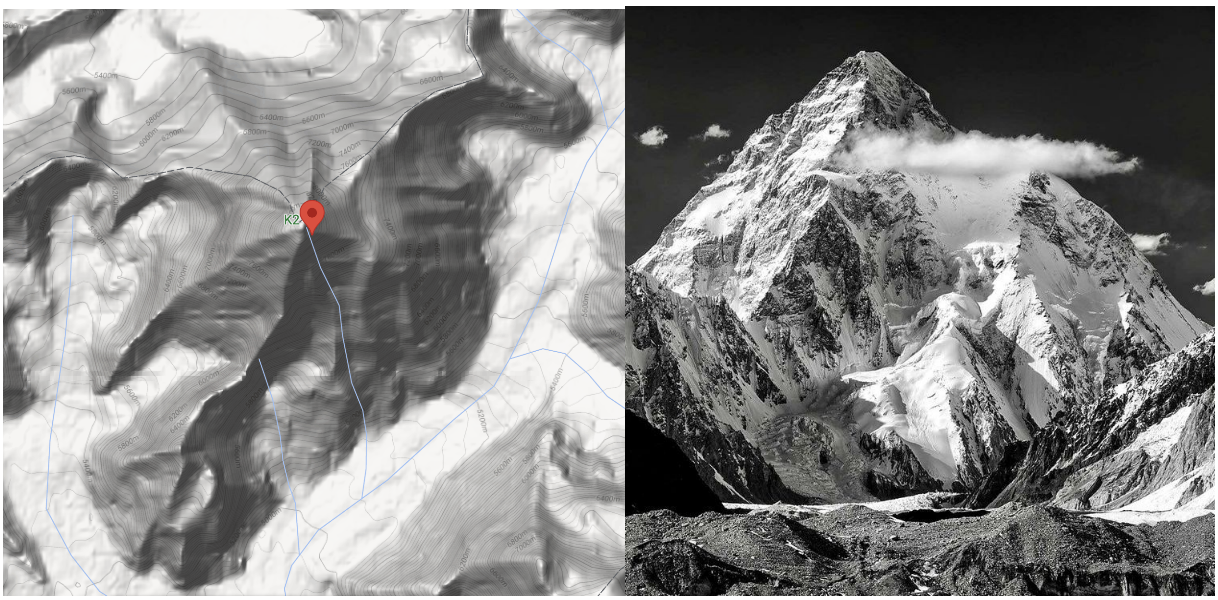
Example 8.13. *Sketch the graph of the functions* $f_{\pm}(x,y) = \pm\sqrt{x^2 + \frac{y^2}{2}}$.



This is a (elliptic) cone.

8.3 Drawing Multivariable Functions: Level Sets/Surfaces/Curves

So far we have been ploting graphs of functions. There is an alternative: level sets or curves. You've probably encountered such drawings in maps: contour/topographic maps are drawings of level curves of some region on Earth. Here is the topographic map of K2 (left) vs the plot/picture of its graph (right):



The topographic map marks different contant height levels on the map with the grey lines: these are the level sets or level curves.

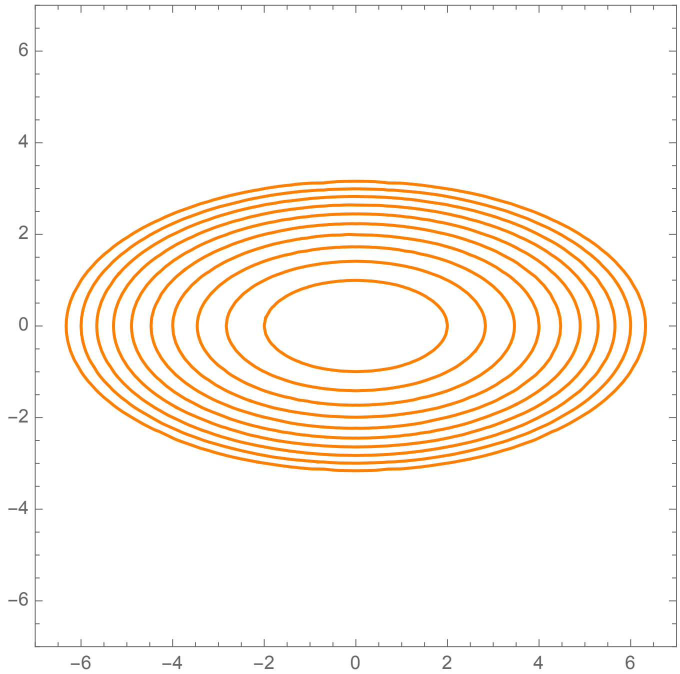
Definition 8.4. *The level sets of a function f of n-variables $(x_1, ..., x_n)$ are the sets*

$$\{(x_1, ..., x_n) : f(x_1, ..., x_n) = k\} \tag{283}$$

for k constant. If f is a function of two variables we call these level curves. If f is a function of three variables we call these level surfaces.

Example 8.14. *Sketch the level curves of the function $f(x, y) = \frac{x^2}{4} + y^2$.*

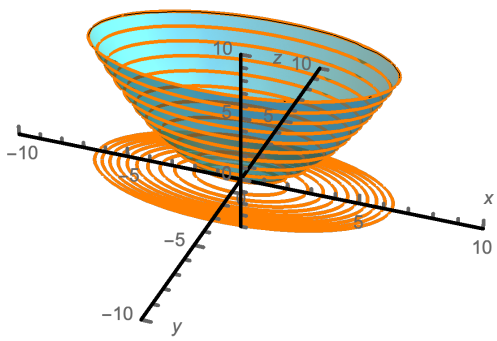
We look at the curves $\frac{x^2}{4} + y^2 = k$ for k constant. For a solution, we must have $k \geq 0$. If $k = 0$ then this is the point $(x, y) = (0, 0)$. If $k > 0$, these are ellipses, which are plotted as follows:



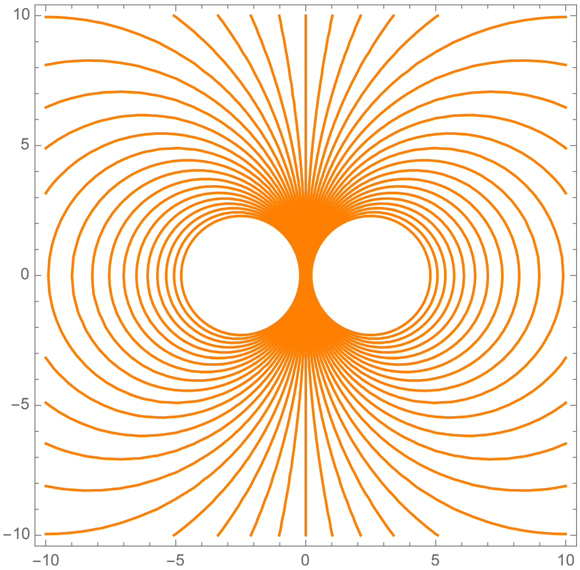
Lets view k as a z coordinate then $f(x, y) = k$ can be rewritten as

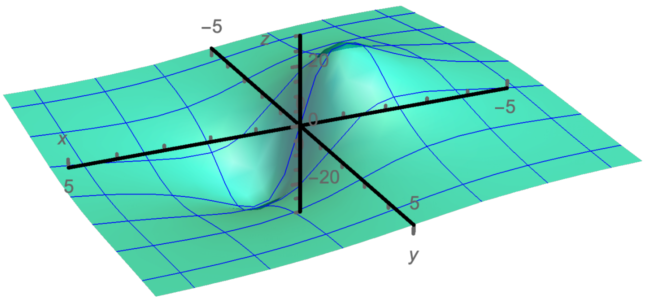
$$\frac{x^2}{4} + y^2 - z = 0 \tag{284}$$

which is the equation of a elliptic paraboloid. One can view the contour plot in the plane as a projection of the curves resulting from intersection the elliptic paraboloid with planes $\{z = k\}$, i.e. the z cross-sections. This is drawn below:

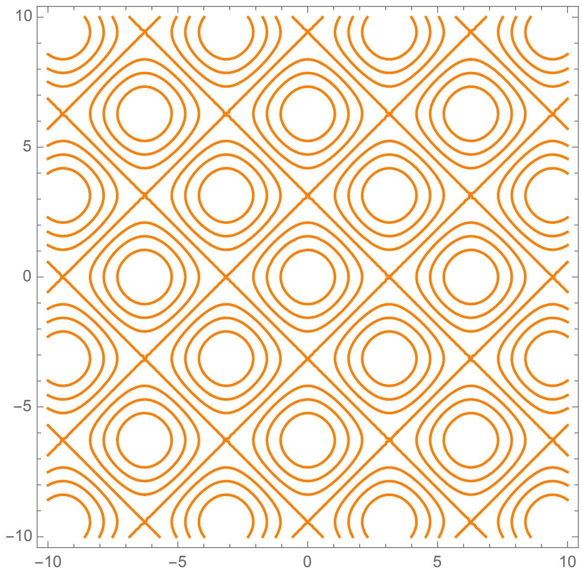


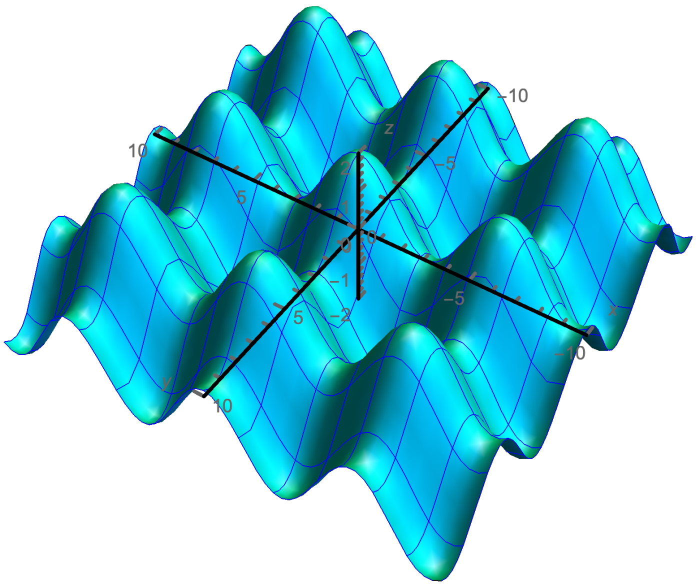
Example 8.15. *The following is a computer generated sketch of the level curves of the function* $f(x,y) = -\frac{50x}{x^2+y^2+1}$ *with the corresponding surface plot of* $(x, y, f(x, y))$*:*





Example 8.16. *The following is a computer generated sketch of the level curves of the function* $f(x,y) = \cos(x) + \cos(y)$ *with the corresponding surface plot of* $(x, y, f(x, y))$*:*





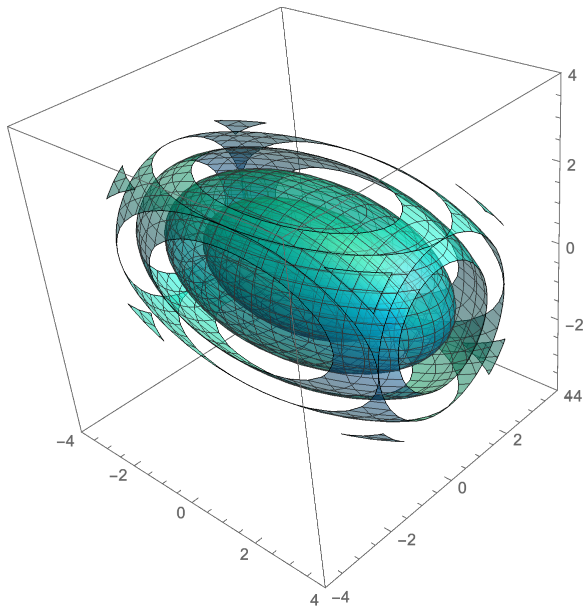
Example 8.17. *A simple example in* $3D$ *would be the level surfaces of*

$$f(x,y,z) = \frac{x^2}{4} + y^2 + z^2. \tag{285}$$

We cannot visualise the graph $\{(x, y, z, f(x, y, z))\}$ *since this requires* 4*-dimensions. However we can look at* $f(x, y, z) = k$ *for* k *constant. This gives,*

$$\frac{x^2}{4k} + \frac{y^2}{k} + \frac{z^2}{k} = 1 \tag{286}$$

for $k > 0$ *(which is nessecary otherwise* $x = y = z = 0$*). This gives concentric ellipsoids:*



8.4 Limits of Multivariable Functions

We want to talk about how functions behave as their variables approach certain values. We need to generalise our notion of limit:

Definition 8.5. *Let* $f : \mathrm{dom}(f) \to \mathbb{R}$ *be a function and* $\mathrm{dom}(f)$ *include all points arbitrarily close to* $\mathbf{a} = (a_1, ..., a_n)$*, i.e. for any* $d > 0$*, there exists a* $\mathbf{x} = (x_1, ..., x_n) \in \mathrm{dom}(f)$ *such that* $||\mathbf{x} - \mathbf{a}|| < d$*. We say that the limit of* f *as* $\mathbf{x}$ *goes to* $\mathbf{a}$ *is* L *if for all* $\epsilon > 0$*, there exists a* $\delta > 0$ *such that if*

$$0 < ||\mathbf{x} - \mathbf{a}|| < \delta \tag{287}$$

and $\mathbf{x} \in \mathrm{dom}(f)$ *then*

$$|f(\mathbf{x}) - L| < \epsilon \tag{288}$$

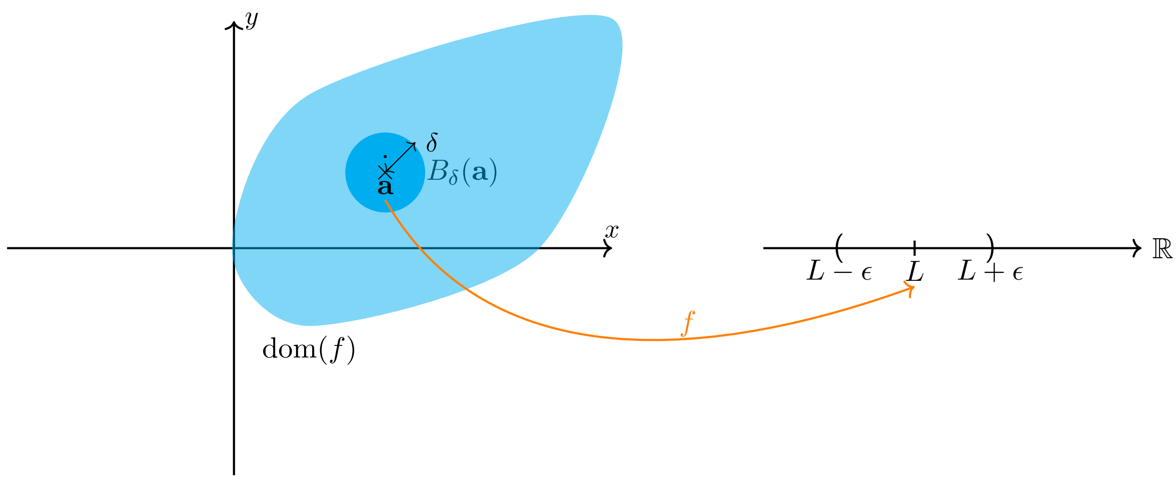
In notation, one writes

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L \tag{289}$$

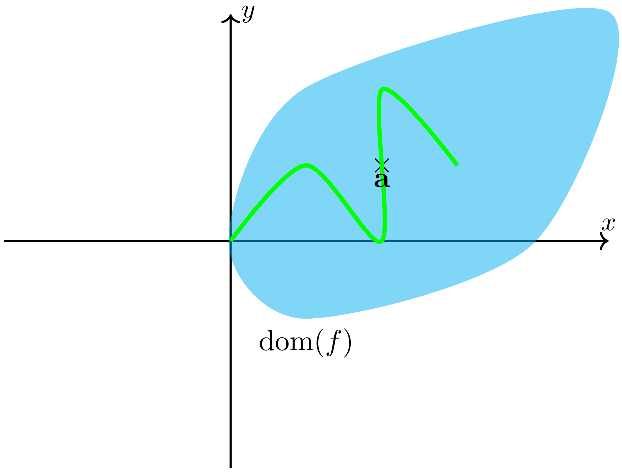
What this says is that given a small interval around L, $I = (L - \epsilon, L + \epsilon)$, then one can find a small (open) ball/disk of radius δ centered at $\mathbf{a}$,

$$B_\delta(\mathbf{a}) \doteq \{\mathbf{x} \in \mathbb{R}^n : ||\mathbf{x} - \mathbf{a}|| < \delta\}, \tag{290}$$

which f maps into I (except possibly $\mathbf{a}$). This is pictured for $\mathbb{R}^2$ below:



When we discussed limits for a function f of single variables we recalled that for the limit $\lim_{x\to a} f(x)$ to exist the above and below limits must exist and agree. There we were constrained to a line to approach the point for which we wanted to investigate the limit of f. In this situation the analogous statement is that approaching $\mathbf{a}$ on **any curve** should give the same limit. **Be careful**, this does not mean that if the limits agree on every line through a point then that is the limit, it must also agree on a parabolic curve, a hyperbolic curve or any weird path you may choose (as long as it lies in the domain of definition). This is illustrated below:



Picking various paths can be very useful for showing that limits do not exist.

8.4.1 Limits Not Existing

These arguments proceed as follows. Assume the limit exists then the limit along any path that passes through the point must exist and it must agree with any other path. Derive a contradiction based upon picking paths.

Example 8.18. *Show that*

$$\lim_{\mathbf{x}\to\mathbf{0}} \frac{x^2-y^2}{x^2+y^2} \tag{291}$$

does not exist.

Assume that

$$\lim_{\mathbf{x}\to\mathbf{0}} \frac{x^2-y^2}{x^2+y^2} \tag{292}$$

exists. Then the multivariable limit must be equal to the limit along the x-axis ($y=0$, $x\neq 0$). So,

$$\lim_{\mathbf{x}\to\mathbf{0}} \frac{x^2-y^2}{x^2+y^2} = \lim_{\substack{x\to 0\\ y=0}} \frac{x^2-y^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{x^2} = \lim_{x\to 0} 1 = 1, \tag{293}$$

along the x-axis.

If the limit exists then the multivariable limit must be equal to the limit along the y-axis ($x=0$, $y\neq 0$). So,

$$\lim_{\mathbf{x}\to\mathbf{0}} \frac{x^2-y^2}{x^2+y^2} = \lim_{\substack{y\to 0\\ x=0}} \frac{x^2-y^2}{x^2+y^2} = \lim_{y\to 0} \frac{-y^2}{y^2} = \lim_{x\to 0}(-1) = -1, \tag{294}$$

along the y-axis. This is a contradiction.

Example 8.19. *Show that*

$$\lim_{\mathbf{x}\to\mathbf{0}} \frac{xy}{x^2+y^2} \tag{295}$$

does not exist.

Assume that

$$\lim_{\mathbf{x}\to\mathbf{0}} \frac{xy}{x^2+y^2} \tag{296}$$

exists. Therefore, the limit along any path through $\mathbf{0}$ *must agree.*

Take a path along the x*-axis, i.e. set* $y = 0$, $x \neq 0$, *then*

$$\lim_{\mathbf{x}\to\mathbf{0}} \frac{xy}{x^2+y^2} = \lim_{\substack{x\to 0\\ y=0}} \frac{xy}{x^2+y^2} = \lim_{\substack{x\to 0\\ y=0}} \frac{0x}{x^2} = 0, \tag{297}$$

along the x*-axis. Take a path along the* y*-axis, i.e. set* $x = 0$, $y \neq 0$, *then*

$$\lim_{\mathbf{x}\to\mathbf{0}} \frac{xy}{x^2+y^2} = \lim_{\substack{y\to 0\\ x=0}} \frac{xy}{x^2+y^2} = 0, \tag{298}$$

along the y*-axis. So far, there is no contradiction. We can now look at other lines through* $(0, 0)$.

Take a path along the the line $y = x$ *then*

$$\lim_{\mathbf{x}\to\mathbf{0}} \frac{xy}{x^2+y^2} = \lim_{\substack{x\to 0\\ y=x}} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}, \tag{299}$$

along the $y = x$ *line. This gives us a contradiction.*

Example 8.20. *Show that*

$$\lim_{\mathbf{x}\to(1,1)} \frac{y-x}{1-y+\ln(x)} \tag{300}$$

does not exist.

Assume that

$$\lim_{\mathbf{x}\to\mathbf{0}} \frac{y-x}{1-y+\ln(x)} \tag{301}$$

exists. Therefore, the limit along any path through $(1, 1)$ *must agree. Be careful: this limit is through* $(1, 1)$ *not* $(0, 0)$*. You must pick a path that goes through this point. In particular,* $x = 0$ *and* $y = 0$ *do not pass through this point.*

First take the path along $y = x$ *then*

$$\lim_{\mathbf{x}\to(1,1)} \frac{y-x}{1-y+\ln(x)} = \lim_{x\to 1} \frac{0}{1-x+\ln(x)} = 0. \tag{302}$$

Consider now the path $y = 1$, *then*

$$\lim_{\mathbf{x}\to(1,1)} \frac{y-x}{1-y+\ln(x)} = \lim_{x\to 1} \frac{1-x}{\ln(x)}. \tag{303}$$

This limit is of the '$0/0$*' form, so we can apply L'Hôpital's rule to try to find the limit:*

$$\lim_{\mathbf{x}\to(1,1)} \frac{y-x}{1-y+\ln(x)} = \lim_{x\to 1} \frac{\frac{d}{dx}(1-x)}{\frac{d}{dx}\ln(x)} = \lim_{x\to 1}(-x) = -1, \tag{304}$$

This gives the desired contradiction.

Example 8.21. *Show that*

$$\lim_{\mathbf{x}\to(1,-1)} \frac{(x-1)(y+1)^2}{(x-1)^2+(y+1)^4} \tag{305}$$

does not exist.

Assume that

$$\lim_{\mathbf{x}\to(1,-1)} \frac{(x-1)(y+1)^2}{(x-1)^2+(y+1)^4} \tag{306}$$

exists. Therefore, the limit along any path through $(1,-1)$ *must agree.*

Let's look at the path $x=1$ *then*

$$\lim_{\mathbf{x}\to(1,-1)} \frac{(x-1)(y+1)^2}{(x-1)^2+(y+1)^4} = \lim_{y\to-1} \frac{(0)(y+1)^2}{(y+1)^4} = 0. \tag{307}$$

Let's check all other lines at once: set $y=m(x-1)-1$. *Then*

$$\lim_{\mathbf{x}\to(1,-1)} \frac{(x-1)(y+1)^2}{(x-1)^2+(y+1)^4} = \lim_{x\to1} \frac{(x-1)m^2}{1+m^4(x-1)^2} = 0. \tag{308}$$

No contradiction so far!

What about a parabola through $(1,-1)$*? Let's set* $x=(y+1)^2+1$. *Then,*

$$\lim_{\mathbf{x}\to(1,-1)} \frac{(x-1)(y+1)^2}{(x-1)^2+(y+1)^4} = \lim_{y\to-1} \frac{(y+1)^4}{2(y+1)^4} = \frac{1}{2} \tag{309}$$

8.5 Properties of Limits

Proposition 8.1 (Limit Properties)**.** *Suppose* f *and* g *are defined on* $D\subseteq\mathbb{R}^n$*, except possibly at some point* $\mathbf{a}\in D$. *Further suppose,*

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}), \qquad \lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x}) \tag{310}$$

exist and $k\in\mathbb{R}$. *Then*

1. $\lim_{\mathbf{x}\to\mathbf{a}}(f(\mathbf{x})+g(\mathbf{x})) = \lim_{\mathbf{x}\to\mathbf{a}} f(x) + \lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x})$.
2. $\lim_{\mathbf{x}\to\mathbf{a}}(f(\mathbf{x})-g(\mathbf{x})) = \lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) - \lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x})$.
3. $\lim_{\mathbf{x}\to\mathbf{a}}(kf(\mathbf{x})) = k\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})$.
4. $\lim_{\mathbf{x}\to\mathbf{a}}(f(\mathbf{x})g(\mathbf{x})) = \lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})\cdot\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x})$.
5. $\lim_{\mathbf{x}\to\mathbf{a}}(f(\mathbf{x})/g(\mathbf{x})) = \lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})/\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x})$ *if* $\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x})\neq 0$.

Proof. Omitted (for those interested: try with an $\epsilon-\delta$ argument). □

Under the conclusions (which you can try to prove with an $\epsilon-\delta$ argument) that, for $\mathbf{x},\mathbf{a}\in\mathbb{R}^n$,

$$\lim_{\mathbf{x}\to\mathbf{a}} x_i = a_i, \qquad \lim_{\mathbf{x}\to\mathbf{a}} c = c, \tag{311}$$

one can see that limit laws imply that limit of any polynomial P can be evaluated by direct substitution:

$$\lim_{\mathbf{x}\to\mathbf{a}} P(\mathbf{x}) = P(\mathbf{a}). \tag{312}$$

Similarly, limit laws imply that limit of any rational function $R=P/Q$ can be evaluated by direct substitution with the caveat that the point $\mathbf{x}=\mathbf{a}$ must be in the domain of definition of R, i.e. Q cannot have a root there.

Let's do some examples:

Example 8.22. *Evaluate*

$$\lim_{(x,y)\to(-2,3)} \frac{3x^2y+1}{x^3y^2-2x}, \tag{313}$$

if it exists.

So, $x^3y^2 - 2x$ *evalutated at* $(-2,3)$ *gives* $-68 \neq 0$. *Therefore,* $\mathbf{x} = (-2,3)$ *is in the domain of the definition of the rational function. Hence, the limit exists and is*

$$\lim_{(x,y)\to(-2,3)} \frac{3x^2y+1}{x^3y^2-2x} = -\frac{37}{68} \tag{314}$$

Example 8.23. *Evaluate*

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}, \tag{315}$$

if it exists.

Consider $y = mx$ *for* $m \in \mathbb{R}$. *Then*

$$\frac{3x^2y}{x^2+y^2} = \frac{3mx}{(1+m^2)} \to 0 \tag{316}$$

as $x \to 0$. *So on any line the limit through the origin is* $(0,0)$. *Along the parabolas* $y = x^2$ *and* $x = y^2$ *one has*

$$\frac{3x^2y}{x^2+y^2} = \frac{3x^2}{2} \to 0, \qquad \frac{3x^2y}{x^2+y^2} = \frac{3y^3}{(1+y^2)} \to 0 \tag{317}$$

as $x \to 0$ *and* $y \to 0$ *respectively.*

Let's attempt to prove the limit exists and is 0. *Let's give ourselves an* $\epsilon > 0$. *What we need to show is that there is a* $\delta > 0$ *such that if* $||\mathbf{x} - \mathbf{0}|| < \delta$ *then* $|{}^{3x^2y}/_{x^2+y^2} - 0| < \epsilon$. *First note that*

$$||\mathbf{x} - \mathbf{0}|| = \sqrt{x^2+y^2} \tag{318}$$

Now,

$$\left|\frac{3x^2y}{x^2+y^2} - 0\right| = \frac{3x^2|y|}{x^2+y^2} \leq 3|y| = 3\sqrt{y^2} \leq 3\sqrt{x^2+y^2}. \tag{319}$$

Take $\delta < \frac{\epsilon}{3}$ *then* $3\sqrt{x^2+y^2} < \epsilon$, *which then gives,*

$$\left|\frac{3x^2y}{x^2+y^2} - 0\right| \leq 3\sqrt{x^2+y^2} < \epsilon. \tag{320}$$

This show that $\lim_{\mathbf{x}\to\mathbf{0}} {}^{3x^2y}/_{x^2+y^2} = 0$.

9 Multivariable Functions II: Continuity, Partial Derivatives and PDE

9.1 Continuity

Now that we have generalised the notion of limits to multivariable functions. We can define what we mean for a multivariable function to be continuous at some point $\mathbf{a} = (a_1, ..., a_n)$.

Definition 9.1 (Continuity)**.** *Let f be a function defined on some subset D of $\mathbb{R}^n$. Then one says that f* **is continuous at $\mathbf{a} \in D$** *if*

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a}). \tag{321}$$

The function $f : D \to \mathbb{R}$ is continuous on D if it is continuous at every $\mathbf{a} \in D$.

In plain terms continuity means that if the point $\mathbf{x}$ changes by a little bit, then $f(\mathbf{x})$ changes by a little bit. This means that a surface that is the graph of a continuous function has no hole or break.

Let's do some examples:

Example 9.1. *Where is*

$$f_1(x, y) = \frac{x^2 + y^2}{x^2 - y^2} \tag{322}$$

continuous? What about the function

$$f_2(x, y) = \begin{cases} 0 & \text{if } y = \pm x \\ \frac{x^2+y^2}{x^2-y^2} & \text{otherwise} \end{cases} \quad ? \tag{323}$$

The function f_1 is continuous where it is defined since it is a rational function. Therefore, its continuous everywhere on $\mathbb{R}^2$ except where $y = \pm x$.

Now f_2 is f_1 except with a modified definition along $y = \pm x$. Note that taking the limit along $x = 0$ gives

$$\lim_{y\to 0}(-1) = -1, \tag{324}$$

which contradicts the definition of continuity since $f(0) = 0$. We need to check the lines $y = \pm x$ too. This means considering $\lim_{(x,y)\to(c,\pm c)} f_2(x, y)$ for c constant. Consider $y = \pm c$, then

$$\lim_{(x,y)\to(c,\pm c)} f_2(x, y) = \lim_{x\to c} \frac{x^2 + c^2}{x^2 - c^2} = \lim_{x\to c} \frac{x^2 + c^2}{(x - c)(x + c)} = \infty, \tag{325}$$

which again contradicts continuity as $f(0) = 0 \neq \infty$. Therefore, f_2 is continuous everywhere on $\mathbb{R}^2$ except where $y = \pm x$.

Example 9.2. *Let*

$$f(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 0) \\ \frac{3x^2y}{x^2+y^2} & \text{otherwise.} \end{cases} \tag{326}$$

Where is f continuous?

We showed in example 8.23 that

$$\lim_{\mathbf{x}\to\mathbf{0}} \frac{3x^2y}{x^2 + y^2} = 0. \tag{327}$$

Since this equals the functions value at that point, f is continuous.

Proposition 9.1. *Suppose $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ is continuous on D and g is a function of a single variable that is continuous on the image of f. Then the composition $h = g \circ f$ defined by*

$$h(\mathbf{x}) = g(f(\mathbf{x})) \tag{328}$$

is continuous on D.

Example 9.3. *Suppose $f(x, y, z) = x^3y + zx$ and $g(x) = \arctan(x)$. The function f is continuous everywhere on $\mathbb{R}^3$ and the function g is continuous on $\mathbb{R}$ then $h = g \circ f$ defined by*

$$h(x, y, z) = \arctan(x^3y + zx) \tag{329}$$

is continuous on all $\mathbb{R}^3$.

Remark 9.1. *When dealing with continuity of multivariable functions one should keep in mind proposition 7.2. For example, that polynomials of many variables are continuous everywhere on $\mathbb{R}^n$ and rational functions are continuous on their domain of definition. Similarly, if constructing compositions based upon the functions of a single variable in proposition 7.2, one can deduce continuity upon this knowledge.*

9.2 Differentiation of Multivariable Functions: Partial Derivatives

Suppose we have a function of two variables $f(x, y)$. We could fix $y = b$ and consider $g(x) = f(x, b)$ as a function of the single variable x. We could look at the limiting process which defines the derivative of $g(x)$ with respect to x at a

$$\frac{dg}{dx}(a) = \lim_{h\to 0} \frac{g(a+h) - g(a)}{h} = \lim_{h\to 0} \frac{f(a+h, b) - f(a, b)}{h}. \tag{330}$$

Similarly, we could fix a in f and consider the derivative of $h(y) = f(a, y)$ with respect to y at b then

$$\frac{dh}{dy}(b) = \lim_{h\to 0} \frac{h(b+h) - g(b)}{h} = \lim_{h\to 0} \frac{f(a, b+h) - f(a, b)}{h}. \tag{331}$$

If these limits exist we have a type of derivative of a multivariable function known as the partial derivative of f with respect to x or y, respectively, at (a, b). One writes:

$$\frac{\partial f}{\partial x}(a, b) = \lim_{h\to 0} \frac{f(a+h, b) - f(a, b)}{h} \tag{332}$$

$$\frac{\partial f}{\partial y}(a, b) = \lim_{h\to 0} \frac{f(a, b+h) - f(a, b)}{h}. \tag{333}$$

In general, one has the following definition for partial derivatives:

Definition 9.2. *Let f be a function of n-variables, U be a subset of $\mathrm{dom}(f)$. Suppose $\mathbf{a} \in U$. The partial derivative of f with respect to the i^{th}-variable x_i at $\mathbf{a}$ is defined as*

$$\frac{\partial f}{\partial x_i}(\mathbf{a}) = \lim_{h\to 0} \frac{f(a_1, ..., a_i + h, ..., a_n) - f(a_1, ..., a_i, ..., a_n)}{h} \tag{334}$$

if the limit on the right-hand side exists.

One could let $\mathbf{a}$ vary in U. If the limits exist then one obtains functions of $\mathbf{x}$, i.e. partial derivative functions. These are often denoted many notations listed below:

- Leibniz: $\frac{\partial f}{\partial x_i}$, $\partial_{x_i} f$ and $\partial_i f$, where the latter is often used by lazy relativists.
- f_{x_i}
- Euler: $D_i f$.

Example 9.4. *Find the partial derivative functions of*

1. $f(x,y) = x^n y^m$ for $m,n \geq 1$.

$$\partial_x(x^n y^m) = nx^{n-1}y^m \qquad \partial_y(x^n y^m) = mx^n y^{m-1}$$

2. $f(x,y) = \frac{y^m}{x^n}$ for $m \geq 1$, $n \geq 1$ and $x \neq 0$.

$$\partial_x \frac{y^m}{x^n} = -n\frac{y^m}{x^{n+1}} \qquad \partial_y \frac{y^m}{x^n} = m\frac{y^{m-1}}{x^n}$$

3. $f(x,y,z) = \ln(xyz)$ for $xyz > 0$.

$$\partial_x \ln(xyz) = \frac{1}{x}, \qquad \partial_y \ln(xyz) = \frac{1}{y}, \qquad \partial_z \ln(xyz) = \frac{1}{z}$$

4. $f(x,y,z) = \ln(x+y+z)$ for $x+y+z > 0$.

$$\partial_{x_i} \ln(x+y+z) = \frac{1}{x+y+z}$$

for $x_i = 1, 2, 3$

5. $f(r,\theta) = r\cos\theta + \sin\theta$.

$$\partial_r(r\cos\theta + \sin\theta) = \cos\theta, \qquad \partial_\theta(r\cos\theta + \sin\theta) = \cos\theta - r\sin\theta.$$

6. $f(x,y) = \arctan(y/x)$ for $x \neq 0$.

$$\partial_x \arctan(y/x) = -\frac{y}{x^2+y^2}, \qquad \partial_y \arctan(y/x) = \frac{x}{x^2+y^2}.$$

7. For $z^3 = 1 - 6xyz - x^3 - y^3$, find $\partial_x z$.

$$\partial_x z^3 = 3z^2\partial_x z = -6yz - 3x^2 \implies \partial_x z = \frac{-6yz - 3x^2}{3z^2}$$

9.3 Higher Derivatives

Let f be a function of two variables and that the partial derivatives of f, $\partial_x f$ and $\partial_y f$, exist in some region D of $\mathbb{R}^2$. We can treat $\partial_x f$ and $\partial_y f$ as functions on D and compute their partial derivatives (if limit above exists). In other words we can consider,

$$\partial_x\partial_x f = \partial_x^2 f = (f_x)_x = f_{xx} = D_1 D_1 f = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} \tag{335}$$

$$\partial_y\partial_y f = \partial_y^2 f = (f_y)_y = f_{yy} = D_2 D_2 f = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} \tag{336}$$

$$\partial_x\partial_y f = \partial_{xy}^2 f = (f_y)_x = f_{yx} = D_1 D_2 f = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} \tag{337}$$

$$\partial_y\partial_x f = \partial_{yx}^2 f = (f_x)_y = f_{xy} = D_2 D_1 f = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} \tag{338}$$

Note that we have separated the cases $\partial_y\partial_x f$ and $\partial_x\partial_y f$. The former means differentiate with respect to x first and then with respect to y, whilst the latter means differentiate with respect to y first and then with respect to x. These **may not** be equal; we will state a theorem below about when these operations commute below. First let's do an example:

Example 9.5. *Let*

$$f(x,y) \doteq x^7 - x^2y^3 + yx + 2. \tag{339}$$

Then,

$$\partial_x\partial_y f = -6xy^2 + 1 = \partial_y\partial_x f. \tag{340}$$

Example 9.6. *Let*

$$f(x,y) \doteq \begin{cases} 0 & \mathbf{x} = \mathbf{0} \\ \frac{xy(x^2-y^2)}{x^2+y^2} & \mathbf{x} \neq \mathbf{0} \end{cases} \tag{341}$$

For $\mathbf{x} \neq \mathbf{0}$, via the quotient rule, we have

$$\partial_x f = \frac{y(x^2-y^2)+xy(2x)}{x^2+y^2} - \frac{xy(x^2-y^2)(2x)}{(x^2+y^2)^2} = \frac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2}. \tag{342}$$

and

$$\partial_y f = \frac{y(x^2-y^2)+xy(2x)}{x^2+y^2} - \frac{xy(x^2-y^2)(2x)}{(x^2+y^2)^2} = -\frac{y^4x+4y^2x^3-x^5}{(x^2+y^2)^2}. \tag{343}$$

For $\mathbf{x} = \mathbf{0}$, we have

$$\partial_x f(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0. \tag{344}$$

Similarly, we have

$$\partial_x f(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0. \tag{345}$$

Lets look at $\partial_x\partial_y f(0,0)$ and $\partial_y\partial_x f(0,0)$:

$$\partial_x\partial_y f = \lim_{h\to 0} \frac{\partial_y f(h,0)-\partial_y f(0,0)}{h} = \lim_{h\to 0} \frac{h-0}{h} = 1 \tag{346}$$

and

$$\partial_y\partial_x f = \lim_{h\to 0} \frac{\partial_x f(0,h)-\partial_y f(0,0)}{h} = \lim_{h\to 0} \frac{-h-0}{h} = -1. \tag{347}$$

Therefore, $\partial_y\partial_x f \neq \partial_x\partial_y f$.

This last example illustrates the failure of the following theorem:

Theorem 9.1. *Let D be some disk in $\mathbb{R}^2$ containing (a,b). Suppose $f : D \to \mathbb{R}$. If $\partial_x\partial_y f$ and $\partial_y\partial_x f$ exists and are continuous on D, then 'partial derivatives commute':*

$$\partial_x\partial_y f(a,b) = \partial_y\partial_x f(a,b). \tag{348}$$

Remark 9.2. *This theorem generalises to more derivatives and to functions of more variables. For example,*

$$\partial_x\partial_y\partial_z f = \partial_y\partial_z\partial_x f = \partial_z\partial_x\partial_y f = \partial_y\partial_x\partial_z f = \partial_x\partial_z\partial_y f = \partial_z\partial_y\partial_x f \tag{349}$$

if these functions are continuous.

9.4 Partial Differential Equations

A partial differential equation (PDE) is a relation between partial derivatives of a multivariable function $f(x_1, ..., x_n)$. For example,

$$\partial_x f = \partial_y f, \qquad \partial_x\partial_y f = 0, \qquad f\partial_z f = 2(\partial_x f)^2. \tag{350}$$

Here f would be 'the unknown' of the equation for which (in an ideal world) we'd like to solve for. Be warned this is not always possible; a solution may not even exist. If it does exist, it may not be unique.

Some famous examples of PDE are listed below (**the last four examples are simply included for interest, you're not expected to fully understand the notation or all comments related to them**):

- **Laplace's equation**:

$$\Delta u \doteq \partial_x^2 u + \partial_y^2 u = 0, \qquad \Delta u \doteq \partial_x^2 u + \partial_y^2 + \partial_z^2 u = 0, \qquad \Delta u \doteq \partial_{x_1}^2 u + ... + \partial_{x_n}^2 u = 0 \tag{351}$$

This first is Laplace's equation in 2-dimensions, the second in 3-dimensions and the last in n-dimensions. Note that in 1-dimension it $u(x)$ must satisfy

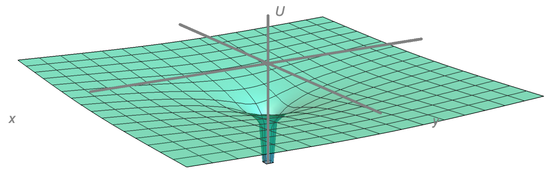
$$\Delta u = \partial_x^2 u = \frac{d^2 u}{dx^2} = 0 \implies u = ax + b \tag{352}$$

for a, b constants in $\mathbb{R}$. The Δ often gets called 'the Laplacian'. Solutions to this equation are called harmonic functions, which crop up everywhere in physics. For example, in gravitation, fluid dynamics, heat conduction, electrostatics.

Lets look at $u(x,y) \doteq \ln(x^2 + y^2)$ for Laplace's equation in 2-dimensions. Now,

$$\partial_x^2 u = -\frac{2(x^2 - y^2)}{(x^2 + y^2)^2}, \quad \partial_y^2 u = \frac{2(x^2 - y^2)}{(x^2 + y^2)^2} \implies \partial_x^2 u + \partial_y^2 u = 0. \tag{353}$$

So $u(x,y)$ is a harmonic function and is plotted below:



This function could describe the electric potential due to a line of unit charge entire z-axis.

- The **heat equation**:

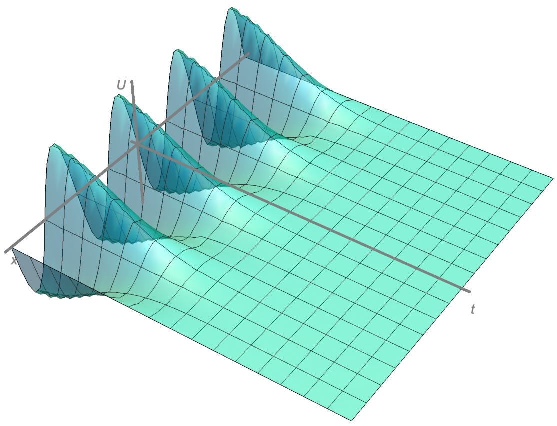
$$\partial_t u - \Delta u = 0 \tag{354}$$

As the name suggests this equation describes how heat is conducted or diffused through a given material. Solutions are called caloric functions. Note that, if u is independent of time, we have Laplace's equation, i.e. if the temperature distribution in some material is not evolving it must be a harmonic function.

Let's look at $u(t,x) \doteq e^{-\kappa^2 t} \sin(\kappa x)$ for the heat equation with the 1-dimensional Laplacian. We have,

$$\partial_t u = -\kappa^2 u, \quad \partial_x^2 u = -\kappa^2 u \implies \partial_t u - \partial_x^2 u = 0. \tag{355}$$

So, $u(t,x)$ is a caloric function and is plotted below:



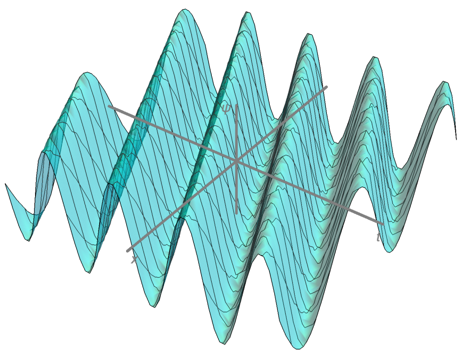
- The **wave equation**:

$$\Box\Psi \doteq -\partial_t^2\Psi + \Delta\Psi = 0. \tag{356}$$

Suppose we take $\Delta\Psi$ as the 1-dimensional Laplacian of $\Psi(t,x)$. Lets look at the function $\Psi(t,x) \doteq \sin(x-t)$. Computing partial derivatives gives

$$\partial_t^2\Psi = -\sin(x-t), \qquad \partial_x^2\Psi = -\sin(x-t) \implies -\partial_t^2\Psi + \partial_x^2\Psi = 0. \tag{357}$$

The function $\Psi(t,x) \doteq \sin(x-t)$ is plotted below:



This is wave type behaviour, hence the name for the equation. This function Ψ could be the a sound wave, a light wave, an ocean wave etc.

- Non-Examinable **Maxwell's equations** for electromagnetism. In the absence of charges and currents these have the form:

$$\mathrm{div}\mathbf{E} = 0 = \mathrm{div}\mathbf{B} \tag{358}$$
$$\mathrm{curl}\mathbf{E} = -\partial_t\mathbf{B} \tag{359}$$
$$\mathrm{curl}\mathbf{B} = \partial_t\mathbf{E}. \tag{360}$$

Here $\mathbf{E}$ and $\mathbf{B}$ are vectors in $\mathbb{R}^3$ representing the electric field and magnetic fields respectively. One can show that these equations imply that

$$-\partial_t^2\mathbf{E} + \Delta\mathbf{E} = 0, \tag{361}$$
$$-\partial_t^2\mathbf{B} + \Delta\mathbf{B} = 0. \tag{362}$$

In otherwords, the electric and magnetic fields satisfy the wave equation and therefore, admit wave-type solutions. This is often why it is said that light (which is electromagnetic radiation) is a wave.

- Non-Examinable The **Navier-Stokes equation** for fluid dynamics.

$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \nu\Delta\mathbf{v} + \mathbf{f}. \tag{363}$$

  Here, $\mathbf{v}$ is the velocity of the fluid. It is a vector in $\mathbb{R}^3$ that depends on time, t and space, $\mathbf{x}$. The function p is the pressure of the fluid and ν is something known as the viscosity (related to thickness of the fluid), $\mathbf{f}$ is some external force. There is a 1 million dollar prize to be awarded to the person who resolves an open problem about the existence of solutions (in some catagory) to this equation. This is one of 7 Millennium Prize Problems put forward by the Clay Mathematics Institute.

- Non-Examinable The **Einstein equation** for general relativity is one of the most celebrated PDE in mathematical physics. It is typically stated as

$$\mathrm{Ric}(g) - \frac{1}{2}\mathrm{Scal}(g)g = 8\pi T. \tag{364}$$

  The unknown in this equation is the 'spacetime metric' g_{ij}, which can be thought of as a collection of functions $g_{ij}(x_0, x_1, x_2, x_3)$ with $1 \leq i, j \leq 4$. It encodes the gravitational field, i.e. it dictates how objects move under gravity. There are g_{ij} that model the gravitational fields of stars, black holes, galaxies, the universe etc.

  The left-hand side of this equation is known as the 'Einstein tensor' and is constructed from objects that should be interpreted as the curvature of spacetime (which is intimately related to properties of the gravitational field). The right-hand side is an object known as the energy-momentum tensor which is an object describing the matter in the universe. So in rough terms the matter content in your universe dictates how the spacetime should curve (how the gravitational field should look) and visa versa.

  Unlike its Newtonian predecessor, one can set $T = 0$ and there are non-trivial solutions. In this case the Einstein equation reduces to

$$\mathrm{Ric}(g) = 0. \tag{365}$$

  This can be written in the form

$$\Box_g(g_{ij}) = N(g, \partial g). \tag{366}$$

  Here, $\Box_g$ is an altered version of the $\Box$ for the wave equation above and $N(g, \partial g)$ denotes a term that involves g_{ij} and $\partial g_{ij}/\partial x_k$ but no higher derivatives.

  When the gravitational field is 'weak' the term $N(g, \partial g)$ can be neglected and $\Box_g$ replaced with the wave equation $\Box$. The functions g_{ij} then satisfy the wave equation, which gave rise to Einsteins famous prediction of gravitational waves in 1916. Direct experimental confirmation of their existence was produced 2016 by the LIGO observations.

- Non-Examinable The **Black–Scholes equation** in mathematical finance:

$$\partial_t V + \frac{1}{2}\sigma^2 S^2 \partial_S^2 V + rS\partial_S V - rV = 0. \tag{367}$$

  Here, V is the price of an option (which is right–or 'option'–to buy a stock at a given predetermined price) as a function of stock price S and time t. The constant r is the risk-free interest rate, and σ is the volatility of the stock. For example, you could have a friend that potentially wants to buy your car at the price of $20k$. They could ask you to sell them the right (or option) to buy your car in the next month for 500. This could vary in time dependent on how desperate your friend is for the car or how much you like your car.

The field of analysis of PDE has been and continues to be a very active field of research in mathematics.

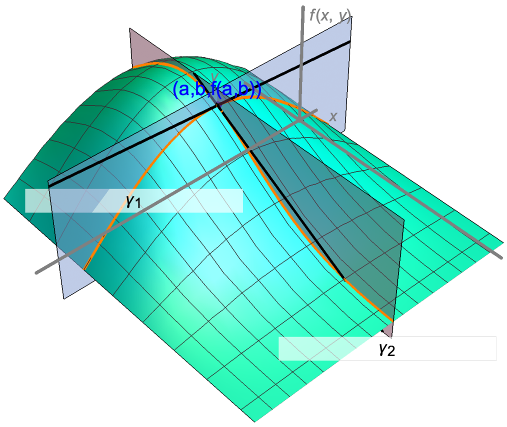
10 Tangent Planes and Linear Approximations

This section is concerned with the local properties of multivariable functions. In particular, how to partial derivatives characterise the behaviour of a multivariable function close to a particular point.

10.1 Tangent Planes

For a function of a single variable $f(x)$, df/dx is interpreted a rate of change with respect to x. Similarly, for a multivariable function $f(x_1, ..., x_n)$, partial derivatives of f can be interpreted as rates of change when all but one variable is fixed. Effectively, this means we're looking at rates of change in certain directions.

Suppose we have a function f of two variables (x, y). Further suppose the partial derivatives of f exists at (a, b), i.e. $\partial_x f(a, b)$ and $\partial_y f(a, b)$ exist. The surface $\{(x, y, z) : z = f(x, y)\}$ near (a, b) is plotted below in cyan.



Note that $y = b$ is a plane in $\mathbb{R}^3$ which intersects the surface in a curve, denoted in the diagram with γ_1. This curve has the equation $g(x) = f(x, b)$ and therefore it's gradient at $x = a$ is

$$\frac{dg}{dx}(a) = \partial_x f(a, b). \tag{368}$$

So, there is a line tangent to the curve γ_1 with gradient $m_1 = \partial_x f(a, b)$ in the plane determined by $y = b$, i.e.

$$z = m_1(x - a) + f(a, b), \tag{369}$$

in $\{(x, y, z) : y = b\}$. Supposing that $m_1 \neq 0$, i.e. $\partial_x f(a, b) \neq 0$ then the symmetric equation of the line is

$$\frac{z - f(a, b)}{\partial_x f(a, b)} = \frac{(x - a)}{1}, \qquad y = b. \tag{370}$$

We can put this into the form for the vector equation of a line as:

$$\mathbf{x}(\lambda) = (a, b, f(a, b)) + \lambda \mathbf{v}_1, \tag{371}$$

with $\mathbf{v}_1 = (1, 0, \partial_x f(a, b))$. This is plotted in black above γ_1. Similarly, $x = a$ is a plane in $\mathbb{R}^3$ which intersects the surface in a curve, denoted in the diagram with γ_2. This curve has the equation $h(y) = f(a, y)$ and therefore it's gradient at $y = b$ is

$$\frac{dh}{dy}(b) = \partial_y f(a, b). \tag{372}$$

So, there is a line tangent to the curve γ_2 with gradient $m_2 = \partial_y f(a, b)$ in the plane determined by $x = a$, i.e.

$$z = m_2(y - b) + f(a, b), \tag{373}$$

We can put this into the form for the vector equation of a line as:

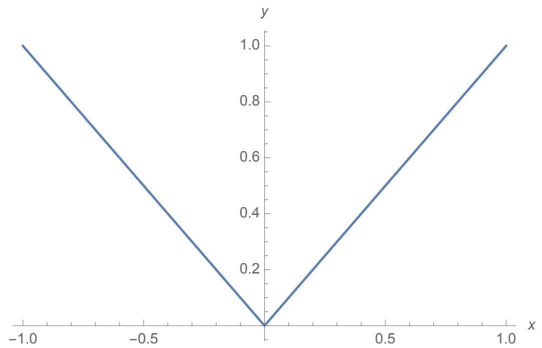
$$\mathbf{x}(\lambda) = (a, b, f(a, b)) + \lambda \mathbf{v}_2. \tag{374}$$

with $\mathbf{v}_2 = (0, 1, \partial_y f(a, b))$. This is plotted in black above γ_2.

What's so special about curves resulting from intersecting with planes parallel to the zy and zx-planes? Well, both nothing and something.

- Nothing: We could have any curve on the surface through $(a, b, f(a, b))$ and look at its tangent line at $(a, b, f(a, b))$, **if** that line exists, which leads to the something...
- Something: The only stipulation about the curves resulting from $\{x = a\}$ and $\{y = b\}$ intersection was that the partial derivatives existed.

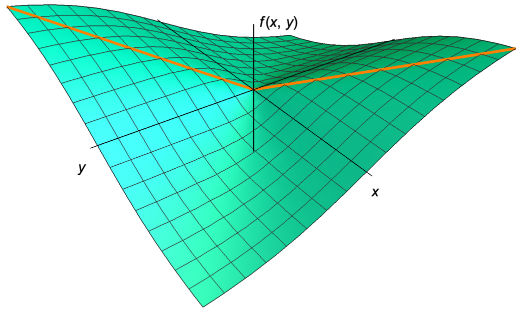
This latter point means that the curves that result from cross-sections $x = a$ and $y = b$ cannot break at (a, b), i.e. be discontinuous, or have corners like $y = |x|$ (which is continuous but not differentiable at $x = 0$) plotted below:



So, the stipulation that $\partial_x f(a, b)$ and $\partial_y f(a, b)$ exist, for the curves resulting from $\{x = a\}$ and $\{y = b\}$ intersection, means that the tangent line is well-defined. In the case of other curves, the tangent line may not be well defined. Consider the function,

$$f(x, y) \doteq \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0). \end{cases} \tag{375}$$

This is plotted below:



Now one can check that

$$\partial_x f(0,0) = 0 = \partial_y f(0,0), \tag{376}$$

Hence, the partial derivatives exist at $(0,0)$, and, therefore, the tangent lines to $(0,0,0)$ in the $\mathbf{i}$ and $\mathbf{j}$ directions exist. However, consider the curve in orange above, which is given by $y = x$. Therefore,

$$\Big(x, x, f(x,x) = \frac{|x|}{\sqrt{2}}\Big), \tag{377}$$

defines a curve in the surface. This has no tangent line at $x = 0$ since $|x|$ is not differentiable at $x = 0$. What's gone wrong here? Well, note that

$$\partial_x f = \frac{y^3}{(x^2+y^2)^{\frac{3}{2}}}, \qquad \partial_y f = \frac{x^3}{(x^2+y^2)^{\frac{3}{2}}} \tag{378}$$

for $(x,y) \neq (0,0)$. Lets focus on $\partial_x f(x,y)$. Take $x = 0$, then

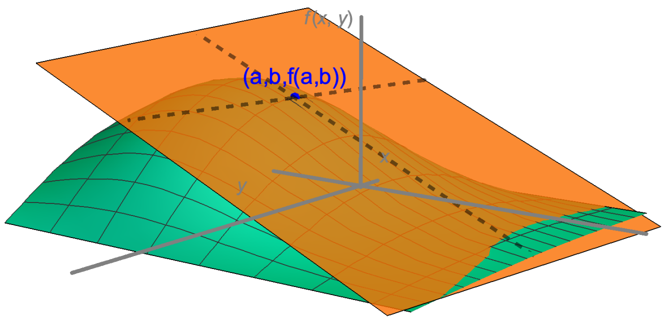
$$\partial_x f(0,y) = \frac{y^3}{(y^2)^{\frac{3}{2}}} = \frac{y^3}{|y|^3} = \begin{cases} 1 & y > 0 \\ -1 & y < 0. \end{cases} \tag{379}$$

Therefore, $\lim_{(x,y)\to(0,0)} \partial_x f(x,y)$ does not exist. In particular, $\partial_x f(x,y)$ is not continuous at $(0,0)$.

This example illustrates the following fact: if the partial derivatives are continuous, then all the tangent lines exist. So when the partial derivatives are continuous, these tangent lines form the **tangent plane** at $(a,b,f(a,b))$, which we will define as the plane which contains the two tangent lines resulting from the intersection with $\{x = a\}$ and $\{y = b\}$, i.e. the plane containing the point $\mathbf{x}_0 = (a,b,f(a,b))$ with normal

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = (-\partial_x f(a,b), -\partial_y f(a,b), 1) \tag{380}$$

This is plotted below:



Recall that our equation for the plane was

$$\langle \mathbf{n}, \mathbf{x} - \mathbf{x}_0 \rangle = 0. \tag{381}$$

Expanding gives

$$\partial_x f(a,b)(x-a) + \partial_y f(a,b)(y-b) - (z - f(a,b)) = 0, \tag{382}$$

which is the equation of the tangent plane at $(a,b,f(a,b))$.

Let's do an example:

Example 10.1. *Take the ellipsoid*

$$\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{49} = 1. \tag{383}$$

Recall that we can solve for z in terms of two functions

$$z = f_{\pm}(x, y) = \pm 7\sqrt{1 - \frac{x^2}{4} - \frac{y^2}{16}}. \tag{384}$$

Lets find the tangent plane at $(1, 1, \frac{7}{4}\sqrt{11})$. To easy notation let $f = f_+$ and let's find $\partial_x f$ and $\partial_y f$:

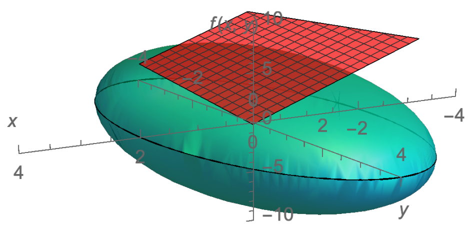
$$\partial_x f = -\frac{7x}{4\sqrt{1 - \frac{x^2}{4} - \frac{y^2}{16}}}, \qquad \partial_y f = -\frac{7y}{16\sqrt{1 - \frac{x^2}{4} - \frac{y^2}{16}}}. \tag{385}$$

So,

$$\partial_x f(1,1) = -\frac{7}{\sqrt{11}}, \qquad \partial_y f(1,1) = -\frac{7}{4\sqrt{11}}. \tag{386}$$

Therefore, the equation of the plane at $(1, 1, \frac{7}{4}\sqrt{11})$ tangent to the ellipsoid is

$$\frac{7}{\sqrt{11}}(x-1) + \frac{7}{4\sqrt{11}}(y-1) + \left(z - \frac{7}{4}\sqrt{11}\right) = 0. \tag{387}$$



Let's return to the non-example above:

Example 10.2. *Consider again the function,*

$$f(x, y) \doteq \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0). \end{cases} \tag{388}$$

As above

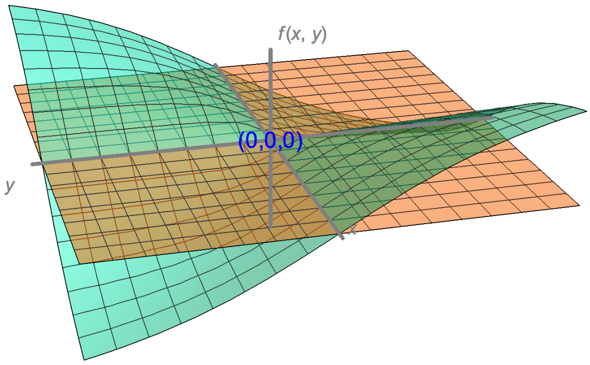
$$\partial_x f = \begin{cases} 0 & \mathbf{x} = (0,0) \\ \frac{y^3}{(x^2+y^2)^{\frac{3}{2}}} & \mathbf{x} \neq (0,0) \end{cases} \tag{389}$$

$$\partial_y f = \begin{cases} 0 & \mathbf{x} = (0,0) \\ \frac{x^3}{(x^2+y^2)^{\frac{3}{2}}} & \mathbf{x} \neq (0,0) \end{cases} \tag{390}$$

which are not continuous at $(0, 0)$. However, since the partial derivatives exist at $(0, 0)$ we can write down an equation for a 'tangent' plane for $f(x, y)$ at $(0, 0)$:

$$\partial_x f(0,0)(x-0) + \partial_y f(0,0)(y-0) - (z - f(0,0)) = 0 \implies z = 0 \tag{391}$$

On the following plot the $\{z = 0\}$ plane is impose on the surface:



One can see that this plane is not really tangent to the surface since it does not just touch at a since point. So the tangent plane was not well defined.

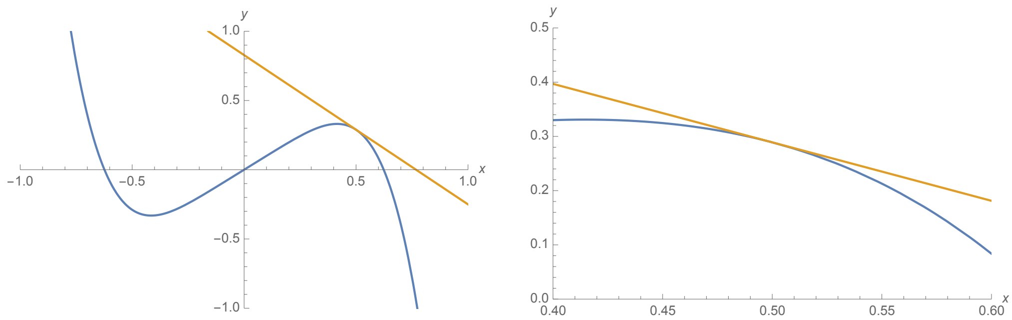
10.2 Linear Approximations of Multivariable Functions

Definition 10.1. *A function f from D (a region in $\mathbb{R}^n$) to $\mathbb{R}$ is called* **linear** *if*

$$f(\mathbf{x}) = b + a_1 x_1 + ... + a_n x_n, \tag{392}$$

where $\mathbf{x} = (x_1, ..., x_n)$ and a_i are real constants for $1 \leq i \leq n$.

Suppose we have a single variable function $f(x)$ which has derivative $f'(a)$ near some point $x = a$. This could look something like the following:



Here we have the function $f(x) = x^7 - 7x^5 + x$ and the line in orange is the tangent line at $(1/2, 37/128)$. Clearly when we have a global plot of the graph this line doesn't approximate $f(x)$ at all well. However, as we 'zoom' in on the graph (i.e. we look locally) the tangent line approximates the function better and better. See the right-hand diagram. Therefore, one would expect that we can approximate the function near a point by the tangent line at that point.

This is the intuition behind **linear approximation** but let's make this a bit more precise. Consider the definition of the derivative

$$\frac{df}{dx}(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{393}$$

and suppose the limit on the right-hand side exists. Let $h = x - a$ then this can be rewritten as

$$\frac{df}{dx}(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}. \tag{394}$$

From the definition of the limit one knows that for all $\epsilon > 0$ that there exists a $\delta > 0$ such that if $0 < |x-a| < \delta$ then

$$\left|\frac{f(x)-f(a)}{x-a} - \frac{df}{dx}(a)\right| < \epsilon. \tag{395}$$

For x close to a (ϵ small and δ small), this means that

$$\frac{f(x)-f(a)}{x-a} = \frac{df}{dx}(a) + \epsilon(x) \tag{396}$$

where $\epsilon(x) \to 0$ as $x \to a$. Rearranging gives

$$f(x) = f(a) + \frac{df}{dx}(a)(x-a) + R(x) \tag{397}$$

where $R(x) = \epsilon(x)(x-a)$, called the remainder, goes to 0 faster than $x-a$ as $x \to a$. This means that for x close to a one can indeed approximate $f(x)$ as

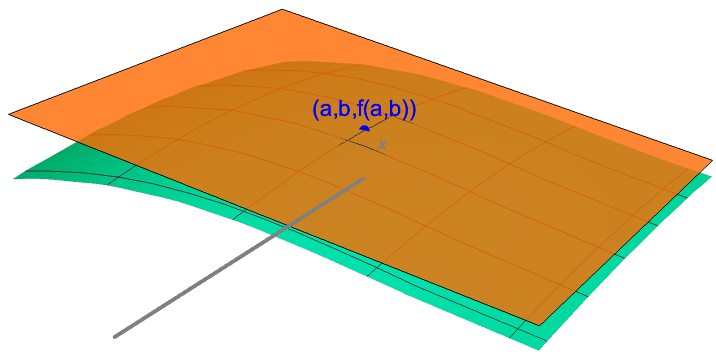
$$f(x) \approx L(x) \doteq f(a) + \frac{df}{dx}(a)(x-a). \tag{398}$$

The function $L(x)$ is a linear function on $\mathbb{R}$ and is known as the **linear approximation** of $f(x)$ at a. Notice that for functions of one variable, the linear approximation only exists if $f'(a)$ exists, i.e. the tangent line exists at $x = a$. In this case it is always a valid approximation.

For a function f of two variables the intuition is similar: if f has **continuous partial derivatives**, the tangent plane at a point $(a, b, f(a, b))$ with equation (from (382))

$$z = f(a,b) + (\partial_x f)(a,b)(x-a) + (\partial_y f)(a,b)(y-b) \tag{399}$$

is well-defined since it contains all tangent lines to the surface at $(a, b, f(a, b))$. In this case, the tangent plane becomes a good approximation of the function near $(a, b, f(a, b))$ see the following diagram:



This means that in analogy with the argument about tangent lines for functions of a single variable we can approximate $f(x, y)$ using (399), i.e.

$$f(x,y) \approx L(x,y) \doteq f(a,b) + \partial_x f(a,b)(x-a) + \partial_y f(a,b)(y-b). \tag{400}$$

The function $L(x, y)$ is known as the **linear approximation** of $f(x, y)$ at (a, b). It is important to stress that even though one can write down the linear approximation in equation (400) when the partial derivatives exist, it is not always a 'good' approximation. It is a **valid approximation** when the partial derivatives are **continuous** at (a, b).

One can generalise the linear approximation to a function f of n-variables $\mathbf{x} = (x_1, ..., x_n)$. In this case, the linear approximation f at $\mathbf{a} = (a_1, ..., a_n)$ is

$$L(\mathbf{x}) = f(\mathbf{a}) + (\partial_{x_1} f)(\mathbf{a})(x_1 - a_1) + ... + (\partial_{x_n} f)(\mathbf{a})(x_n - a_n). \tag{401}$$

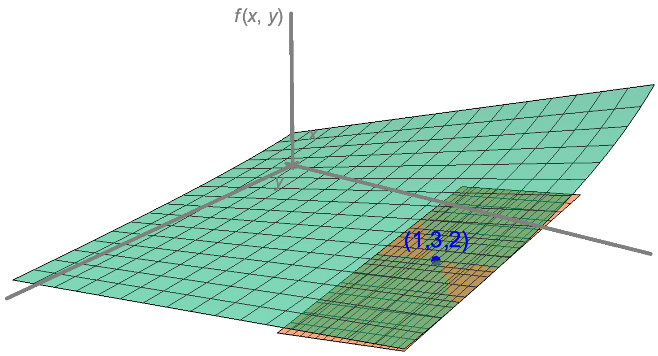
This is a **valid approximation** for $\mathbf{x}$ close to $\mathbf{a}$ when the partial derivatives $\partial_{x_i} f$ are **continuous** at $\mathbf{a}$ for $1 \le i \le n$.

Example 10.3. *Lets find the linear approximation of* $f(x, y) = \frac{1+y}{1+x}$ *at* $(1, 3)$.

Lets compute the first partial derivatives for $x > -1$

$$\partial_x f = -\frac{1+y}{(1+x)^2}, \qquad \partial_y f = \frac{1}{1+x}. \tag{402}$$

Note that the partial derivatives in equation (402) are both rational functions and, therefore, continuous on their domain of definition. In particular at $(1, 3)$. *Hence, the tangent plane at* $(1, 3)$ *is well-defined and the linear approximation is valid as shown by the following plot:*



At $(1, 3)$,

$$\partial_x f(1, 3) = -1, \quad \partial_y f(1, 3) = \frac{1}{2}. \tag{403}$$

Therefore, the linear approximation of $f(x, y)$ *at* $(1, 3)$ *is*

$$L(x, y) = f(1, 3) + \partial_x f(1, 3)(x - 1) + \partial_y f(1, 3)(y - 3) = 2 - (x - 1) + \frac{1}{2}(y - 3). \tag{404}$$

Example 10.4. *Lets find the linear approximation of* $f(x, y) = xe^{xy}$ *at* $(1, 0)$.

Note that $f(x, y)$ *is well defined on all* $\mathbb{R}^2$ *and therefore, we can freely compute the partial derivatives*

$$\partial_x f = (1 + xy)e^{xy}, \qquad \partial_y f = x^2 e^{xy} \tag{405}$$

which are continuous everywhere on $\mathbb{R}^2$ *since these functions are products of polynomials and the exponential. Therefore, the linear approximation is good at* $(1, 0)$*:*

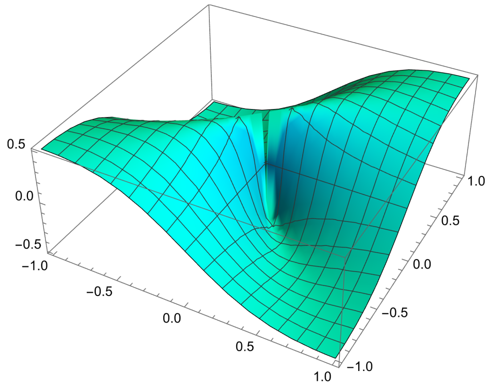
$$L(x, y) = f(1, 0) + \partial_x f(1, 0)(x - 1) + \partial_y f(1, 0)(y - 1) = 1 + (x - 1) + y = x + y. \tag{406}$$

Let's illustrate how the linear approximation fails for non-continuous partial derivatives with the following example:

Example 10.5. *Consider the function,*

$$f(x,y) \doteq \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases} \tag{407}$$

This is plotted below:



One can compute (this is a good exercise):

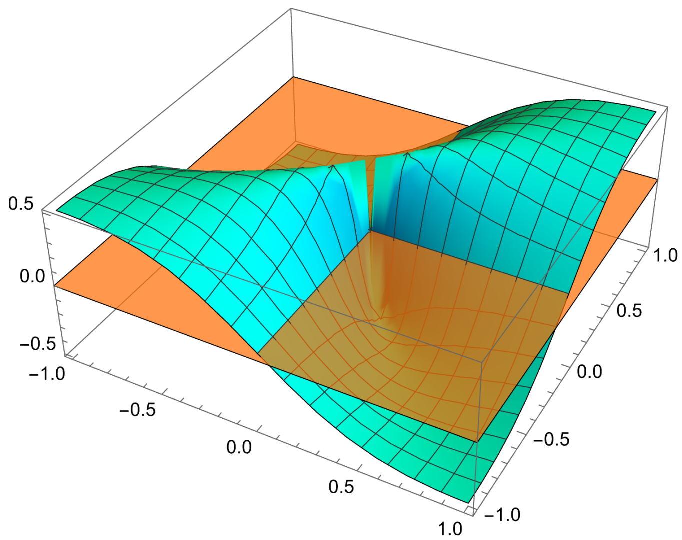
$$\partial_x f = \begin{cases} 0 & \mathbf{x} = (0,0) \\ -\frac{y(x^2-y^2)}{(x^2+y^2)^2} & \mathbf{x} \neq (0,0) \end{cases} \tag{408}$$

$$\partial_y f = \begin{cases} 0 & \mathbf{x} = (0,0) \\ \frac{x(x^2-y^2)}{(x^2+y^2)^2} & \mathbf{x} \neq (0,0) \end{cases} \tag{409}$$

Since the partial derivatives exist at $(0,0)$ we can use equation (400) to write down a linear approximation for $f(x,y)$ at $(0,0)$:

$$L(x,y) = f(0,0) + \partial_x f(0,0)(x-0) + \partial_y f(0,0)(y-0) = 0. \tag{410}$$

However, the partial derivatives are not continuous (you should check this) and therefore, the tangent plane was not well defined (and therefore, the linear approximation is not a good one). One can see this from the following plot:



11 Differentiability for Multivariable Functions

This section is about what it means for a multivariable to be differentiable, not just *partial* differentiable.

11.1 Motivation and Definition

Lets try to reverse engineer the computation above where we derived the linear approximation for a function $f(x)$ of a single variable from the definition of differentiability. In other words, let's derive another condition that a function of two variables must satisfy for the linear approximation to be valid. This will lead us to a definition of what it means for a function of many variables to be differentiable, **not** just partial differentiable.

To this end, suppose $x = a + h_1$ and $y = b + h_2$ for h_1, h_2 small. So, we have $h_1 = x - a$ and $h_2 = y - b$. To ease notation define

$$\mathbf{h} = (h_1, h_2), \tag{411}$$

which has the property $||\mathbf{h}|| \to 0$ as $(x, y) \to (a, b)$. Suppose that the linear approximation in equation (400) is valid for (x, y) close to (a, b). Therefore, one has

$$f(a + h_1, b + h_2) = f(a, b) + \partial_x f(a, b)h_1 + \partial_y f(a, b)h_2 + R(h_1, h_2) \tag{412}$$

where our remainder $R(h_1, h_2) = R(x - a, y - b)$ goes to zero faster than h_1 or h_2, i.e.

$$f(a + h_1, b + h_2) = f(a, b) + \partial_x f(a, b)h_1 + \partial_y f(a, b)h_2 + \varepsilon(h_1, h_2)||\mathbf{h}|| \tag{413}$$

where we have written our remainder $R(h_1, h_2) = \varepsilon(h_1, h_2)||\mathbf{h}||$ with $\varepsilon(h_1, h_2) \to 0$ as $||\mathbf{h}|| \to 0$. Therefore,

$$\frac{|f(a + h_1, b + h_2) - f(a, b) - \langle \nabla f(a, b), \mathbf{h}\rangle|}{||\mathbf{h}||} \leq |\varepsilon(h_1, h_2)|. \tag{414}$$

where we've introduced the **vector** of partial derivatives

$$\nabla f \doteq (\partial_x f, \partial_y f) = \partial_x f\mathbf{i} + \partial_y f\mathbf{j}, \tag{415}$$

which is called the **gradient** of f.

Remark 11.1. *In* $\mathbb{R}^n$,

$$\nabla f \doteq (\partial_{x_1} f, \partial_{x_2} f, ..., \partial_{x_n} f) = \partial_{x_1} f\mathbf{e}_1 + \partial_{x_2} f\mathbf{e}_2 + ... + \partial_{x_n} f\mathbf{e}_n. \tag{416}$$

One makes the following definition about differentiability for multivariable functions,

Definition 11.1 (Differentiability)**.** *Let* $f : \mathbb{R}^n \to \mathbb{R}$*. The function* f *is said to be differentiable at* $\mathbf{x}_0 \in \mathbb{R}^n$ *if there exists a* $\mathbf{v} \in \mathbb{R}^n$ *such that*

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{|f(\mathbf{x}_0 + \mathbf{h}) - f(\mathbf{x}_0) - \langle \mathbf{v}, \mathbf{h}\rangle|}{||\mathbf{h}||} = 0. \tag{417}$$

The condition that a multivariable function is differentiable at (a, b) implies that the linear approximation of a the function at (a, b) is valid, just like the single variable case! Note that previously we said that the linear approximation at (a, b) was valid/good if the partial derivatives of the function at (a, b) were continuous. Therefore, we expect some relation between (continuous) partial differentiability and differentiability.

11.2 Relation to Partial Differentiability

Proposition 11.1. *If f is differentiable at $\mathbf{x}_0$ then this implies all partial derivatives exist at $\mathbf{x}_0$ and $\mathbf{v} = \nabla f$.*

Proof. Essentially this comes from looking at $\mathbf{h} = (h, 0)$ and $\mathbf{h} = (0, h)$ respectively for $\partial_x f$ and $\partial_y f$. Explicitly, suppose

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{|f(\mathbf{x}_0+\mathbf{h}) - f(\mathbf{x}_0) - \langle \mathbf{v}, \mathbf{h}\rangle|}{||\mathbf{h}||} = 0. \tag{418}$$

then since this is a multivariable limit, the limit along all paths through $\mathbf{h} = \mathbf{0}$ must exist and be equal. Therefore, the limit along the path $\mathbf{h} = (h, 0)$ must exist and be equal to zero:

$$\lim_{h\to 0} \frac{|f(x_0+h, y_0) - f(x_0, y_0) - v_1 h|}{|h|} = 0. \tag{419}$$

We can rewrite this as

$$\lim_{h\to 0} \Big|\frac{f(x_0+h, y_0) - f(x_0, y_0)}{h} - v_1\Big| = 0. \tag{420}$$

Unpacking the definition of the limit this says that for any number $\epsilon > 0$ there exists another number $\delta > 0$ such that if $0 < |h - 0| < \delta$ then

$$\Big|\Big|\frac{f(x_0+h, y_0) - f(x_0, y_0)}{h} - v_1\Big| - 0\Big| < \epsilon \tag{421}$$

or equivalently

$$\Big|\frac{f(x_0+h, y_0) - f(x_0, y_0)}{h} - v_1\Big| < \epsilon. \tag{422}$$

This says that

$$\lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h} = v_1, \tag{423}$$

i.e. the limit on the left-hand side exists and is v_1. At this point recall that the limit on the left-hand side is the definition of $\partial_x f(x_0, y_0)$:

$$\partial_x f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}. \tag{424}$$

Therefore, $\partial_x f(x_0, y_0)$ exists and $v_1 = \partial_x f(x_0, y_0)$. A similar argument shows that $\partial_x f(x_0, y_0)$ and $v_2 = \partial_x f(x_0, y_0)$. □

The converse is **not** true: all partial derivatives can exist but f may not be differentiable. For example

Example 11.1. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by*

$$f(x, y) \doteq \begin{cases} \frac{y^3}{x^2+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0). \end{cases} \tag{425}$$

One can check that the gradient at $(0, 0)$ is

$$\nabla f|_{(0,0)} = (0, 1). \tag{426}$$

If f is differentiable then the limit of

$$g(\mathbf{h}) \doteq \frac{|f(\mathbf{h}) - f(\mathbf{0}) - \langle \mathbf{v}, \mathbf{h}\rangle|}{||\mathbf{h}||} \tag{427}$$

should vanish as $\mathbf{h} \to 0$ *for* $\mathbf{v} = \nabla f(0,0)$. *One can compute that*

$$g(\mathbf{h}) = \frac{\left|\frac{h_2^3}{h_1^2+h_2^2} - h_2\right|}{\sqrt{h_1^2+h_2^2}} = \frac{|h_2|h_1^2}{(h_1^2+h_2^2)^{\frac{3}{2}}}. \tag{428}$$

Take $h_2 = h_1$ *then*

$$g(h_1,h_1) = \frac{|h_1|^3}{2\sqrt{2}|h_1|^3} = \frac{1}{2\sqrt{2}} \neq 0. \tag{429}$$

So, we've argued that for the linear approximation to be valid our function must be differentiable and we've showed that partial differentiability does not imply differentiability. So what use is partial differentiability then? Well, as you might expect, if the partial derivatives are continuous then partial differentiability implies differentiability. We simply state the following theorem, which is probably the most useful theorem to know regarding differentiability of multivariable functions:

Theorem 11.1. *If the partial derivatives* $\partial_x f$ *and* $\partial_y f$ *exist near* (a,b) *and are continuous at* (a,b), *then* f *is differentiable at* (a,b).

Remark 11.2. *In practise you're unlikely to ever prove that a function is differentiable directly using definition 11.1. You will use theorem 11.1.*

We will not prove theorem 11.1 in general but we will illustrate it with an example:

Example 11.2. *Show that* $f(x,y) = \frac{1+y}{1+x}$ *is differentiable at* $(1,3)$.

So the partial derivatives are for $x > -1$

$$\partial_x f = -\frac{1+y}{(1+x)^2}, \qquad \partial_y f = \frac{1}{1+x}. \tag{430}$$

Note that these partial derivatives are continuous on their domains of definition since they are rational functions. Further $(1,3)$ *is in the domain of definition so the partial derivatives are continuous at* $(1,3)$, *hence the function is differentiable at* $(1,3)$ *by theorem 11.1.*

Let's show directly that f *is differentiable at* $(1,3)$. *For* f *to be differentiable at* $(1,3)$,

$$\lim_{\mathbf{h}\to 0} \frac{|f(1+h_1, 3+h_2) - f(1,3) - \langle \mathbf{v}, \mathbf{h}\rangle|}{||\mathbf{h}||} = 0 \tag{431}$$

for $\mathbf{v} = \nabla f(1,3)$. *Evalutating the partial derivatives at* $(1,3)$ *gives:*

$$\partial_x f(1,3) = -1, \quad \partial_y f(1,3) = \frac{1}{2} \implies \nabla f = (-1, 1/2). \tag{432}$$

Define for notational simplicity:

$$g(\mathbf{h}) \doteq \frac{|f(1+h_1, 3+h_2) - f(1,3) - \langle \nabla f(1,3), \mathbf{h}\rangle|}{||\mathbf{h}||}. \tag{433}$$

Evaluating:

$$g(\mathbf{h}) = \frac{\left|\frac{4+h_2}{2+h_1} - 2 + h_1 - \frac{1}{2}h_2\right|}{\sqrt{h_1^2+h_2^2}} = \frac{|h_1|\left|h_1 - \frac{1}{2}h_2\right|}{|2+h_1|\sqrt{h_1^2+h_2^2}}. \tag{434}$$

For any $\epsilon > 0$, *we want to show that there is a* $\delta > 0$ *such that if* $||\mathbf{h}|| < \delta$ *then*

$$|g(\mathbf{h})| < \epsilon. \tag{435}$$

So, note that (the triangle inequality on $\mathbb{R}$ tells us)

$$\left|h_1 - \frac{1}{2}h_2\right| \leq |h_1| + \frac{1}{2}|h_2|. \tag{436}$$

Now $|h_1|, |h_2| \leq ||\mathbf{h}||$, so

$$\left|h_1 - \frac{1}{2}h_2\right| \leq \frac{3}{2}||\mathbf{h}||. \tag{437}$$

Hence,

$$|g(\mathbf{h})| \leq \frac{|h_1|}{|2+h_1|}. \tag{438}$$

You can see that this goes to zero for $h_1 \to 0$ in the region

$$\{(h_1, h_2) \in \mathbb{R}^2 : h_2 > -2\}. \tag{439}$$

We have that $|h_1| \leq ||\mathbf{h}|| < \delta$ and we would like

$$\frac{1}{|2+h_1|} \leq 1 \tag{440}$$

so that $|g(\mathbf{h})| < |h_1| \leq ||\mathbf{h}||$. The inequality (440) *is true if $|h_1| < 1$. So pick $\delta = \min(1, \delta_2)$ for δ_2 to be chosen. Therefore,*

$$|h_1| < ||\mathbf{h}|| < \min(\delta_2, 1) \leq 1, \tag{441}$$

and

$$|2+h_1| > |2-|h_1|| \geq 1 \implies \frac{1}{|2+h_1|} < \frac{1}{|2-|h_1||} \leq 1. \tag{442}$$

Therefore,

$$|g(\mathbf{h})| \leq \frac{|h_1|}{|2+h_1|} < |h_1| < \min(\delta_2, 1) \tag{443}$$

if one now picks $\delta_2 = \epsilon$ then

$$|g(\mathbf{h})| < \epsilon. \tag{444}$$

12 The Chain Rule

The chain rule provides a way to differentiate a composite function. You are probably familiar with this for functions of a single variable. This section is about how to generalise this to multivariable functions.

12.1 Single Variable Functions

Proposition 12.1. *Suppose f is a differentiable function of a single variable t and suppose $t = g(x)$ for a variable x and g is differentiable. Then the composite function $f(g(x))$ or $(f \circ g)(x)$ satisfies the chain rule for differentiation with respect to x:*

$$\frac{d(f \circ g)}{dx} = \frac{df}{dg}\frac{dg}{dx}, \qquad \frac{d(f \circ g)}{dx} = \frac{df}{dt}\frac{dt}{dx}, \qquad \frac{d(f \circ g)}{dx} = \frac{df}{dt}\frac{dg}{dx} \tag{445}$$

or explicitly in with arguments

$$\frac{d(f \circ g)}{dx}(a) = \frac{df}{dt}(g(a))\frac{dg}{dx}(a). \tag{446}$$

Proof.(Sketch). This can be proved in the following way. Suppose f and g can be differentiated at $g(a)$ and a respectively. Assume that $g(x) \neq g(a)$ when $x \neq a$ (you can deal with the case $g(x) = g(a)$ for x close to a by noting that then $g'(a) = 0$ and prove that $(f \circ g)'(a) = 0$). Then

$$\frac{d(f \circ g)}{dx}(a) = \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a} = \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \frac{g(x) - g(a)}{x - a}. \tag{447}$$

Now by the assumption that the derivatives of f and g exist one has

$$\Big(\lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)}\Big)\Big(\lim_{x \to a} \frac{g(x) - g(a)}{x - a}\Big) = \frac{df}{dg}(g(a))\frac{dg}{dx}(a). \tag{448}$$

From the if the product of the limits exist then the limit of the product exists and therefore,

$$\frac{d(f \circ g)}{dx}(a) = \Big(\lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)}\Big)\Big(\lim_{x \to a} \frac{g(x) - g(a)}{x - a}\Big) = \frac{df}{dg}(g(a))\frac{dg}{dx}(a). \tag{449}$$

□

Remark 12.1. *Often one is sloppy and does not explicity keep the composite function in all expressions, i.e. it is understood when writing $\frac{df}{dx}$ we mean the derivative of the composition,*

$$\frac{df}{dx} = \frac{d(f \circ g)}{dx}.$$

Example 12.1. *Suppose $f(x) = 3x^2$ and $x = g(t) = 2t^2 - 1$. Find $\frac{df}{dt}$.*

We compute using the chain rule

$$\frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt} = (6x)(4t) = 24t(2t^2 - 1). \tag{450}$$

Example 12.2. *Suppose $f(x) = e^{7x}$ and $x = g(t) = \frac{1}{7}\ln(t)$ for $t > 0$. Find $\frac{df}{dt}$.*

We compute using the chain rule

$$\frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt} = (7e^{7x})\Big(\frac{1}{7t}\Big) = 1. \tag{451}$$

12.2 Multivariable Functions: A First Step

Suppose we have a differentiable function of two variables, $f : D \to \mathbb{R}$ where D is a subset of $\mathbb{R}^2$. So f takes (x, y) and evalutates $f(x, y)$. Suppose $x = g(t)$ and $y = h(t)$ are differentiable, so that one has a single variable function in reality. How does one compute df/dt?

Proposition 12.2. *Let f be a differentiable function of two variables (x, y) and $\mathbf{g}(t) = (g_1(t), g_2(t))$ such that g_1 and g_2 are differentiable. Then the composite function $f \circ \mathbf{g}$ which is a map from $\mathbb{R}$ to $\mathbb{R}$ satisfies*

$$\frac{d(f \circ \mathbf{g})}{dt} = \partial_x f \frac{dx}{dt} + \partial_y f \frac{dy}{dt}, \qquad \frac{d(f \circ \mathbf{g})}{dt} = \partial_x f \frac{dg_1}{dt} + \partial_y f \frac{dg_2}{dt}. \tag{452}$$

or if you wish to be explicit in your arguments

$$\frac{d(f \circ \mathbf{g})}{dt}(a) = \partial_x f(\mathbf{g}(a)) \frac{dg_1}{dt}(a) + \partial_y f(g(a), h(a)) \frac{dg_2}{dt}(a) = \langle \nabla f(\mathbf{g}(a)), \frac{d\mathbf{g}}{dt}(a) \rangle. \tag{453}$$

Remark 12.2. *Again, one is often sloppy and drops the explicity dependence on the composite function, i.e.*

$$\frac{df}{dt} = \frac{d(f \circ \mathbf{g})}{dt}.$$

Example 12.3. *Let*

$$f(x, y) = xy \tag{454}$$

with $x = \sin(t)$ and $y = \cos(t)$. We can compute

$$\frac{df}{dt} = (y)(\cos t) + (x)(-\sin(t)) = \cos^2(t) - \sin^2(t). \tag{455}$$

Example 12.4. *Let*

$$f(x, y) = x^2 y + 3xy^4 \tag{456}$$

with $x = \sin(2t)$ and $y = \cos(t)$. Find $\frac{df}{dt}(t = 0)$. We can compute

$$\frac{df}{dt} = (2xy + 3y^4)(2\cos(2t)) + (x^2 + 12xy^3)(-sin(t)). \tag{457}$$

Let's be lazy and not simplify. We can just find $(x(0), y(0)) = (0, 1)$ to give

$$\left.\frac{df}{dt}\right|_{t=0} = 6. \tag{458}$$

12.3 Multivariable Functions: Adding Complexity

One could complicate things further and have $x = g(s, t)$ and $y = h(s, t)$. In this case we have the following chain rule for the partial derivatives of f. We will need a vector-valued multivariable function $\mathbf{g} : D \subseteq \mathbb{R}^2 \to \mathbb{R}^2$, i.e. $\mathbf{g}(s, t) = (g_1(s, t), g_2(s, t))$.

Proposition 12.3. *Let f be a differentiable function of two variables (x, y) andf $\mathbf{g} : D \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be differentiable, i.e. $\mathbf{g}(s, t) = (g_1(s, t), g_2(s, t))$ such that g_1 and g_2 are differentiable. Then the composite function $f \circ \mathbf{g}$ which maps $\mathbb{R}^2$ to $\mathbb{R}$ satisfies the chain rule,*

$$\frac{\partial(f \circ \mathbf{g})}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial g_1}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial g_2}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}, \tag{459}$$

$$\frac{\partial(f \circ \mathbf{g})}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial g_1}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial g_2}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}, \tag{460}$$

or if you want to be explicit in arguments at $(s, t) = \mathbf{a} = (a, b)$,

$$\frac{\partial(f \circ \mathbf{g})}{\partial s}(\mathbf{a}) = \frac{\partial f}{\partial x}(\mathbf{g}(\mathbf{a}))\frac{\partial g_1}{\partial s}(\mathbf{a}) + \frac{\partial f}{\partial y}(\mathbf{g}(\mathbf{a}))\frac{\partial g_2}{\partial s}(\mathbf{a}) = \left\langle \nabla f(\mathbf{g}(\mathbf{a})), \frac{\partial \mathbf{g}}{\partial s}(\mathbf{a}) \right\rangle \tag{461}$$

$$\frac{\partial(f \circ \mathbf{g})}{\partial t}(\mathbf{a}) = \frac{\partial f}{\partial x}(\mathbf{g}(\mathbf{a}))\frac{\partial g_1}{\partial t}(\mathbf{a}) + \frac{\partial f}{\partial y}(\mathbf{g}(\mathbf{a}))\frac{\partial g_2}{\partial t}(\mathbf{a}) = \left\langle \nabla f(\mathbf{g}(\mathbf{a})), \frac{\partial \mathbf{g}}{\partial t}(\mathbf{a}) \right\rangle. \tag{462}$$

Example 12.5. *Suppose $F(x,y)=xy$ and $x=st$ and $y=\ln(st)$ with $st>0$ then*

$$\partial_s F = yt + \frac{x}{st}t = yt + \frac{x}{s} = t\ln(st)+t, \qquad \partial_t F = s\ln(st)+s \tag{463}$$

Example 12.6. *Let $f(x,y)=e^x\sin(y)$ and $x=st^2$, $y=s^2t$.*

$$\partial_s f = e^x\sin(y)t^2 + e^x\cos(y)2st = e^{st^2}t^2\sin(s^2t)+2e^{st^2}st\sin(s^2t) \tag{464}$$

$$\partial_t f = 2ste^x\sin(y)+s^2e^x\cos(y) = 2ste^{st^2}\sin(s^2t)+s^2e^{t^2s}\cos(s^2t). \tag{465}$$

Example 12.7. *Let's use the chain rule to show that the Laplacian*

$$\Delta f = \partial_x^2 f + \partial_y^2 f \tag{466}$$

can be written in polar coordinates, $x(r,\theta)=r\cos\theta$, $y(r,\theta)=r\sin\theta$ as

$$\Delta f = \frac{1}{r}\partial_r(r\partial_r f)+\frac{1}{r^2}\partial_\theta^2 f. \tag{467}$$

We will assume that mixed partial derivatives commute: $\partial_x\partial_y f = \partial_y\partial_x f$.

The function f here is a function of x and y which depend on r and θ when we change coordinates. So $x=x(r,\theta)$ and $y=y(r,\theta)$ and our $\mathbf{x}:\mathbb{R}^2\setminus\{0\}\to\mathbb{R}^2$ is

$$\mathbf{x}(r,\theta)=(r\cos\theta, r\sin\theta).$$

So, using the chain rule (suppressing the composition)

$$\frac{\partial f}{\partial r}(r,\theta)=\langle\nabla_{\mathbf{x}}f,\frac{\partial \mathbf{x}}{\partial r}\rangle = \frac{\partial f}{\partial x}(x(r,\theta),y(r,\theta))\frac{\partial x}{\partial r}(r,\theta)+\frac{\partial f}{\partial y}(x(r,\theta),y(r,\theta))\frac{\partial y}{\partial r}(r,\theta), \tag{468}$$

$$\frac{\partial f}{\partial r}(r,\theta)=\frac{\partial f}{\partial x}(x(r,\theta),y(r,\theta))\cos\theta+\frac{\partial f}{\partial y}(x(r,\theta),y(r,\theta))\sin\theta. \tag{469}$$

Call

$$g_1(r,\theta)=\frac{\partial f}{\partial x}(x(r,\theta),y(r,\theta)) \qquad g_2(r,\theta)=\frac{\partial f}{\partial y}(x(r,\theta),y(r,\theta)) \tag{470}$$

then

$$\frac{\partial f}{\partial r}(r,\theta)=\cos\theta g_1(r,\theta)+\sin\theta g_2(r,\theta). \tag{471}$$

Lets compute now

$$\frac{\partial^2 f}{\partial r^2}(r,\theta)=\cos\theta\frac{\partial g_1}{\partial r}(r,\theta)+\sin\theta\frac{\partial g_2}{\partial r}(r,\theta). \tag{472}$$

We can reuse equation (469) with $f\mapsto g_1$ and g_2 to deduce:

$$\begin{aligned}\frac{\partial^2 f}{\partial r^2}(r,\theta) &= \cos^2\theta\frac{\partial g_1}{\partial x}(x(r,\theta),y(r,\theta))+\sin^2\theta\frac{\partial g_2}{\partial y}(x(r,\theta),y(r,\theta)) \\ &\quad+\sin\theta\cos\theta\Big[\frac{\partial g_1}{\partial y}(x(r,\theta),y(r,\theta))+\frac{\partial g_2}{\partial x}(x(r,\theta),y(r,\theta))\Big]\end{aligned} \tag{473}$$

Plugging back in $g_1=\partial_x f$ and $g_2=\partial_y f$ gives (suppressing arguments):

$$\frac{\partial^2 f}{\partial r^2}=\cos^2\theta\frac{\partial^2 f}{\partial x^2}+\sin^2\theta\frac{\partial^2 f}{\partial y^2}+2\sin\theta\cos\theta\frac{\partial^2 f}{\partial x\partial y}. \tag{474}$$

If you do the same with $\partial_\theta^2 f$ you will find

$$\partial_\theta^2 f = r^2\sin^2\theta\frac{\partial^2 f}{\partial x^2} + r^2\cos^2\theta\frac{\partial^2 f}{\partial y^2} - 2r^2\sin\theta\cos\theta\frac{\partial^2 f}{\partial x\partial y} \\ - r\cos\theta\partial_x f - r\sin\theta\partial_y f. \tag{475}$$

If you now compute

$$\frac{1}{r}\partial_r(r\partial_r f) + \frac{1}{r^2}\partial_\theta^2 f = \partial_r^2 f + \frac{1}{r}\partial_r f + \frac{1}{r^2}\partial_\theta^2 f, \tag{476}$$

you will find

$$\frac{1}{r}\partial_r(r\partial_r f) + \frac{1}{r^2}\partial_\theta^2 f = \partial_x^2 f + \partial_y^2 f \tag{477}$$

using $\sin^2\theta + \cos^2\theta = 1.$

12.4 Multivariable Functions: Generality

Let's do the general case:

Proposition 12.4. *Let f be a differentiable function of n-variables $x_1, ..., x_n$. Suppose that $x_1, ..., x_n$ are given by differentiable functions of m-variables $y_1, ..., y_m$, i.e. $\mathbf{x} = \mathbf{g}(y_1, ..., y_m)$ where $\mathbf{g} : \mathbb{R}^m \to \mathbb{R}^n$. Then $(f \circ \mathbf{g}) : \mathbb{R}^m \to \mathbb{R}$ satisfies the chain rule,*

$$\frac{\partial f}{\partial y_i} = \sum_{k=1}^{n}\frac{\partial f}{\partial x_k}\frac{\partial x_k}{\partial y_i} = \Big\langle \nabla f, \frac{\partial \mathbf{x}}{\partial y_i}\Big\rangle, \qquad \frac{\partial f}{\partial y_i} = \sum_{k=1}^{n}\frac{\partial f}{\partial g_k}\frac{\partial g_k}{\partial y_i}, \tag{478}$$

for all $i = 1, ..., m$. Explicitly at $\mathbf{a} \in \mathbb{R}^m$ with arguments and composition

$$\frac{\partial(f\circ\mathbf{g})}{\partial y_i}(\mathbf{a}) = \Big\langle \nabla f(\mathbf{g}(\mathbf{a})), \frac{\partial \mathbf{g}}{\partial y_i}(\mathbf{a})\Big\rangle = \sum_{k=1}^{n}\frac{\partial f}{\partial x_k}\Big|_{(g_1(a_1,...,a_n),...,g_m(a_1,...,a_n))}\frac{\partial x_k}{\partial y_i}\Big|_{(a_1,...,a_n)}. \tag{479}$$

Example 12.8. *Suppose $u(x, y, z) = x^4y + y^2z^3$, where $x = rse^t$ and $y = rs^2e^{-t}$ and $z = r^2s\sin(t)$. Find $\partial_s u$ at $(r, s, t) = (2, 1, 0)$.*

Using the chain rule one has

$$\partial_s u = \partial_x u\partial_s x + \partial_y u\partial_s y + \partial_z u\partial_s z \tag{480}$$
$$= 4x^3yre^t + (x^4 + 2yz^3)2rse^{-t} + 3yz^2r^2\sin(t). \tag{481}$$

Evaluating at $(r, s, t) = (2, 1, 0)$ or equivalently $(x, y, z) = (2, 2, 0)$ one has

$$\partial_s u(2, 1, 0) = 192. \tag{482}$$

Example 12.9. *Suppose $u(t, r) = \sin(t - kr)$ for a constant k, $r = ||\mathbf{x}||$ where $\mathbf{x} = (x, y, z)$ and $t = t$. Compute:*

$$k^2\partial_t^2 u - \partial_z^2 u - \partial_y^2 u - \partial_x^2 u. \tag{483}$$

Computing directly gives

$$\partial_z u = \partial_t u\partial_z t + \partial_r u\partial_z r = -k\cos(t - kr)\frac{z}{||\mathbf{x}||}. \tag{484}$$

Taking a second derivative using the product rule gives

$$\partial_z^2 u = -k(\partial_z\cos(t - kr))\frac{z}{||\mathbf{x}||} - k\cos(t - kr)\partial_z\Big(\frac{z}{||\mathbf{x}||}\Big). \tag{485}$$

Using the chain rule for multivariable functions on the first term and the chain rule for functions of a single variable on the second term gives,

$$\partial_z^2 u = -\frac{k^2 z^2}{||\mathbf{x}||^2}\sin(t-kr) + \frac{k(z^2 - ||\mathbf{x}||^2)\cos(t-kr)}{||\mathbf{x}||^3} \tag{486}$$

The function is symmetric under $z \leftrightarrow x$ and $z \leftrightarrow y$, so

$$\partial_x^2 u = -\frac{k^2 x^2}{||\mathbf{x}||^2}\sin(t-kr) + \frac{k(x^2 - ||\mathbf{x}||^2)\cos(t-kr)}{||\mathbf{x}||^3} \tag{487}$$

$$\partial_y^2 u = -\frac{k^2 y^2}{||\mathbf{x}||^2}\sin(t-kr) + \frac{k(y^2 - ||\mathbf{x}||^2)\cos(t-kr)}{||\mathbf{x}||^3}. \tag{488}$$

Additionally,

$$\partial_t^2 u = -\sin(t-kr). \tag{489}$$

Therefore,

$$k^2\partial_t^2 u - \partial_z^2 u - \partial_y^2 u - \partial_x^2 u = 0. \tag{490}$$

12.5 Implicit Functions

One cannot always solve $F(x,y) = 0$ for y. We then view F as an implicit definition of y as a function of x, i.e.

$$F(x, y(x)) = 0. \tag{491}$$

How do we find $\frac{dy}{dx}$?

Proposition 12.5. *Let y be an implicitly defined function of x by*

$$F(x,y) = 0. \tag{492}$$

Suppose F and y are differentiable and $\partial_y F \neq 0$ then

$$\frac{dy}{dx} = -\frac{\partial_x F}{\partial_y F}. \tag{493}$$

Proof. Writing $F_2(x) = F(x, y(x)) = 0$ and using the chain rule gives

$$\frac{dF_2}{dx} = 0 = \partial_x F + \partial_y F \frac{dy}{dx}. \tag{494}$$

□

This generalises to the case where z is an implicitly defined function of (x,y) through

$$F(x,y,z) = 0, \tag{495}$$

i.e.

$$F_2(x,y) = F(x,y,z(x,y)) = 0 \tag{496}$$

Proposition 12.6. *Suppose F and z are differentiable and $\partial_z F \neq 0$ then*

$$\frac{\partial z}{\partial x} = -\frac{\partial_x F}{\partial_z F}, \qquad \frac{\partial z}{\partial y} = -\frac{\partial_y F}{\partial_z F}. \tag{497}$$

Proof. This follows from the chain rule applied to F_2, i.e.

$$0 = \partial_x F_2 = \partial_x F \cancelto{1}{\partial_x x} + \partial_z F \partial_x z. \tag{498}$$

□

Example 12.10. *Suppose* $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1 = 0$*. Find* $\partial_x z$ *and* $\partial_y z$*.*

$$\partial_x F = 3x^2 + 6yz, \quad \partial_y F = 3y^2 + 6yz, \quad \partial_z F = 6xy + 3z^2. \tag{499}$$

Therefore,

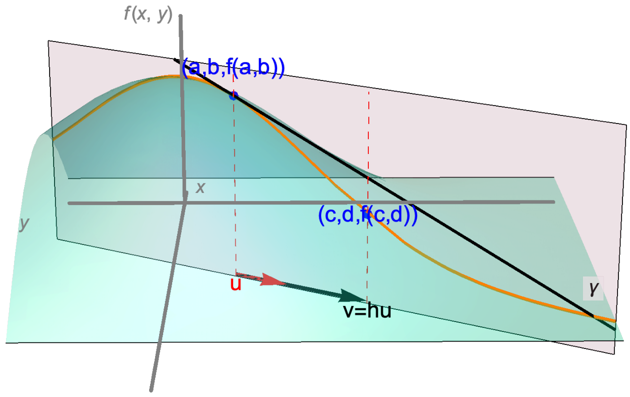
$$\partial_x z = -\frac{x^2 + 2yz}{2xy + z^2}, \quad \partial_x y = -\frac{y^2 + 2xz}{2xy + z^2}. \tag{500}$$

13 Directional Derivatives and the Gradient Vector

As the name suggests directional derivatives are derivatives of a function f in a direction specified by a vector $\mathbf{u}$.

13.1 Introduction and Definition

Suppse f is a function of two variables (x, y) and $\mathbf{u} = (u_1, u_2)$ is a **unit** vector in the xy-plane. The surface defined by the equation $z = f(x, y)$ is plotted below:



Let p be a point on this surface with Cartesian coordinates $\mathbf{x}_0 = (a, b, f(a, b))$. Consider intersecting the surface with the plane containing $\mathbf{x}_0$, $\mathbf{u}$ and $\mathbf{k}$ (the unit vector in the z-direction), i.e. the plane with equation,

$$\langle \mathbf{n}, \mathbf{x} - \mathbf{x}_0 \rangle, \quad \mathbf{n} = \mathbf{u} \times \mathbf{k}. \tag{501}$$

This produces a curve γ lying in the surface defined by $z = f(x, y)$ along which $\mathbf{u}$ points. Let q be a points on this curve with Cartesian coordinates $(c, d, f(c, d))$. The projections of p and q to the xy-plane have Cartesian coordinates (a, b) and (c, d) respectively. The displacement vector $\mathbf{v}$ (the black arrow on the above diagram) from the projection of p to the projection of q is proportional to $\mathbf{u}$ by a positive scalar multiple, i.e.

$$\mathbf{v} = h\mathbf{u} = (hu_1, hu_2). \tag{502}$$

Consider the new function

$$g(h) = \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h}. \tag{503}$$

This ratio is the change in height in z as you move along γ over the change in length in the xy-plane. Taking the limit of g as $h \to 0$ gives the rate of change of f in the direction of $\mathbf{u}$. We make the following definition:

Definition 13.1. *Let f be a function of two vartiables (x, y). Then the directional derivative of f at (a, b) in the direction of the unit vector $\mathbf{u} = (u_1, u_2)$ is*

$$D_{\mathbf{u}} f(a, b) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h}, \tag{504}$$

if the limit on the right-hand side exists.

Remark 13.1. *This generalises to n-variables*

$$D_{\mathbf{u}} f(a_1, ..., a_n) = \lim_{h \to 0} \frac{f(a_1 + hu_1, a_2 + hu_2, ..., a_n + hu_n) - f(a_1, a_2, ..., a_n)}{h}, \tag{505}$$

Example 13.1. *Note that even all directional derivatives existing does not imply differentiablility. For example, let*

$$f(x,y) \doteq \begin{cases} \frac{x^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases} \tag{506}$$

Now,

$$D_{\mathbf{u}} f(0,0) = \lim_{h\to 0} \frac{f(hu_1, hu_2) - f(0,0)}{h} = \lim_{h\to 0} \frac{(hu_1)^3}{h[(hu_1)^2 + (hu_2)^2]} = \frac{u_1^3}{u_1^2 + u_2^2}. \tag{507}$$

So all directional derivatives exist at $(0,0)$. In particular, by take $\mathbf{u} = (1,0)$ and $\mathbf{u} = (0,1)$, one finds $\nabla f = (1,0)$.

So, to check the differentiablility of f one has to check the following limit vanishes:

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{|f(h_1,h_2) - f(0,0) - \langle \nabla f(0,0), \mathbf{h}\rangle|}{||\mathbf{h}||} = \lim_{\mathbf{h}\to\mathbf{0}} \frac{|h_1|h_2^2}{(h_1^2+h_2^2)^{\frac{3}{2}}}. \tag{508}$$

Take $h_1 = mh_2$ then

$$\frac{|h_1|h_2^2}{(h_1^2+h_2^2)^{\frac{3}{2}}} = \frac{|m||h_2|^3}{((m^2+1)h_2^2)^{\frac{3}{2}}} = \frac{|m|}{(m^2+1)^{\frac{3}{2}}} \neq 0. \tag{509}$$

Therefore, the limit in definition differentiablility does not vanish and hence, f is not differentiable.

13.2 The Relation Between Directional Derivatives and the Gradient Vector

If f is differentiable then we have a nice relation to the gradient vector above:

Proposition 13.1. *Suppose f is a differentiable function of n-variables $(x_1, ..., x_n)$ and $\mathbf{u}$ is a unit vector in $\mathbb{R}^n$. Then all directional derivatives exist and*

$$D_{\mathbf{u}} f = \langle \nabla f, \mathbf{u}\rangle. \tag{510}$$

Proof. (Non-Examinable) We will prove this in $\mathbb{R}^2$. If f is differentiable one has

$$\lim_{\mathbf{h}\to 0} \frac{|f(a_1+h_1, a_2+h_2) - f(a_1,a_2) - \langle \nabla f, \mathbf{h}\rangle|}{||\mathbf{h}||} = 0. \tag{511}$$

Set $\mathbf{h} = h\mathbf{u}$ where $\mathbf{u}$ is unit. Then one has

$$\lim_{h\to 0} \left| \frac{f(a_1+hu_1, a_2+hu_2) - f(a_1,a_2)}{h} - \langle \nabla f, \mathbf{u}\rangle \right| = 0, \tag{512}$$

since $||\mathbf{h}|| = h||\mathbf{u}|| = h$. Now this say that for all $\epsilon > 0$, there exists a $\delta > 0$ such that if $|h - 0| < \delta$ then

$$\left| \left| \frac{f(a_1+hu_1, a_2+hu_2) - f(a_1,a_2)}{h} - \langle \nabla f, \mathbf{u}\rangle \right| - 0 \right| < \epsilon, \tag{513}$$

which is equivalent to

$$\left| \frac{f(a_1+hu_1, a_2+hu_2) - f(a_1,a_2)}{h} - \langle \nabla f, \mathbf{u}\rangle \right| < \epsilon, \tag{514}$$

which by definition says that

$$\lim_{h\to 0} \frac{f(a_1+hu_1, a_2+hu_2) - f(a_1,a_2)}{h} = \langle \nabla f, \mathbf{u}\rangle. \tag{515}$$

□

Example 13.2. *Find the directional derivative of the function $f(x,y) = x^3 - 3xy + 4y^2$ in the direction $\mathbf{u} = (1, \frac{1}{2})$. Evaluate $D_{\hat{\mathbf{u}}} f(1,2)$.*

Let's now compute the partial derivatives:

$$\partial_x f = 3x^2 - 3y \tag{516}$$
$$\partial_y f = -3x + 8y. \tag{517}$$

These are continuous and therefore f is differentiable.

Since f is differentiable one has from proposition 13.1,

$$D_{\hat{\mathbf{u}}} f = \langle \nabla f, \hat{\mathbf{u}} \rangle \tag{518}$$

So let's make $\mathbf{u}$ a unit vector. Its length is $||\mathbf{u}|| = \frac{\sqrt{5}}{2}$. Therefore, $\hat{\mathbf{u}} = (2/\sqrt{5}, 1/\sqrt{5})$ and

$$D_{\hat{\mathbf{u}}} f = \langle (3x^2 - 3y, -3x + 8y), (2/\sqrt{5}, 1/\sqrt{5}) \rangle = 2/\sqrt{5}(3x^2 - 3y) + 1/\sqrt{5}(-3x + 8y) \tag{519}$$
$$= \frac{6}{\sqrt{5}} x^2 - \frac{3}{\sqrt{5}} x + \frac{2}{\sqrt{5}} y. \tag{520}$$

Evaluating at $(1,2)$ gives

$$D_{\mathbf{u}} f(1,2) = \frac{7}{\sqrt{5}}. \tag{521}$$

13.3 Properties of the Gradient Vector

We have already seen one interesting property of the gradient vector. Namely for differentiable functions,

$$D_{\mathbf{u}} f = \langle \nabla f, \mathbf{u} \rangle. \tag{522}$$

Another interesting property follows from considering level sets/curves/surfaces. Suppose we consider a level curve γ of a continuous function of two variables $f(x,y)$, i.e. we look at

$$\{(x,y) \in \mathbb{R}^2 : f(x,y) = k, k = \text{const}\}. \tag{523}$$

Suppose the level curve γ can expressed as a differentiable vector-valued function $\mathbf{r}(t) = (x = r_1(t), y = r_2(t))$. So we have

$$g(t) = f(r_1(t), r_2(t)) = k \tag{524}$$

for all t. We note that $dg/dt = 0$ and suppose $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist. We now compute $df(r_1(t), r_2(t))/dt$ using the chain rule:

$$0 = \frac{\partial f}{\partial x}\frac{dr_1}{dt} + \frac{\partial f}{\partial y}\frac{dr_2}{dt} = \langle \nabla f, \mathbf{r}'(t) \rangle. \tag{525}$$

The vector $\mathbf{r}'(t)$ is tangent to the curve γ. Hence, if $\nabla f \neq \mathbf{0}$, it must be orthogonal to γ, i.e. **the gradient vector is orthogonal to the level curves**. Note that this generalises to level surfaces and general level sets.

The final interesting property of the gradient vector that we will mention here is that $\nabla f(\mathbf{x})$ points in the direction of maximum rate of increase of f at $\mathbf{x}$, and that maximum rate of change is $||\nabla f(\mathbf{x})||$. Precisely:

Proposition 13.2. *Suppose f is a differentiable of n-variables. The maximum value of the directional derivative $D_{\mathbf{u}} f(\mathbf{x})$ is $||\nabla f(\mathbf{x})||$ and it occurs when the unit vector $\mathbf{u}$ is in the same direction as ∇f.*

Proof. From propositions 13.1 and 3.2 one has

$$D_{\mathbf{u}} f = \langle \nabla f, \mathbf{u} \rangle = ||\mathbf{u}|| ||\nabla f|| \cos\theta = ||\nabla f|| \cos\theta. \tag{526}$$

The left-hand side is maximised when $\theta = 0$. Hence, $D_{\mathbf{u}} f$ has maximimum $||\nabla f||$ and this occurs when $\theta = 0$, i.e. when $\mathbf{u}$ and ∇f are in the same direction. □

14 Extrema: Maxima and Minima

14.1 Review of Single Variables

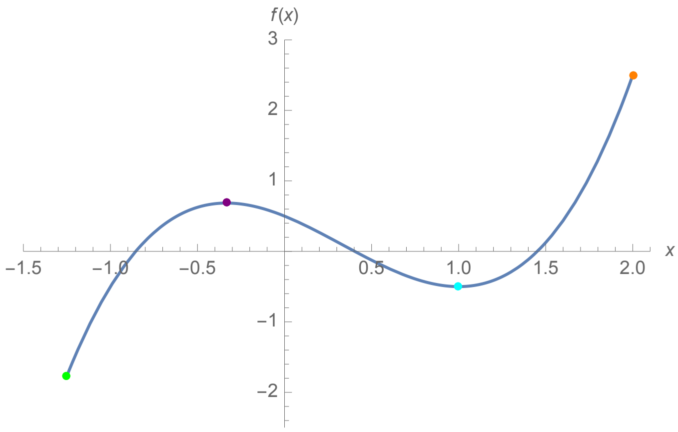
Here we briefly recall the notions associated to extrema of functions of a single variable.

Definition 14.1 (Maxima/Minima/Extrema). *Let $f : D \to \mathbb{R}$ where D is a subset of $\mathbb{R}$. We have the following notions of maxima and minima (collectively known as extrema):*

- *The point $a \in D$ is said to be a local maximum for f if there exists an $\epsilon > 0$ such that $f(x) \leq f(a)$ for all $x \in (a - \epsilon, a + \epsilon)$.*
- *The point $a \in D$ is said to be a local minimum for f if there exists an $\epsilon > 0$ such that $f(x) \geq f(a)$ for all $x \in (a - \epsilon, a + \epsilon)$.*
- *The point $a \in D$ is said to be a global maximum for f if $f(x) \leq f(a)$ for all $x \in D$.*
- *The point $a \in D$ is said to be a global minimum for f if $f(x) \geq f(a)$ for all $x \in D$.*

One can place the word strict in front of these notions if the inequality is strict.

This definition is illustrated with the following picture:



Here we have a function on the interval $D = [-\frac{5}{4}, 2]$.

- The point in purple is a local max.
- The point in cyan is a local min.
- The point in orange is a global max.
- The point in green is a global min.

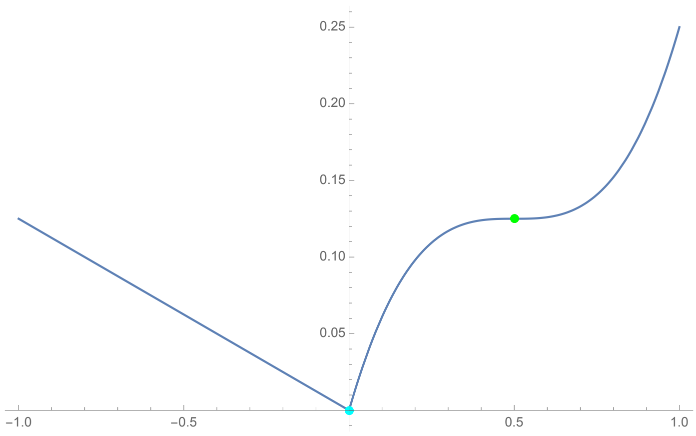
Recall also the following definition:

Definition 14.2 (Critical/Stationary Point). *Let $\mathrm{dom}(f)$ be a subset of $\mathbb{R}$. A stationary point of a function $f : \mathrm{dom}(f) \to \mathbb{R}$ is a number $a \in \mathbb{R}$ such that $f'(a) = 0$. A critical point $a \in \mathbb{R}$ of f is a stationary point or a point where $f'(a)$ does not exist.*

Example 14.1. *Let*

$$f(x) = \begin{cases} -\frac{1}{8}x & x < 0 \\ \left(x - \frac{1}{2}\right)^3 + \frac{1}{8} & x \geq 0. \end{cases} \tag{527}$$

This is plotted below:



This function is continuous on $\mathbb{R}$: observe that $\lim_{x\to 0} f(x) = 0 = f(0)$. For $x \neq 0$, the functions derivative is the following

$$f'(x) = \begin{cases} -\frac{1}{8} & x < 0 \\ 3\left(x - \frac{1}{2}\right)^2 & x > 0. \end{cases} \tag{528}$$

Therefore is has a stationary point at $x = \frac{1}{2}$ (plotted in green). For $x = 0$ then

$$\lim_{h\to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h\to 0^+} \frac{(h - \frac{1}{2})^3 + \frac{1}{8} - 0}{h} = \frac{3}{4}. \tag{529}$$

$$\lim_{h\to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h\to 0^-} \frac{-\frac{1}{8}h - 0}{h} = -\frac{1}{8}. \tag{530}$$

Therefore, $f'(0)$ doesn't exist, which means its a critical point, shown in cyan in the figure above.

14.1.1 Local Extrema

The following theorem of Fermat gives a nessecary condition for finding a local min/max for differentiable functions via stationary points:

Theorem 14.1 (Interior Extremum Theorem)**.** *Suppose f has a local minimum or maximum at a. If f is differentiable at a then $f'(a) = 0$, i.e. a is a stationary point.*

This can be used to attempt to identify local maximums and minimums for functions that are differentiable. You should think that it narrows down the points we need to examine as potential local min/max. For example,

Example 14.2. *Let $f(x) = x^3 - 2x^2 + 4$. The function f is differentiable everywhere with*

$$f'(x) = 3x^2 - 4x. \tag{531}$$

To find the stationary points we consider $3x^2 - 4x = 0$, which gives the stationary points $x = 0, x = \frac{4}{3}$.

What this doesn't tell us is whether the stationary points are local maximum/minimums or something else. There are a few ways to check, the most common is the second derivative test:

Proposition 14.1 (Second Derivative Test)**.** *Suppose f'' is continuous near a. If a is a stationary point*

- *and $f''(a) > 0$, then f has a local min at a.*
- *and $f''(a) < 0$, then f has a local max at a.*
- *and if $f''(a) = 0$, then the test is inconclusive.*

Let's return to our example

Example 14.3. *Let $f(x) = x^3 - 2x^2 + 4$. We've found*

$$f'(x) = 3x^2 - 4x. \tag{532}$$

You can compute that

$$f''(x) = 6x - 4, \tag{533}$$

which is continuous. So, evaluting at the stationary points $x = 0, x = \frac{4}{3}$ gives

$$f''(0) = -4 < 0 \implies (0, f(0)) \text{ is a max.} \tag{534}$$

$$f''(\frac{4}{3}) = 4 > 0 \implies (\frac{4}{3}, f(\frac{4}{3})) \text{ is a min..} \tag{535}$$

What's so inconclusive about the case $f''(0) = 0$? Here's a little bit of context:

Definition 14.3. *Suppose f is a function of a single variable x which is twice differentiable with continuous second derivative on its domain $\text{dom}(f)$. A point $a \in \text{dom}(f)$ is said to be a point of inflection of $f''(x)$ changes sign at a, i.e. $f''(a) = 0$, $f''(x) > 0$ ($f''(x) < 0$) for $x < a$ and $f''(x) < 0$ ($f''(x) > 0$) for $x > a$.*

Example 14.4. *Let's look at three examples:*

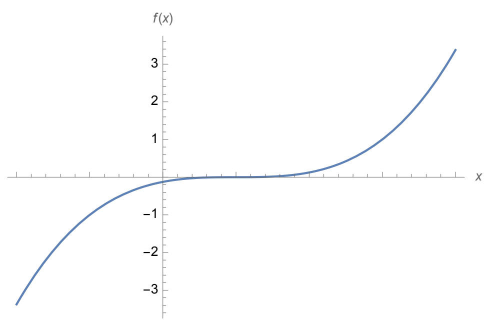
1. *Let*

$$f(x) = (x - \frac{1}{2})^3. \tag{536}$$

Then

$$f'(x) = 3(x - \frac{1}{2})^2, \qquad f''(x) = 6(x - \frac{1}{2}). \tag{537}$$

*So, $f'(x) = 0$ implies $x = \frac{1}{2}$ is a stationary point, with $f''(\frac{1}{2}) = 0$. This is in fact a point of inflection (and **not** a maxima or minima) as shown in the following plot:*



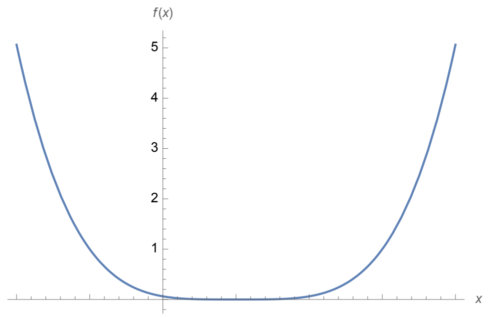
2. *Let*

$$f(x) = (x - \frac{1}{2})^4. \tag{538}$$

Then

$$f'(x) = 4(x - \frac{1}{2})^3, \qquad f''(x) = 12(x - \frac{1}{2})^2. \tag{539}$$

So, $f'(x) = 0$ implies $x = \frac{1}{2}$ is a stationary point, with $f''(\frac{1}{2}) = 0$. However this is not a point of inflection since $f''(x) \geq 0$ for all x. In fact $x = \frac{1}{2}$ is a minima:



3. *Finally, a point where $f'' = 0$ does not need to be a stationary point. For example, take*

$$f(x) = -\sin(x). \tag{540}$$

This gives

$$f'(x) = -\cos(x), \qquad f''(x) = \sin(x). \tag{541}$$

We have $f''(n\pi) = 0$ for n integer and $f''(x)$ changes sign at $n\pi$, i.e. $n\pi$ is a point of inflection. However, $f'(n\pi) = -1(\pm 1)^n \neq 0$. So, it is not a stationary point and certainly not a maxima or minima of $-\sin(x)$.

14.1.2 Global Extrema

The following theorem characterises when one can find an absolute maxima or minima for a function:

Theorem 14.2 (Extreme Value Theorem). *Suppose $f : [a, b] \to \mathbb{R}$ is continuous. Then f attains an absolute maximum $f(c)$ and absolute minimum $f(d)$ for some $c, d \in [a, b]$.*

Remark 14.1. *Relax closedness of the interval or continuity and this theorem is not true.*

To go from local maximum/minimums to absolute maximum/minimums of a continuous function f on a closed interval $[a, b]$ we have the following method:

- Find the critical points of f in (a, b) by considering f': call them $c_1, ..., c_m \in \mathbb{R}$ and evaluate f at $c_1, ..., c_m$, i.e. compute $f(c_1), ..., f(c_m)$.
- Evaluate $f(a)$ and $f(b)$.
- The absolute maximum of f on $[a, b]$ is then

$$\max(f(a), f(b), f(c_1), ..., f(c_m)). \tag{542}$$

- The absolute minimum of f on $[a, b]$ is then

$$\min(f(a), f(b), f(c_1), ..., f(c_m)). \tag{543}$$

Example 14.5. *Take $f(x) = |x|$ on $[-1, 1]$. One has the following derivative*

$$f'(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \\ \text{undefined} & x = 0. \end{cases} \tag{544}$$

Hence, there are no stationary points but there is a critical point at $x = 0$, for which $f(0) = 0$. We can evaluate at the end points $f(-1) = 1 = f(1)$. Therefore, the absolute max of f is

$$\max(1, 0) = 1, \tag{545}$$

and the absolute min of f is

$$\max(1, 0) = 0. \tag{546}$$

14.2 Local Extrema of Functions of Two Variables

We can generalise our notion of local max/min to 2-variables. Effectively, a local max is a point $\mathbf{a}$ in $\mathbb{R}^2$ such that $f(\mathbf{a})$ is greater than f as any point 'near' $\mathbf{a}$. Similarly, a local min is a point $\mathbf{a}$ in $\mathbb{R}^2$ such that $f(\mathbf{a})$ is less than f as any point 'near' $\mathbf{a}$. We formalise this as follows:

Definition 14.4 (Local Maxima/Minima)**.** *Let $f : D \to \mathbb{R}$ where D is a subset of $\mathbb{R}^2$. We have the following notions of maxima and minima:*

- *The point $\mathbf{a} \in D$ is said to be a local maximum for f if there exists an $\epsilon > 0$ such that $f(\mathbf{x}) \leq f(\mathbf{a})$ for all $\mathbf{x} \in \{\mathbf{x} \in \mathbb{R}^2 : ||\mathbf{x} - \mathbf{a}|| < \epsilon\}$.*
- *The point $\mathbf{a} \in D$ is said to be a local minimum for f if there exists an $\epsilon > 0$ such that $f(\mathbf{x}) \geq f(\mathbf{a})$ for all $\mathbf{x} \in \{\mathbf{x} \in \mathbb{R}^2 : ||\mathbf{x} - \mathbf{a}|| < \epsilon\}$.*

One can place the word strict in front of these notions if the inequality is strict.

The notions of critical and stationary points generalise as:

Definition 14.5 (Critical/Stationary Point)**.** *Let $\mathrm{dom}(f)$ be a subset of $\mathbb{R}^2$. A stationary point of a function $f : \mathrm{dom}(f) \to \mathbb{R}$ is a point $\mathbf{a} \in \mathbb{R}$ such that $\partial_x f(\mathbf{a}) = 0$ and $\partial_y f(\mathbf{a}) = 0$. A critical point $\mathbf{a} \in \mathbb{R}^2$ of f is a stationary point or a point where $\partial_x f(\mathbf{a})$ and/or $\partial_y f(\mathbf{a})$ does not exist.*

We have the generalisation of theorem 14.1:

Theorem 14.3. *Suppose f has a local max/min at (a, b) and $\partial_x f(a, b)$ and $\partial_y f(a, b)$ exist. Then (a, b) is a stationary point.*

Proof. If f has a local max/min at (a, b) then $g(x) = f(x, b)$ has a local max/min at a. By theorem 14.1 $g'(a) = 0$, which implies $\partial_x f(a, b) = 0$. Similarly, if f has a local max/min at (a, b) then $h(y) = f(a, y)$ has a local max/min at b then by theorem 14.1 $h'(b) = 0$, which implies $\partial_y f(a, b) = 0$. □

The generalisation of the second derivative test for single variable functions is

Proposition 14.2. *Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that (a, b) is a stationary point of f. Further, let*

$$\mathcal{D}(a, b) = \left| \begin{pmatrix} \partial_x^2 f(a, b) & \partial_{xy}^2 f(a, b) \\ \partial_{xy}^2 f(a, b) & \partial_y^2 f(a, b) \end{pmatrix} \right| = \partial_x^2 f(a, b) \partial_y^2 f(a, b) - [\partial_{xy}^2 f(a, b)]^2. \tag{547}$$

The we have the following cases:

- *If $\mathcal{D}(a, b) > 0$ and $\partial_x^2 f(a, b) > 0$, then $f(a, b)$ is a local min.*
- *If $\mathcal{D}(a, b) > 0$ and $\partial_x^2 f(a, b) < 0$, then $f(a, b)$ is a local max.*
- *If $\mathcal{D}(a, b) < 0$, then $f(a, b)$ is a saddle point.*
- *If $\mathcal{D}(a, b) = 0$, then the test is inconclusive.*

Example 14.6. *Suppose $f(x, y) = axy - bx^2 - cy^2$ with $a \neq 0$ and $b, c > 0$. This function has partial derivatives*

$$\partial_x f = ay - 2bx, \qquad \partial_y f = ax - 2cy. \tag{548}$$

To find the stationary points we look for $\partial_x f = 0$ and $\partial_y f = 0$. One can solve the first of these for $x = \frac{a}{2b} y$ which gives

$$\left(\frac{a^2}{2b} - 2c \right) y = 0. \tag{549}$$

So either $a^2 - 4bc = 0$ or $y = 0$. So one has two cases:

1. *If $a^2 - 4bc \neq 0$ then $(x, y) = (0, 0)$ is a stationary point.*
2. *If $a^2 - 4bc = 0$ then $(x, y) = (\frac{a}{2b}y, y)$ is a line of stationary points.*

Let's find the second deritivatives:

$$\partial_x^2 f = -2b, \qquad \partial_y^2 f = -2c, \qquad \partial_{xy}^2 f = a. \tag{550}$$

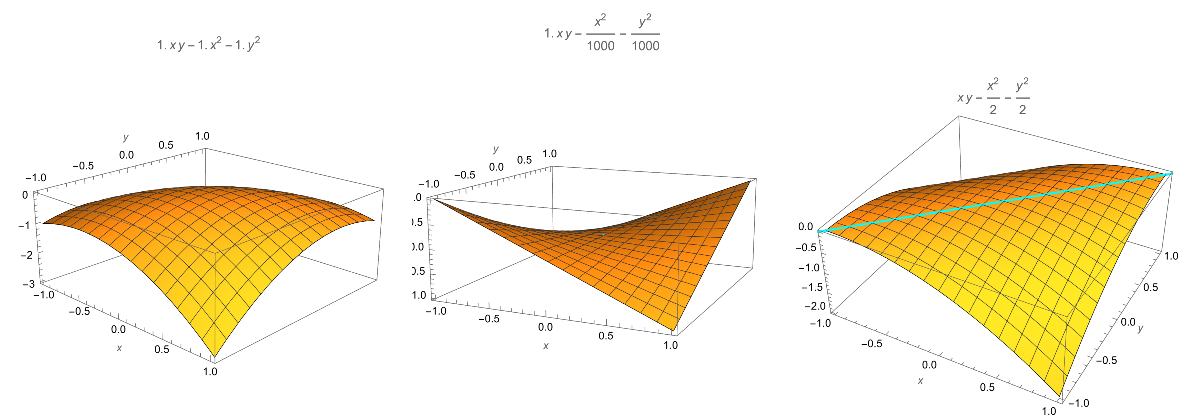
So

$$\mathcal{D} = \partial_x^2 f \partial_y^2 f - (\partial_{xy}^2 f)^2 = 4bc - a^2. \tag{551}$$

Let's compute our cases:

1. *If $a^2 - 4bc < 0$ then $(x, y) = (0, 0)$ is a stationary point, $\mathcal{D}(0, 0) > 0$ and $\partial_x^2 f(0, 0) < 0$ so we have a maximum at $(0, 0)$ by the second derivative test.*
2. *If $a^2 - 4bc > 0$ then $(x, y) = (0, 0)$ is a stationary point, $\mathcal{D}(0, 0) < 0$, so we have a saddle point at $(0, 0)$ by the second derivative test.*
3. *If $a^2 - 4bc = 0$ then $(x, y) = (\frac{a}{2b}y, y)$ is a line of stationary points, but the second derivative test is inconclusive. One can show that this is a line of maximums.*

Here is the plots of the three cases (1,2,3 from left to right):



Example 14.7. *Let*

$$f(x, y) = \frac{x^2 + y^2 + xy - \frac{1}{2}x}{x^2 + y^2}. \tag{552}$$

Note that its domain of definition is $\{(x, y) : x, y \neq 0\}$. To look for stationary points in its domain we compute:

$$\partial_x f = \frac{(2y - 1)(y^2 - x^2)}{(x^2 + y^2)^2}, \qquad \partial_y f = \frac{x(x^2 + y - y^2)}{(x^2 + y^2)^2} \tag{553}$$

So,

$$\partial_x f = 0 \implies y = \frac{1}{2} \qquad y = \pm x. \tag{554}$$

Therefore,

$$\partial_y f(x, \frac{1}{2}) = \frac{4x}{1 + x^2}, \implies x = 0 \tag{555}$$

and

$$\partial_y f(x, \pm x) = \frac{x(x^2 \pm x - x^2)}{2x^2} = \pm \frac{1}{4x^2} \neq 0. \tag{556}$$

So, $(x,y) = (0, \frac{1}{2})$ *is a stationary point of* f*.*

Lets check its second derivative

$$\partial_x^2 f = \frac{x(2y-1)(x^2-3y^2)}{(x^2+y^2)^3}, \qquad \partial_y^2 f = \frac{x^3(1-6y)+xy^2(2y-3)}{(x^2+y^2)^3}, \tag{557}$$

and

$$\partial_{xy}^2 f = -\frac{x^4+3x^2(1-2y)y+(y-1)y^3}{(x^2+y^2)^3}. \tag{558}$$

So,

$$\mathcal{D}\Big(0, \frac{1}{2}\Big) = -4 < 0. \tag{559}$$

Therefore, $(x,y) = (0, \frac{1}{2})$ *is a saddle point.*

Example 14.8. *Suppose*

$$f(x,y) = \arctan(xy). \tag{560}$$

Note the following trick if you forget the derivative of $\arctan$*. Let* $w(z) = \arctan(z)$*. Therefore, from the chain rule*

$$z = \tan(w(z)) \implies 1 = \frac{dz}{dz} = \frac{d\tan(w)}{dw}\frac{dw}{dz}. \tag{561}$$

Now,

$$\tan(w) = \frac{\sin(w)}{\cos(w)}. \tag{562}$$

So, from the product rule

$$\frac{d}{dw}\tan(w) = \frac{\frac{d}{dw}\sin(w)}{\cos(w)} + \sin(w)\frac{d}{dw}\frac{1}{\cos(w)} = 1 + \frac{\sin^2(w)}{\cos^2(w)} = 1 + \tan^2(w). \tag{563}$$

Therefore,

$$\frac{dz}{dz} = 1 = (1+\tan^2(z))\frac{dw}{dz} \implies \frac{dw}{dz} = \frac{1}{1+z^2}. \tag{564}$$

So,

$$\frac{d}{dz}\arctan(z) = \frac{1}{1+z^2}. \tag{565}$$

So, returning to the maxima and minima of $\arctan(xy)$ *we can search for stationary points by considering*

$$\partial_x f = \frac{y}{1+(xy)^2}, \qquad \partial_y f = \frac{x}{1+(xy)^2} \tag{566}$$

using the chain rule. The only stationary point is $(x,y) = (0,0)$ *and there are no critical points. Let's attempt the second derivative test,*

$$\partial_x^2 f = -\frac{2y^3x}{(1+(xy)^2)^2}, \qquad \partial_y^2 f = -\frac{2x^3y}{(1+(xy)^2)^2}, \qquad \partial_{xy}^2 f = \frac{1-x^2y^2}{(1+x^2y^2)^2}. \tag{567}$$

So,

$$\mathcal{D}(0,0) = -1 < 0. \tag{568}$$

Therefore, we have a saddle point at $(x,y) = (0,0)$*.*

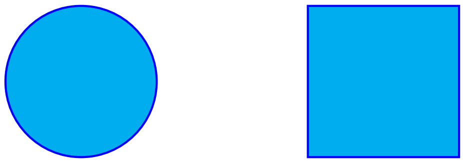
14.3 Global Extrema of Functions of Two Variables

Definition 14.6 (Global Maxima/Minima)**.** *Let $f : D \to \mathbb{R}$ where D is a subset of $\mathbb{R}^2$. We have the following notions of maxima and minima:*

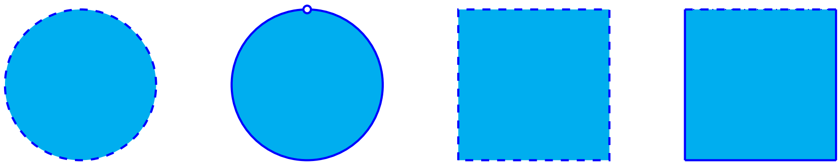
- *The point $\mathbf{a} \in D$ is said to be a global maximum for f if $f(\mathbf{x}) \leq f(\mathbf{a})$ for all $\mathbf{x} \in D$.*
- *The point $\mathbf{a} \in D$ is said to be a global minimum for f if $f(\mathbf{x}) \geq f(\mathbf{a})$ for all $\mathbf{x} \in D$..*

One can place the word strict in front of these notions if the inequality is strict.

What happens to the extreme value theorem 14.2 for function of two variables. We need a notion of closedness for sets in $\mathbb{R}^2$ to replace the closed interval $[a, b]$. Effectively, what one requires is that the set in $\mathbb{R}^2$ contains all points on its boundary. For example,



would be closed where as,



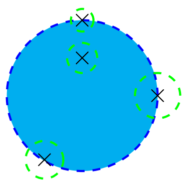
would be not be closed. For this course you can associated the term **closed** with **contains all boundary points**.

Non-Examinable: If you're interested in making this precise read on. The relevant definition to capture the idea of boundary points is this:

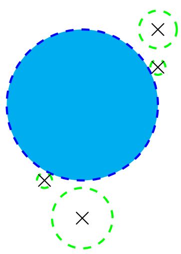
Definition 14.7. *Let S be a set in $\mathbb{R}^2$, $\mathbf{x} \in \mathbb{R}^2$ is a* **limit point** *of S if for all $\epsilon > 0$ there exists $\mathbf{y} \in S$ such that*

$$\mathbf{y} \in \{(x, y) \in \mathbb{R}^2 : ||\mathbf{x} - \mathbf{y}|| < \epsilon\}. \tag{569}$$

The following would be a limit points of S



The following would **not** be a limit point of S:



We then characterise closedness with the condition that all boundary points must be included:

Definition 14.8. *A set S is* **closed** *if S contains all its limit points.*

Something slightly problematic is that $\mathbb{R}$ as a subset of $\mathbb{R}^2$ (i.e. the x-axis say) is closed. The property we need to capture from the extreme value theorem for single variable functions is the fact we were using an **interval**, which is finite or 'bounded'.

Definition 14.9. *Let S be a set in $\mathbb{R}^2$. The set S is bounded if S is contained in a disk/ball of radius $R < \infty$:*

$$B_R = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq R^2\}. \tag{570}$$

Theorem 14.4. *If f is continuous on a closed and bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.*

Why is this theorem important you may ask? Well, if we have a continuous function on a closed and bounded set D we know that the global max/min are attained. If the set was not closed or bounded the function could **approach** a maximum but never attain it. So we know the function attains its global max/min in the set D. At this point we know the local max/min are contained in the stationary points of the function, if those lie in D then these are good candidates for global max/min. We have two other places to check: 1. the critical points, since the function can have odd behaviour there (think $|x|$ has a minimum at $x = 0$ but not stationary points), 2. the boundary, the value on the boundary could be greater than all these points.

So, to go from local maximum/minimums to absolute maximum/minimums of a continuous function f on a closed and bounded set D we have the following method:

- Find the critical points of f in D by considering $\partial_x f, \partial_y f$: call them $\mathbf{c}_1, ..., \mathbf{c}_m \in \mathbb{R}^2$ and evaluate f at $\mathbf{c}_1, ..., \mathbf{c}_m$, i.e. compute $f(\mathbf{c}_1), ..., f(\mathbf{c}_m)$.
- Evaluate f on the boundary of D.
- The absolute maximum of f on D is then the maximum value from steps 1 and 2.
- The absolute minimum of f on D is then the minimum value from steps 1 and 2.

Example 14.9. *Find the absolute maximum and minimum values of the function*

$$f(x, y) = -x^2 + 7xy + 1 + 3y \tag{571}$$

on the square $\{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 1, -1 \leq y \leq 1\}$.

Note that we have a polynomial, so we have continuity everywhere. Let's find the critical points:

$$\partial_x f = -2x + 7y, \qquad \partial_y f = 7x + 3. \tag{572}$$

Therefore, $x = -\frac{3}{7}$ for $\partial_y f = 0$ and $y = -\frac{6}{49}$ for $\partial_x f = 0$. Now one can evaluate $f(-\frac{3}{7}, -\frac{6}{49}) = \frac{40}{49}$.

Let's evaluate on the boundary which is given by the four sets

$$\{(x, y) \in \mathbb{R}^2 : x = 1 \text{ and } -1 \leq y \leq 1\} \tag{573}$$
$$\{(x, y) \in \mathbb{R}^2 : x = -1 \text{ and } -1 \leq y \leq 1\} \tag{574}$$
$$\{(x, y) \in \mathbb{R}^2 : y = 1 \text{ and } -1 \leq x \leq 1\} \tag{575}$$
$$\{(x, y) \in \mathbb{R}^2 : y = -1 \text{ and } -1 \leq x \leq 1\}. \tag{576}$$

1. *$x = 1$ then $f(1, y) = 10y$, which is an increasing function from -10 at $y = -1$ to 10 at $y = 1$.*
2. *$x = -1$ then $f(-1, y) = -4y$, which is an decreasing function from 4 at $y = -1$ to -4 at $y = 1$.*

3. *$y = 1$ then $f(x, 1) = -x^2 + 7x + 4$, which is 10 at $x = 1$ and -4 at $x = -1$ and has a maximum of at $x = \frac{7}{2} > 1$ (so is not in our domain).*

4. *$y = -1$ then $f(x, 1) = -x^2 - 7x - 2$, which is 4 at $x = -1$ and -10 at $x = 1$ and has a maximum at $x = -\frac{7}{2} < -1$ (so is not in our domain).*

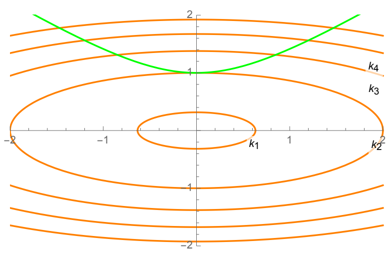
Therefore, our absolute maximum is 10 at $(x, y) = (1, 1)$ and absolute minimum is -10 at $(-1, -1)$.

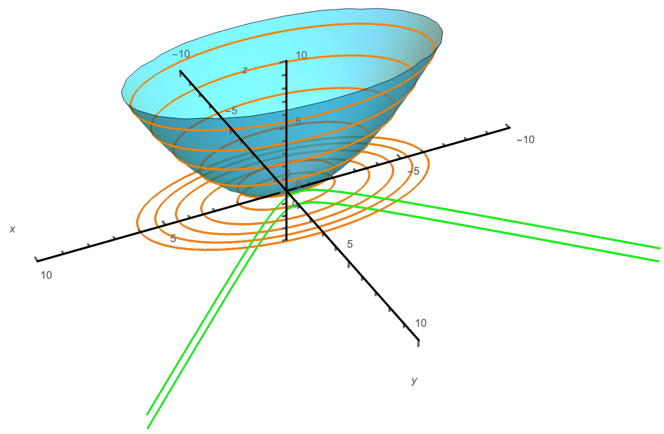
15 Optimisation: Lagrange Multipliers

In the previous section 14, we studied maximums and minimums of functions. If the function represents some physical quantity, say a heat distribution in a room, then the maximum and minimum come with a physical interpretation also, the point with highest and lowest temperture respectively. You could imagine that you want to know where these maximums and minimums are in a physical problem to help you with a decision, i.e. if its a cold day you want to sit at the hottest place in the room. On a less personal level: suppose you work for a renewable energy company and you want to build a new wind farm. You could collect data on where the wind speeds around the world. This could be modeled with a function s which depends on the longditude and lattitude (x, y). Finding the maximum is then important to know where to place your windmills. However, you may not want to place them anywhere in the world. You could have a constraint on where they can be coming from where the energy needs to be, the cost of transport of that energy to the consumer and perhaps where people live. This would be a **contrained** optimisation problem: find the maximum (or minimum) of a function with a constraint. This is where the method of Lagrange multipliers comes into play: find the maximum or minimum of a function $f(x, y, z)$ given a constraint that $g(x, y, z) = k$ where k is a constant.

15.1 Illustration of the Idea

Let's think about finding extreme values of a function of two variables $f(x, y)$ given a constraint $g(x, y) = k$. So $(x, y, f(x, y))$ determines a surface in space $\mathbb{R}^3$. The constraint is a level curve of g, i.e. a curve in the plane. We can plot these together in the plane $\mathbb{R}^2$, i.e. we can plot the level curve of g along with a collection of level curves of f, $f(x, y) = k_1, ..., k_n$. This is plotted below for $f(x, y) = \frac{x^2}{2} + y^2$ and $g(x, y) = x^2 - y^2 = 1$ and $k_1 = \frac{1}{10}, k_2 = 1, k_3 = \frac{19}{10}, k_4 = \frac{14}{5}....$





We can see here that we've rephrased the problem of minimisation under the constraint as: which is the lowest level curve of $f(x, y)$ which touches the level curve given by the constraint? We can see this is the level curve of f that is tangent to the level curve $g(x, y) = k$, i.e. the ellipse that touches the hyperbola on the y-axis. You can also see this from the neighbouring picture in $3D$.

It turns out that above is the general principle: the extreme values of $f(x, y)$ under the constraint $g(x, y) = k$ occur when the level set of $f(x, y)$ is **tangent** to the level curve $g(x, y) = k$. You should think that if the level curve of g crosses a level curve of f there is a lower level curve of f that g touches. Therefore, the lowest level curve of f selected is the one which $g = k$ just touches and no more, i.e. it should be tangent.

15.2 Derivation of the Method

Suppose we want to find the extreme values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$.

- An extreme point for $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ must lie of the surface S defined by $g(x, y, z) = k$. Let this extreme point be denoted $\mathbf{x}_0 = (x_0, y_0, z_0)$.
- Suppose we take a curve γ in S with equation $\mathbf{r}(t) = (r_1(t), r_2(t), r_3(t))$ and at t_0, $\mathbf{r}(t_0) = (x_0, y_0, z_0)$. So it passes through the extreme point of f subject to the constraint $g(x, y, z) = k$.
- Denote $h(t) = f(\mathbf{r}(t)) = f(r_1(t), r_2(t), r_3(t))$. Note that since $\mathbf{r}(t)$ constrains the inputs of f to the surface S, then the extreme value of f subject to the constraint $g(x, y, z) = k$ occurs at the extreme values of h, i.e. when the derivative of h vanishes. We know that the extreme value of f subject to the constraint $g(x, y, z) = k$ occurs at (x_0, y_0, z_0) so

$$h'(t_0) = 0. \tag{577}$$

- Using the chain rule gives

$$0 = h'(t_0) = \partial_x f(x_0, y_0, z_0)r_1'(t_0) + \partial_y f(x_0, y_0, z_0)r_2'(t_0) + \partial_z f(x_0, y_0, z_0)r_3'(t_0) \tag{578}$$
$$= \langle \nabla f(\mathbf{x}_0), \mathbf{r}'(t_0)\rangle. \tag{579}$$

  So $\nabla f(\mathbf{x}_0)$ is orthogonal to $\mathbf{r}'(t_0)$. Note that the curve determined by $\mathbf{r}$ was an arbitrary curve in S that passes through $\mathbf{x}_0$. So this has to be true for any such curve and, therefore, $\nabla f(\mathbf{x}_0)$ is orthogonal to the level surface S determined by the constraint $g(\mathbf{x}) = k$.
- As shown in section 13.3, the gradient of g is orthogonal to the level surfaces of g, if it does not vanish, i.e. provided $\nabla g \neq \mathbf{0}$.
- This shows that $\nabla f(\mathbf{x}_0)$ and $\nabla g(\mathbf{x}_0)$ must be parallel, i.e.

$$\nabla f(\mathbf{x}_0) = \lambda \nabla g(\mathbf{x}_0) \tag{580}$$

  for $\lambda \in \mathbb{R}$ and $\nabla g(\mathbf{x}_0) \neq \mathbf{0}$. The number λ is ascribed the name the Lagrange Multiplier. The equation (580) is what we want to solve to find the extreme values of f under the constraint $g(x, y, z) = k$.

This is a sketch of the derivation of the equation. In practise what you need to do is the following:

The method of Lagrange Multipliers: Suppose you wish to find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ under the assumption that these extreme values exist and $\nabla g \neq \mathbf{0}$ on the surface $\{g(x, y, z) = k\}$. Then one executes the following algorithm:

1. Find all values of x, y, z and λ such that

$$\nabla f = \lambda \nabla g \quad \text{and} \quad g(x, y, z) = k. \tag{581}$$

2. Evaluate f at all points (x, y, z, λ) that result from the first step. The largest of these is the maximum of f subject to the constraint and the smallest is the minimum of f subject to the constraint.

15.3 Examples

Example 15.1. *Find the extreme values of*

$$f(x, y) = x^2 + 2y^2$$

subject to the constraint

$$g(x, y) = x^2 + y^2 = 1$$

.

Let's compute the gradient of f and g. We have

$$\nabla f = (2x, 4y), \qquad \nabla g = (2x, 2y). \tag{582}$$

So we want to solve the system of equations

$$2x = 2\lambda x \tag{583}$$
$$4y = 2\lambda y \tag{584}$$
$$g(x, y) = x^2 + y^2 = 1. \tag{585}$$

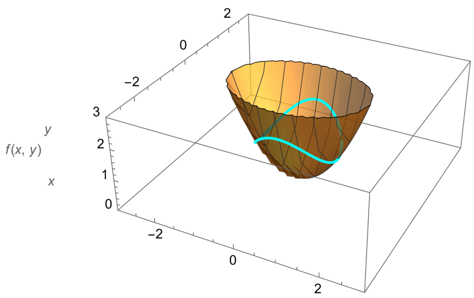
So the first has solutions $x = 0$ or $\lambda = 1$.

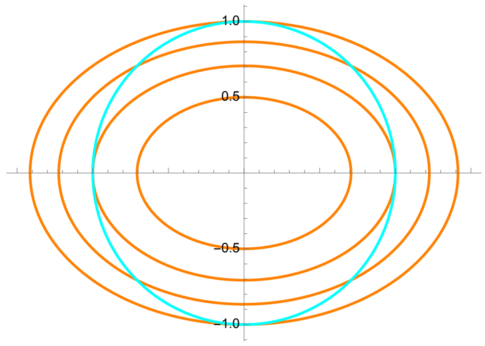
1. If $x = 0$ then $y = \pm 1$ from the constaint $y^2 = 1 - x^2$. The second equation then enforces $\lambda = 2$.
2. If $\lambda = 1$ then $y = 0$ from the second equation. This enforces $x = \pm 1$ from the constraint.

Therefore we have 4 solutions $(x, y, \lambda) = (0, 1, 2)$, $(x, y, \lambda) = (0, -1, 2)$, $(x, y, \lambda) = (1, 0, 1)$ and $(x, y, \lambda) = (-1, 0, 1)$. Evaluating f at these points gives

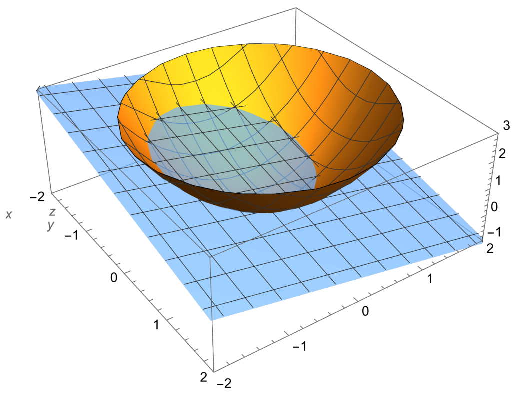
$$f(0, 1) = 2, \quad f(0, -1) = 2, \quad f(1, 0) = 1, \quad f(-1, 0) = 1. \tag{586}$$

So we have the maximum of f subject to the constraint g is 2 at $(0, \pm 1)$ and the minimum of f subject to the constraint g is 1 at $(\pm 1, 0)$.





Example 15.2. *The plane $x + y + 2z = 2$ intersects the elliptic paraboloid $z = x^2 + y^2$ in the an ellipse. Find the points on this ellipse nearest and farthest the from origin $(0, 0, 0)$.*



First, notice that the distance from the origin is given by

$$d(x, y, z) = \sqrt{x^2 + y^2 + z^2} \tag{587}$$

Note that minimising and maximising the distance from the origin is equivalent to maximising/minimising the distance squared. Second, note that we can solve for z here using the plane equation:

$$z = 1 - \frac{1}{2}x - \frac{1}{2}y. \tag{588}$$

Therefore, we want to minimise/maximise

$$f(x,y) = d\big(x, y, 1 - \frac{1}{2}x - \frac{1}{2}y\big)^2 = x^2 + y^2 + \big(1 - \frac{1}{2}x - \frac{1}{2}y\big)^2. \tag{589}$$

We need to find an equation for our constraint, which is to lie on the ellipse determined by the intersection of the paraboloid and the plane. To do this we can substitute in the plane equation solved for z:

$$1 - \frac{1}{2}x - \frac{1}{2}y = x^2 + y^2 \implies g(x,y) = x^2 + y^2 + \frac{1}{2}x + \frac{1}{2}y = 1. \tag{590}$$

Therefore, by the method of Lagrange multipliers, we need to find (x, y, λ) such that

$$\nabla f = \lambda \nabla g \tag{591}$$
$$g(x,y) = 1 \tag{592}$$

Computing the gradients we have

$$\nabla f = \Big(-1 + \frac{5}{2}x + \frac{1}{2}y, -1 + \frac{1}{2}x + \frac{5}{2}y\Big), \tag{593}$$
$$\nabla g = \Big(2x + \frac{1}{2}, 2y + \frac{1}{2}\Big). \tag{594}$$

So, equating $\nabla f = \lambda \nabla g$ gives

$$-1 + \frac{5}{2}x + \frac{1}{2}y = \lambda\Big(2x + \frac{1}{2}\Big) \tag{595}$$
$$-1 + \frac{1}{2}x + \frac{5}{2}y = \lambda\Big(2y + \frac{1}{2}\Big). \tag{596}$$

Solving the second for x gives

$$x = 2 + \lambda + (4\lambda - 5)y. \tag{597}$$

Substituting this into the first gives

$$(\lambda - 1)(2 + \lambda + (4\lambda - 6)y) = 0. \tag{598}$$

So either $\lambda = 1$ or $2 + \lambda + (4\lambda - 6)y = 0$. If $\lambda \neq \frac{3}{2}$ then the second can be solved for y

$$y = -\frac{2 + \lambda}{4\lambda - 6}. \tag{599}$$

If $\lambda = \frac{3}{2}$ then $2 + \lambda = 0$, i.e. $\lambda = -2$ which is a contradiction. So either

$$y = -\frac{2 + \lambda}{4\lambda - 6} \quad \textit{or} \quad \lambda = 1. \tag{600}$$

If $\lambda = 1$ then from (597) $x = 3 - y$. Therefore, from the constraint

$$g(3 - y, y) = (3 - y)^2 + y^2 + \frac{1}{2}(3 - y) + \frac{1}{2}y = 1. \tag{601}$$

So, y has to be a root of the polynomial:

$$2y^2 - 6y + \frac{19}{2} = 0 \implies 2\Big(y - \frac{3}{2}\Big)^2 + \frac{10}{2} = 0 \tag{602}$$

which has no solution.

If $y = -\frac{2+\lambda}{4\lambda-6}$ *then*

$$x = -\frac{2+\lambda}{4\lambda - 6} \tag{603}$$

from (597). *Then plugging* x, y *into the constraint gives that* λ *must satisfy:*

$$\frac{9\lambda^2 - 27\lambda + 8}{4(3-2\lambda)^2} = 0 \implies \lambda = \frac{1}{3}, \quad \text{or} \quad \lambda = \frac{8}{3}. \tag{604}$$

This then gives $x = y = \frac{1}{2}$ *or* $x = y = -1$. *We can then solve for* z *via the plane equation this gives two points in* $\mathbb{R}^3$*:*

$$(x, y, z) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \quad \text{or} \quad (x, y, z) = (-1, -1, 2). \tag{605}$$

To figure out the max/min we can simply plug into the distance:

$$d(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})^2 = \frac{3}{4}, \qquad d(-1, -1, 2)^2 = 6. \tag{606}$$

Therefore, distance is maximise at $(-1, -1, 2)$ *and minimised* $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

Example 15.3. *Find the max/min of* $f(x, y) = x^2 + y^2 + 4x - 4y$ *subject to the constraint* $x^2 + y^2 \leq 9$.

This problem can be dealt with using Lagrange multipliers by introducting a dummy variable to encode the constraint with an equality:

$$x^2 + y^2 \leq 9 \implies x^2 + y^2 - 9 \leq 0. \tag{607}$$

We can write this as

$$9 - x^2 - y^2 = z^2 \tag{608}$$

since, for $x^2+y^2-9 \leq 0$, $9-x^2-y^2$ *has to be some positive number. Therefore, we want to max/minimise*

$$f(x, y, z) = x^2 + y^2 + 4x - 4y, \tag{609}$$

subject to the constraint

$$g(x, y, z) = 9 - x^2 - y^2 - z^2 = 0. \tag{610}$$

Computing gradients gives

$$\nabla f = (2x + 4, 2y - 4, 0), \qquad \nabla g = (-2x, -2y, -2z). \tag{611}$$

Therefore, the system resulting from the Lagrange multiplier method gives

$$2x + 4 = -2\lambda x \tag{612}$$
$$2y - 4 = -2\lambda y \tag{613}$$
$$0 = -2\lambda z \tag{614}$$

Therefore, $\lambda = 0$ *or* $z = 0$. *If* $\lambda = 0$ *then* $x = -2$ *and* $y = 2$. *If* $z = 0$ *then*

$$x = \frac{-2}{1+\lambda}, \qquad y = \frac{2}{1+\lambda}. \tag{615}$$

Substituting into our constraint gives

$$9 - \frac{8}{(1+\lambda)^2} = 0 \implies \lambda = -1 \pm \frac{2\sqrt{2}}{3}, \quad (616)$$

and, therefore,

$$(x, y, z) = (-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}, 0), \qquad (x, y, z) = (\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}, 0). \quad (617)$$

Evalutating f at these points gives a maximum of $9 + 12\sqrt{2}$ at $(x, y) = (\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}})$ and a minimum at $(x, y) = (-2, 2)$.

Example 15.4. *Here's an example of what can go wrong if $\nabla g = \mathbf{0}$.*

Consider optimising the function $f(x, y) = x$ subject to the constraint $g(x, y) = y^2 + x^4 - x^3 = 0$. We now compute

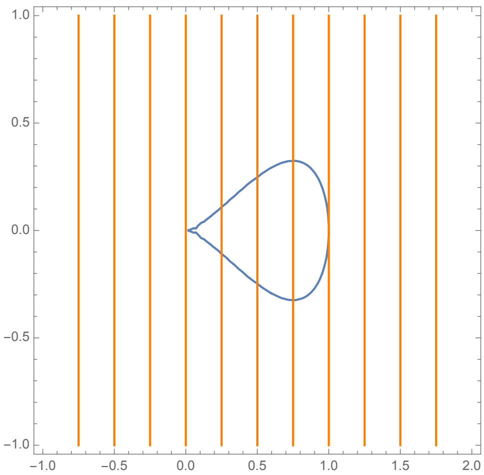
$$\nabla f = (1, 0) = \lambda \nabla g = \lambda(4x^3 - 3x^2, 2y). \quad (618)$$

Note that $\lambda \neq 0$ otherwise we have $1 = 0$ which is a contradiction. So,

$$y = 0 \qquad \lambda(4x^3 - 3x^2) = 1 \qquad x^4 - x^3 = 0.$$

The second tells us that $x = 0$ or $x = 1$. Note that, naively $x = 0$ is a contradiction since $1 \neq 0$. However, note that $\nabla g(0, 0) = \mathbf{0}$, so this is outside our method, so we should check what happens here! In the case where $x = 1$, $\lambda = 1$ and we have the solution where $(x, y, \lambda) = (1, 0, 1)$. So we have an extrema candidate at $(1, 0)$ of $f(1, 0) = 1$.

However, note that in the case $(x, y) = 0$ then $\nabla g = (0, 0)$, so this is outside the realm of our method. Let's look at the level curve $g = 0$ (blue) with the level curves of f (orange) superimposed (increasing from left to right) and see whats gone wrong:



We see here that the level curve of g has a cusp at $(0, 0)$ and this is in fact a minima for f subject to the constraint. The failure of the method is that our level curve $g = 0$ has no tangent vector at $(0, 0)$.

Having looked at the problem geometrically we have maxima at $(1, 0)$ of 1 and a minima at $(0, 0)$ of $f(0, 0) = 0$.

16 Integration

16.1 The Riemann Integral for Single Variable Functions

Let f be a real-valued function defined everywhere on an interval $[a, b]$. We partition $[a, b]$ by a sequence of numbers

$$a = x_0 < x_1 < ... < x_i < ... < x_n = b$$

into n sub-intervals $[x_i, x_{i+1}]$ with $i = 0, 1, ..., n-1$. For simplicity, make them of equal width (this is not strictly necessary)

$$\Delta x = x_{i+1} - x_i = \frac{b-a}{n}$$

For $i = 0, ..., n-1$, let we take a distinguished sample point of each subinterval $t_i \in [x_i, x_{i+1}]$.

The **Riemann sum** of f with respect to the partition $\{x_i\}_{i=0}^{n}$ and the sample points $\{t_i\}_{i=0}^{n-1}$ is

$$\sum_{i=0}^{n-1} f(t_i)\Delta x.$$

We say that the Riemann integral of f is s on $[a, b]$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that for any partition $\{x_i\}_{i=1}^{n}$ with any sample points $\{t_i\}_{i=0}^{n-1}$ with $\Delta x < \delta$ then

$$\Big| \sum_{i=0}^{n-1} f(t_i)\Delta x - s \Big| < \epsilon.$$

One can think of this in terms of the limit

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} f(t_i)\Delta x = s$$

for any partition $\{x_i\}_{i=1}^{n}$ with any sample points $\{t_i\}_{i=0}^{n-1}$. If this limit exists then we say f is (Riemann) integrable on $[a, b]$ and write

$$\int_a^b f(x)dx = s. \tag{619}$$

Remark 16.1. *One often interperates the Riemann integral as the difference of the areas where $f \geq 0$ and $f \leq 0$.*

Typically, one does not evaluate this limiting proceedure directly but rather employs the fundamental theorem of calculus to compute these integrals:

Theorem 16.1. *Suppose that $F : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b) with $F' = f$ on (a, b) then*

$$F(b) - F(a) = \int_a^b f(y)dy. \tag{620}$$

16.2 The Riemann Integral in Many Variables

16.2.1 Definition

Let f be a real-valued function defined everywhere on an closed rectangle $[a, b] \times [c, d]$. We partition $[a, b]$ by a sequence of $m + 1$-numbers

$$a = x_0 < x_1 < ... < x_i < ... < x_m = b$$

into m sub-intervals $[x_i, x_{i+1}]$ with $i = 0, 1, ..., n-1$ and partition $[c, d]$ by a sequence of $n+1$-numbers

$$c = y_0 < y_1 < ... < y_i < ... < y_n = d$$

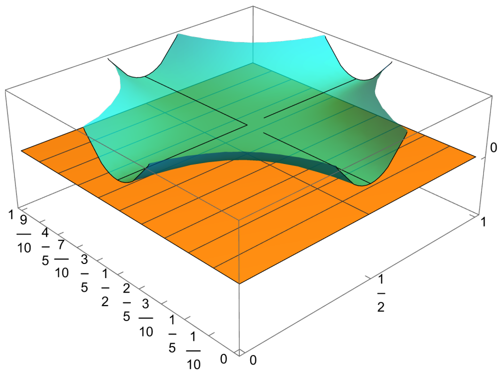
into n sub-intervals $[y_i, y_{i+1}]$ with $i = 0, 1, ..., n-1$. For simplicity, make them of equal width

$$\Delta x = x_{i+1} - x_i = \frac{b-a}{m}, \qquad \Delta y = y_{i+1} - y_i = \frac{d-c}{n}$$

Now we construct mn sub-rectangles

$$R_{ij} = [x_i, x_{i+1}] \times [y_i, y_{i+1}] \qquad i = 0, ..., m-1, j = 0, ..., n-1$$

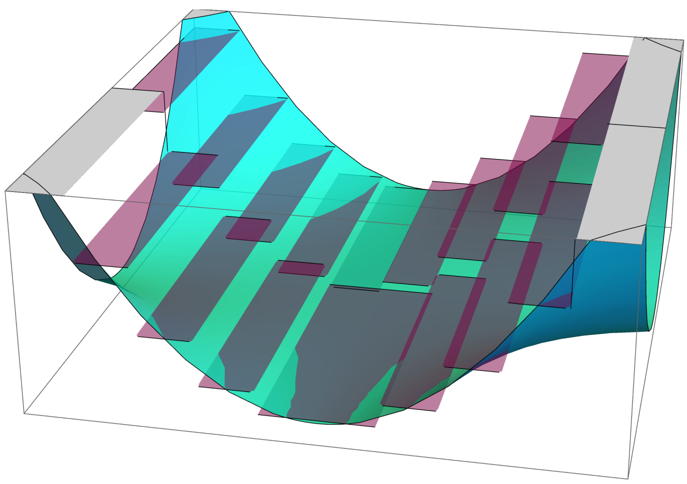
which cover $[a, b] \times [c, d]$. Below is a partition on the rectangle $[0, 1]$ into the rectangles of side $\Delta y = \frac{1}{10}$ and $\Delta x = \frac{1}{2}$.



For i and j, we take a distinguished sample point $(\check{x}_{ij}, \check{y}_{ij})$ of each rectangle R_{ij}. We construct the Riemann sum

$$\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} f(\check{x}_{ij}, \check{y}_{ij}) \Delta x \Delta y \tag{621}$$

This means we are summing over the functions value at some point in the rectangle and multiplying this by the the rectangles area, i.e. we are producing a volume in $\mathbb{R}^3$. See for example the following figure:



We say that the Riemann integral of f on $[a, b] \times [c, d]$ is s if for all $\epsilon > 0$ there exists $\delta > 0$ such that for any partition into rectangles R_{ij} with any sample points $\{(\check{x}_{ij}, \check{y}_{ij})\}$ with $\sqrt{\Delta x^2 + \Delta y^2} < \delta$ then

$$\left| \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} f(\check{x}_{ij}, \check{y}_{ij}) \Delta x \Delta y - s \right| < \epsilon.$$

Again, one can think of this in terms of the limit

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} f(\check{x}_{ij},\check{y}_{ij})\Delta x\Delta y = s$$

for any partition into rectangles R_{ij} with any sample points $\{(\check{x}_{ij},\check{y}_{ij})\}$. If this limit exists then we say f is (Riemann) integrable on $R=[a,b]\times[c,d]$ and write

$$\int_R f(x,y)dxdy = s. \tag{622}$$

Remark 16.2. *If $f(x,y)\geq 0$ then this integral is the volume of the region between the xy-plane and the surface determined by $z=f(x,y)$.*

We can generalise the Riemann integral to functions of n-variables in the following manner. Let f be a real-valued function defined everywhere on a closed box in $\mathbb{R}^n$:

$$R=[a_1,b_1]\times[a_2,b_2]\times ...\times[a_n,b_n]. \tag{623}$$

We divide this box in the $m_1...m_n$ sub-boxes $R_{i_1...i_n}$ of side

$$\Delta\mathrm{vol}_n=\Delta x_1\Delta x_2...\Delta x_n, \qquad \Delta x_i=\frac{b_i-a_i}{m_i} \tag{624}$$

which are disjoint and sum to give R. Let $\{\check{\mathbf{x}}_{i_1...i_n}\}$ be the set of distinguished sample points for each $R_{i_1...i_n}$. The Riemann sum is then

$$\sum_{i_1=0}^{m_1-1}...\sum_{i_n=0}^{m_n-1} f(\check{\mathbf{x}}_{i_1...i_n})\Delta\mathrm{vol}_n. \tag{625}$$

If the limit

$$\lim_{m_1\to\infty}...\lim_{m_n\to\infty}\sum_{i_1=0}^{m_1-1}...\sum_{i_n=0}^{m_n-1} f(\check{\mathbf{x}}_{i_1...i_n})\Delta\mathrm{vol}_n$$

exists for any partition into sub-boxes $R_{i_1...i_n}$ with any sample points $\{\check{\mathbf{x}}_{i_1...i_n}\}$ then we say f is Riemann integrable on R and write

$$\int_R f(\mathbf{x})dx_1...dx_n=\lim_{m_1\to\infty}...\lim_{m_n\to\infty}\sum_{i_1=0}^{m_1-1}...\sum_{i_n=0}^{m_n-1} f(\check{\mathbf{x}}_{i_1...i_n})\Delta\mathrm{vol}_n.$$

We will use the notation $\mathrm{dvol}_n=dx_1...dx_n$ or dvol if the dimension is clear.

In practise this definition is rather useless; you must take a limit over all partitions with all samplings! However, if you know the function is Riemann integrable, then the Riemann sum can give produce a value arbitrarily close to the Riemann integral provided you take a fine enough division into sub-boxes. This is precisely what a computer does when you ask it to numerically integrate! For example, try 'NIntegrate' in Mathematica.

16.2.2 Iterated Integrals

To compute the integral of a function f of a single variable we typically use the **fundamental theorem of calculus**, rather than the definition of the Riemann integral explicitly, i.e. we find an antiderivative $F(x)$ such that $F'=f$. Suppose, we are integrating f from a to $x<\infty$ then

$$\int_a^x f(t)dt=\int_a^x F'(t)dt=F(x)-F(a).$$

For example, the fundamental theorem tells us that

$$\int_a^x \cos(t)dt = \int_a^x \frac{d}{dt}\sin(t)dt = \sin(x).$$

For functions of more than one variable we can use the fundamental theorem of calculus on a **partial integral**. For a function f of two variables (x, y), the partial integrals on the rectangle $[a, b] \times [c, d]$ are defined by

$$g(x) = \int_c^d f(x, y)dy, \qquad h(y) = \int_a^b f(x, y)dx. \tag{626}$$

Provided $f(x, y)$ with fixed x or y is a continuous function on $[a, b]$ or $[c, d]$ respectively, these *functions* can be computed using the fundamental theorem of calculus. For example, let $f(x, y) = \cos(xy)$ on $[0, \pi] \times [0, \pi]$ then

$$g(x) = \int_0^\pi \cos(xy)dy = \int_0^\pi \partial_y\Big(\frac{\sin(xy)}{x}\Big)dy = \frac{\sin(\pi x)}{x}. \tag{627}$$

In general, for a function f of n-variables $(x_1, ..., x_n)$, the partial integrals on the box $[a_1, b_1] \times ... \times [a_n, b_n]$ are defined by

$$g_i(x_1, ..., \hat{x}_i, ..., x_n) = \int_{a_i}^{b_i} f(x_1, ..., x_i, ..., x_n)dx_i, \tag{628}$$

where $(x_1, ..., \hat{x}_i, ..., x_n)$ denotes $(x_1, ..., x_n)$ without the i-th coordinate.

Remark 16.3. *The fundamental theorem of calculus tells us that partial integration and partial differentiation are inverses.*

We can keep partially integrating in each direction to construct the iterated integrals. For a function f of two variables (x, y), the iterated integrals on the rectangle $[a, b] \times [c, d]$ are defined by

$$\int_a^b \int_c^d f(x, y)dydx = \int_a^b \Big[\int_c^d f(x, y)dy\Big]dx, \quad \int_c^d \int_a^b f(x, y)dxdy = \int_c^d \Big[\int_a^b f(x, y)dx\Big]dy. \tag{629}$$

Typically, one can use the fundamental theorem of calculus to evaluate such integrals.

Example 16.1. *Iteratively integrate $f(x, y) = xy$ on $[0, 1] \times [0, 1]$.*

$$\int_0^1 \int_0^1 xydxdy = \int_0^1 \frac{1}{2}x^2y|_{x=0}^{x=1}dy = \int_0^1 \frac{1}{2}ydy = \frac{1}{4}.$$

$$\int_0^1 \int_0^1 xydydx = \int_0^1 \frac{1}{2}xy^2|_{y=0}^{y=1}dx = \int_0^1 \frac{1}{2}xdx = \frac{1}{4}.$$

So the iterated integrals agree!

Example 16.2. *Iteratively integrate $f(x, y) = \frac{x^2-y^2}{(x^2+y^2)^2}$ on $[0, 1] \times [0, 1]$. We note that*

$$\partial_x \frac{1}{x^2 + y^2} = -\frac{2x}{(x^2 + y^2)^2}$$

which is close to an antiderivative, we need one more power of x in the numerator. Let's try

$$\partial_x \frac{x}{x^2 + y^2} = \frac{1}{x^2 + y^2} - \frac{2x^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = -f(x, y).$$

So,

$$f(x,y)=\partial_x\Big(-\frac{x}{x^2+y^2}\Big)\qquad f(x,y)=\partial_y\Big(\frac{y}{x^2+y^2}\Big).$$

Therefore,

$$\int_0^1\int_0^1\frac{x^2-y^2}{(x^2+y^2)^2}dxdy=\int_0^1\int_0^1\partial_x\Big(-\frac{x}{x^2+y^2}\Big)dxdy=-\int_0^1\frac{1}{1+y^2}dy=-\arctan(y)\Big|_0^1=-\frac{\pi}{4}$$

and

$$\int_0^1\int_0^1\frac{x^2-y^2}{(x^2+y^2)^2}dydx=\int_0^1\int_0^1\partial_x\Big(\frac{y}{x^2+y^2}\Big)dydx=\int_0^1\frac{1}{1+x^2}dx=\arctan(x)\Big|_0^1=\frac{\pi}{4}$$

So the iteratred integral disagree!

In the above example we have two different answers depending on the order of integration! Is either the Riemann integral? In this case no, the function is not Riemann integrable. Now, the question is when/how are these related to the Riemann integral defined above for functions of two/many variables. The answer comes from Fubini's theorem (which can be with more relaxed assumptions than appear here)

Theorem 16.2. *(Fubini) Assume f be an Riemann integrable function on the rectangle $R=[a,b]\times[c,d]$ and suppose*

$$\int_a^b f(x,y)dx,\quad \int_c^d f(x,y)dy,\quad \int_a^b\int_c^d f(x,y)dydx,\quad \int_c^d\int_a^b f(x,y)dxdy$$

exist. Then

$$\int_R f(x,y)dxdy=\int_a^b\int_c^d f(x,y)dydx=\int_c^d\int_a^b f(x,y)dxdy. \tag{630}$$

More generally, assume f be a Riemann integrable function on the box

$$R=[a_1,b_1]\times...\times[a_n,b_n]$$

and

$$\int_{a_1}^{b_1} f(x_1,...,x_n)dx_1,...,\int_{a_1}^{b_1}\int_{a_2}^{b_2} f(x_1,...,x_n)dx_1,...,\int_{a_1}^{b_1}...\int_{a_n}^{b_n} f(x_1,...,x_n)dx_1...dx_n$$

exist, where the ... denotes all partial integrations all the way up to all $n!$ iterated integrals. Then

$$\int_R f(\mathbf{x})\mathrm{dvol}_n=\int_{a_1}^{b_1}...\int_{a_n}^{b_n} f(\mathbf{x})dx_1...dx_n=... \tag{631}$$

where the ... denote any permutation of integrals.

Remark 16.4. *The assumptions of Fubini's theorem are satisfied for bounded functions on R which have discontinuous on a finite number of continuous curves. In particular, a continuous function on a box is bounded and have no discontiniuities, so Fubini applies.*

Example 16.3. *Integrate $f(x,y,z)=x^2yz^4$ over $[0,1]\times[0,2]\times[-1,1]$.*

The function is continuous so Fubini applies. Therefore,

$$\begin{aligned}\int_D x^2yz^4dxdydz&=\int_0^1\int_0^2\int_{-1}^1 x^2yz^4dzdydx=\int_0^1\int_2\frac{1}{5}x^2y(1^5-(-1)^5)dydx\\&=\int_0^1\int_0^2\frac{2}{5}x^2ydzdy=\int_0^1\int_0^2\frac{2}{5}x^2ydydx=\int_0^1\int_0^2\frac{1}{5}x^2(2^2-0)dx\\&=\int_0^1\frac{4}{5}x^2dx=\frac{4}{15}.\end{aligned}$$

Example 16.4. *Evaluate:*

$$\int_R y\sin(xy)dxdy \tag{632}$$

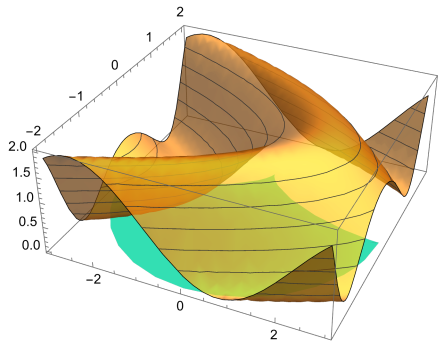
for $R = [1,2]\times[0,\pi]$.

Again, the function is continuous so Fubini applies

$$\int_R y\sin(xy)dxdy = \int_0^\pi \int_1^2 y\sin(xy)dxdy = \int_0^\pi (\cos(y)-\cos(2y))dy = \sin(y) - \frac{1}{2}\sin(2y)\Big|_{y=0}^{\pi} = 0.$$

16.3 Integrating over General Regions

For functions of more than one variable we can integrate over more than just boxes. For example



To define an integral of f over a general bounded region D (can be contained in a ball) we define a new function

$$g(\mathbf{x}) = \begin{cases} f(\mathbf{x}) & \mathbf{x}\in D \\ 0 & \mathbf{x}\in\mathbb{R}^n\setminus D. \end{cases} \tag{633}$$

Then we integrate g over a large rectangle/box R which contains D. If g is Riemann integrable over R then we say that f in Riemann integrable over D with Riemann integral given by

$$\int_D f(\mathbf{x})\mathrm{dvol}_n = \int_R g(\mathbf{x})\mathrm{dvol}_n. \tag{634}$$

Note that effectively we are cutting off our function outside the region D we wish to integrate over which will often result in a jump discontinuity. However, if f is continuous on D and the boundary of the region D is 'well-behaved' this is usually not a problem and

$$\int_D f(\mathbf{x})\mathrm{dvol}$$

will exist. We will examine two special cases below.

16.3.1 Normal Domains

Definition 16.1. *(Normal Domain in* $\mathbb{R}^2$*) A domain* $D\subset\mathbb{R}^2$ *of integration is* **normal** *if* D *can be written as the region between the graphs of two continuous functions. More precisely,* D *can be written in one of the following two ways:*

$$D = \{(x,y) : a\le x\le b, g_1(x)\le y\le g_2(x)\}$$

or

$$D = \{(x,y) : c\le y\le d, h_1(y)\le x\le h_2(y)\}$$

for g_1, g_2, h_1 *and* h_2 *continuous.*

To evaluate an integrals over a normal domain D, we choose a rectangle $R = [a, b] \times [c, d]$ containing D and define g as in equation (633). Then we have

$$\int_D f(x, y)dxdy = \int_R g(x, y)dxdy. \tag{635}$$

Let's assume that f is continuous so that g is bounded and is only discontinuous on the boundary of D (which will be continuous by definition). Using Fubini one has

$$\int_D f(x, y)dxdy = \int_a^b \int_c^d g(x, y)dydx = \int_c^d \int_a^b g(x, y)dxdy. \tag{636}$$

Let's suppose that

$$D = \{(x, y) : a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

then

$$g(x, y) = 0 \qquad \text{if} \qquad y > g_2(x) \text{ or } y < g_1(x). \tag{637}$$

Therefore,

$$\int_D f(x, y)dxdy = \int_a^b \int_c^d g(x, y)dydx = \int_a^b \int_{g_1(x)}^{g_2(x)} g(x, y)dydx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)dydx. \tag{638}$$

Similarly if

$$D = \{(x, y) : c \le y \le d, h_1(y) \le x \le h_2(y)\}$$

then

$$\int_D f(x, y)dxdy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)dxdy. \tag{639}$$

Example 16.5. *Lets integrate $x \sin y$ on D bounded by $y = 0$, $y = x^2$ and $x = 1$. The function is continuous so we can evaluate iterated integrals. Now the domain is normal since:*

$$D = \{(x, y) \in \mathbb{R}^2 : 0 \le x \le 1, 0 \le y \le x^2\}$$

Therefore,

$$\begin{aligned} \int_D f \mathrm{dvol} &= \int_0^1 \int_0^{x^2} x \sin(y)dydx = \int_0^1 -x\cos(y)\Big|_0^{x^2} dx \\ &= \int_0^1 \Big[-x\cos(x^2) + x\Big]dx = \Big[-\frac{1}{2}\sin(x^2) + \frac{1}{2}x^2\Big]\Big|_0^1 = \frac{1}{2}(1 - \sin(1)). \end{aligned}$$

Definition 16.2. *(Normal Domain in $\mathbb{R}^3$) Let $\Pi_x(D), \Pi_y(D), \Pi_z(D)$ denote the projections of a set D on to the yz, xz or xy-plane respectively. A domain $D \subset \mathbb{R}^3$ of integration is* **normal** *if D can be written as the region between the graphs of two continuous functions u_1 and u_2. More precisely, D can be written in one of the following three ways:*

$$D = \{(x, y, z) : (x, y) \in \Pi_z(D), u_1(x, y) \le z \le u_2(x, y)\}$$

or

$$D = \{(x, y, z) : (y, z) \in \Pi_x(D), u_1(y, z) \le z \le u_2(y, z)\}$$

or

$$D = \{(x, y, z) : (x, z) \in \Pi_y(D), u_1(x, z) \le z \le u_2(x, z)\}.$$

for u_1, u_2 continuous.

As above, with appropriate assumptions (say f is continuous on D) we can then write

$$\int_D f(\mathbf{x})\mathrm{dvol}_3 = \int_{\Pi_z(D)} \Big[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dz\Big]\mathrm{dvol}_2. \tag{640}$$

If additionally $\Pi_z(D)$ is normal then, under appropriate (say $\int f dz$ is continuous on $\Pi_z(D)$) assumptions,

$$\int_D f(\mathbf{x})\mathrm{dvol}_3 = \int_a^b \Big[\int_{g_1(y)}^{g_2(y)} \Big[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dz\Big]dy\Big]dx. \tag{641}$$

Example 16.6. *Let's integrate $f(x,y) = xy$ over a domain D which lies below $1+x+y$ and above the region in xy plane bounded by the curves $y = \sqrt{x}$ $y = 0$ and $x = 1$.*

We have that

$$D = \{(x,y,z) : 0 \le x \le 1, 0 \le y \le \sqrt{x}, 0 \le z \le 1+x+y\}$$

$$\int f\mathrm{dvol} = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} xydzdydx = \int_0^1 \int_0^{\sqrt{x}} xyz\Big|_0^{1+x+y} dydx = \int_0^1 \int_0^{\sqrt{x}} xy(1+x+y)dydx$$
$$= \int_0^1 \Big[\frac{1}{2}xy^2 + \frac{1}{2}x^2y^2 + \frac{1}{3}xy^3\Big]|_0^{\sqrt{x}} dx = \int_0^1 \Big[\frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{3}x^{\frac{5}{2}}\Big]dx = \frac{1}{6} + \frac{1}{8} + \frac{2}{21} = \frac{65}{168}.$$

16.3.2 Changing the Order of Integration

If we wish to integrate a continuous function on $D \subset \mathbb{R}^2$ we can do so in two ways by Fubini's theorem. It is often the case that one way is much harder than the other. Let us suppose that our domain is normal and in fact can be written in either way stated in definition 16.1. The integral can be computed in two ways:

$$\int_D f(x,y)dxdy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx. \tag{642}$$

One is often very hard; it may not even be expressable in terms of elementary functions. The other can be easy, for example:

Example 16.7.

$$\int_D \sin(y^2)\mathrm{dvol} \tag{643}$$

with $D = \{\mathbf{x} \in \mathbb{R}^2 : x \in [0,1], y \in [x,1]\}$.

So the function is the composition of two continuous functions so therefore is continuous. Hence, we can apply Fubini! Now for the iterated integrals we in theory could want to compute

$$\int_x^1 \sin(y^2)dy$$

Although it is integrable (it is continuous), it cannot be expressed as an elementary function, e.g. rational function, trignometric function, exponential, logarithm etc. This integral arises so much that it has a name:

$$S(x) = \int_0^x \sin(y^2)dy$$

is called the Fresnel integral and arises in optics. It does have a power series expansion which converges for all x:

$$S(x) = \sum_{n=0}^{\infty} \frac{x^{4n+3}}{(2n+1)!(4n+3)}.$$

The point is that this integral is problematic, especially if one wish to compute an iterated integral with it.

The assumptions of Fubini are satisfied so we can change the order of integration. So we can integrate x first. Note that region we are integrating over can be written as

$$D = \{\mathbf{x} \in \mathbb{R}^2 : y \in [0,1], x \in [0,y]\}$$

Therefore,

$$\int_D f \mathrm{dvol} = \int_0^1 \int_0^y \sin(y^2) dx dy = \int_0^1 y \sin(y^2) dy = -\frac{1}{2}\cos(y^2)\Big|_0^1 = \frac{1}{2}(1 - \cos(1)).$$

16.4 Properties

Proposition 16.1. *Assume that f and g are Riemann integrable on $D \subset \mathbb{R}^n$ and let $k \in \mathbb{R}$. Then*

1. *$f(\mathbf{x}) + kg(\mathbf{x})$ is Riemann integrable with*

$$\int_D (f(\mathbf{x}) + kg(\mathbf{x}))\mathrm{dvol} = \int_D f(\mathbf{x})\mathrm{dvol} + k\int_D g(\mathbf{x})\mathrm{dvol}.$$

2. *if $f(\mathbf{x}) \leq g(\mathbf{x})$ for all $\mathbf{x} \in D$ then*

$$\int_D f(\mathbf{x})\mathrm{dvol} \leq \int_D g(\mathbf{x})\mathrm{dvol}. \tag{644}$$

3. *if $D = D_1 \cup D_2$ where D_1 and D_2 are disjoint (they may share a boundary)*

$$\int_D f(\mathbf{x})\mathrm{dvol} = \int_{D_1} f(\mathbf{x})\mathrm{dvol} + \int_{D_2} f(\mathbf{x})\mathrm{dvol} \tag{645}$$

4. *If $f = 1$ then*

$$\int_D 1\mathrm{dvol} = \mathrm{Vol}(D). \tag{646}$$

5. *if $f(\mathbf{x})$ is bounded, i.e. there exists c and C such that*

$$c \leq f(\mathbf{x}) \leq C$$

one has

$$c\mathrm{Vol}(D) \leq \int_D f(\mathbf{x})\mathrm{dvol} \leq C\mathrm{Vol}(D). \tag{647}$$

Example 16.8. *Let D be the region contained in the ellipse:*

$$\frac{1}{2}x^2 + y^2 = 1$$

Let's estimate the function $e^{\sin x \cos y}$ on D. First, $e^{\sin x \cos y}$ is the composition of continuous functions and therefore is Riemann integrable on D. We note that

$$-1 \leq \sin(x)\cos(y) \leq 1.$$

So,

$$\frac{1}{e} \leq f(x,y) \leq e.$$

Therefore,

$$\frac{\mathrm{V}(D)}{e} \leq \int_D f\mathrm{dvol} \leq e\mathrm{Vol}(D).$$

Now, lets work out the volume integral as a bonus.

$$\mathrm{Vol}(D) = \int_D 1\mathrm{dvol}.$$

Note that the domain can be split into two normal regions

$$D_1 = \left\{\mathbf{x} \in \mathbb{R}^2 : -\sqrt{2} \leq x \leq \sqrt{2}, 0 \leq y \leq \sqrt{1 - \frac{1}{2}x^2}\right\}$$
$$D_2 = \left\{\mathbf{x} \in \mathbb{R}^2 : -\sqrt{2} \leq x \leq \sqrt{2}, 0 \geq y \geq -\sqrt{1 - \frac{1}{2}x^2}\right\}.$$

So,

$$\mathrm{Vol}(D_1) = \int_{-\sqrt{2}}^{\sqrt{2}} \int_0^{\sqrt{1-\frac{1}{2}x^2}} 1dydx = \int_{-2}^{2} \sqrt{1 - \frac{1}{2}x^2}dx = \frac{\pi}{\sqrt{2}}$$
$$\mathrm{Vol}(D_2) = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{1-\frac{1}{2}x^2}}^{0} 1dydx = \int_{-2}^{2} \sqrt{1 - \frac{1}{2}x^2}dx = \frac{\pi}{\sqrt{2}}$$

where one uses the trignometric subsitution $x = \sqrt{2}\sin\theta$. Therefore,

$$\mathrm{Vol}(D) = \frac{2\pi}{\sqrt{2}}.$$

16.5 Changing Variables

An extremely useful tool for evaluating integrals in changing variables. If we trying to integrate a function of one variable x from a to b, we could find it useful to find a **one-to-one** substitution $u = g(x)$, i.e. it maps precisely one $x \in [a, b]$ to each $u \in \mathbb{R}$, see the following figure: If it is one-to-one which we can invert for $x = x(u)$ and write

$$\int_a^b f(x)dx = \int_{u(a)}^{u(b)} f(x(u))\frac{dx}{du}du. \tag{648}$$

Example 16.9. *Lets evaluate*

$$\int_0^1 \frac{1}{\sqrt{a^2 - x^2}}dx \tag{649}$$

with $|a| \geq 1$. Set $x = a\sin\theta$. Then

$$dx = a\cos\theta d\theta \implies \int_0^1 \frac{1}{\sqrt{a^2 - x^2}}dx = \int_0^{\arcsin(\frac{1}{a})} \frac{a\cos\theta}{|a\cos\theta|}d\theta = \int_0^{\arcsin(\frac{1}{a})} \mathrm{sign}(a\cos\theta)d\theta.$$

We note that

$$-\frac{\pi}{2} \leq \arcsin\left(\frac{1}{a}\right) \leq \frac{\pi}{2} \implies a\cos\theta \geq 0$$

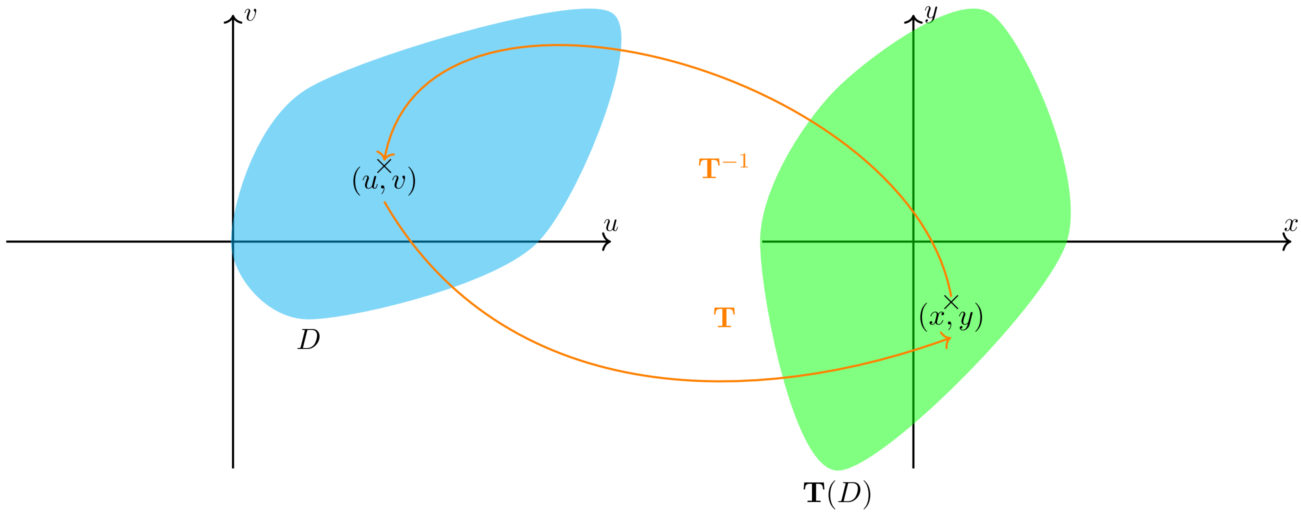
$$\int_0^1 \frac{1}{\sqrt{a^2 - x^2}}dx = \int_0^{\arcsin(\frac{1}{a})} d\theta = \arcsin(\frac{1}{a})$$

The same holds true for multiple integrals.

We'll start with $\mathbb{R}^2$. Let's suppose we have a function of two variables that gives out two numbers $\mathbf{T}:\mathbb{R}^2\to\mathbb{R}^2$, i.e.

$$\mathbf{T}(u,v)=(x,y), \tag{650}$$

i.e. $x=x(u,v)$ and $y=y(u,v)$. The picture you should have in mind is



We will assume that $\mathbf{T}$ is a C^1-mapping[11]; this means $x(u,v)$ and $y(u,v)$ have to have continuous first partial derivatives. Additionally, we will assume that the mapping is **one-to-one**: every pair (u,v) is mapped to a *unique* pair (x,y). This means that one can invert the transformation $\mathbf{T}$ for an inverse function $\mathbf{T}^{-1}$.

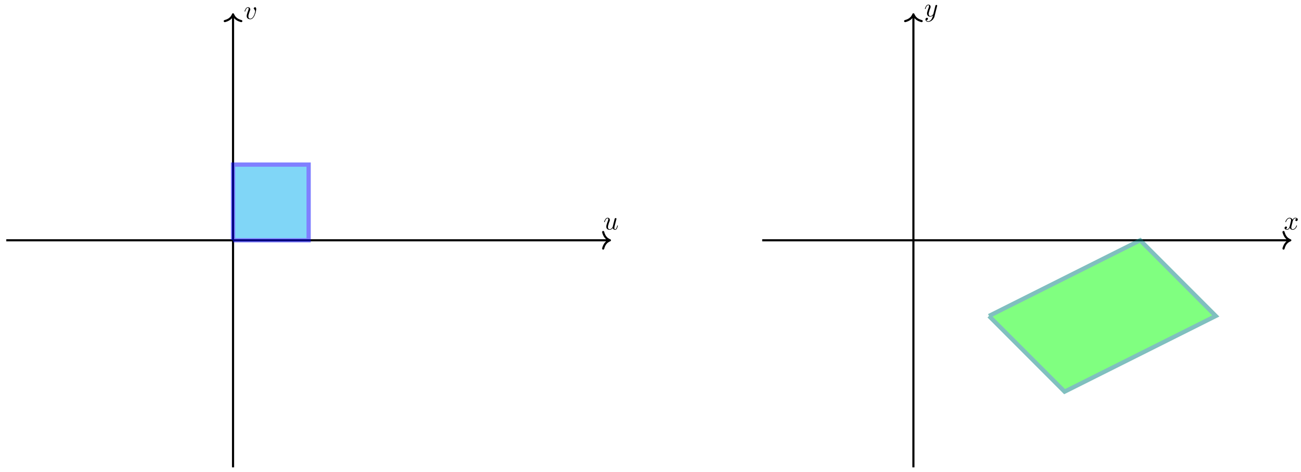
Example 16.10. *Let $\mathbf{T}$ be defined by the* **linear transformation** *$x=u+2v+1$ and $y=v-u-1$. Find the image of the square $[0,1]\times[0,1]$ in the uv-plane.*

We note that

- $(u,v)=(0,v)$ *goes to* $(x,y)=(2v+1,v-1)$ *which means* $y=\frac{1}{2}x-\frac{3}{2}$.
- $(u,v)=(u,0)$ *goes to* $(x,y)=(u+1,-u-1)$ *which means* $y=-x$.
- $(u,v)=(1,v)$ *goes to* $(x,y)=(2+2v,v-2)$ *which means* $y=\frac{1}{2}x-3$
- $(u,v)=(u,1)$ *goes to* $(x,y)=(3+u,-u)$ *which means* $y=-x+3$.

Therefore, the region is contained in the parallelogram:

[11] A C^k-mapping $\mathbf{T}$ means that x and y have to have continuous partial derivatives up to order k.



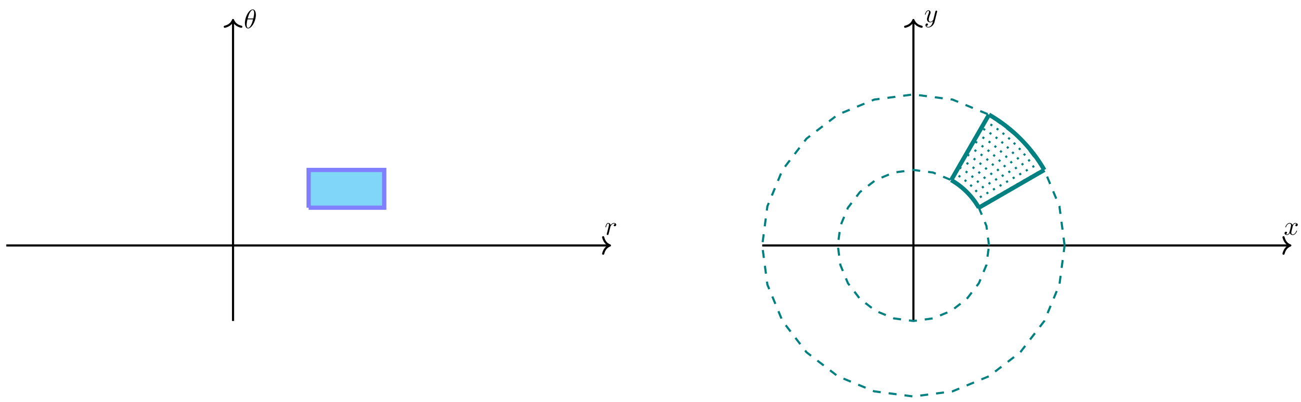
Any linear mapping:

$$x = au + bv + c, \qquad y = du + ev + f \quad \Leftrightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} c \\ f \end{pmatrix}$$

will take a rectangle to a parallelogram.

Example 16.11. *Let* $\mathbf{T}$ *be defined by* $x = r\cos\theta$ *and* $y = r\sin\theta$*. Find the image of the square* $[1,2]\times[\frac{\pi}{6},\frac{\pi}{3}]$ *in the* xy*-plane.*

Note that $x^2 + y^2 = r^2$ *so that at fixed* r *we get an arc of length* $\frac{\pi}{6}$ *between* $[\frac{\pi}{6},\frac{\pi}{3}]$*. Hence,*



Note that for $x(r,\theta) = r\cos\theta$ *and* $y(r,\theta) = r\sin\theta$ *the linear approximation is valid. These are differentiable functions, so for* (r,θ) *close to* (r_0,θ_0) *one has*

$$x(r,\theta) = r_0\cos\theta_0 + \cos\theta_0(r - r_0) - r_0\sin\theta_0(\theta - \theta_0) + \mathrm{err}$$
$$y(r,\theta) = r_0\sin\theta_0 + \sin\theta_0(r - r_0) + r_0\cos\theta_0(\theta - \theta_0) + \mathrm{err}$$

where err *is a small error term. Therefore, for* (r,θ) *close to* (r_0,θ_0) *the transformation is approximately linear.*

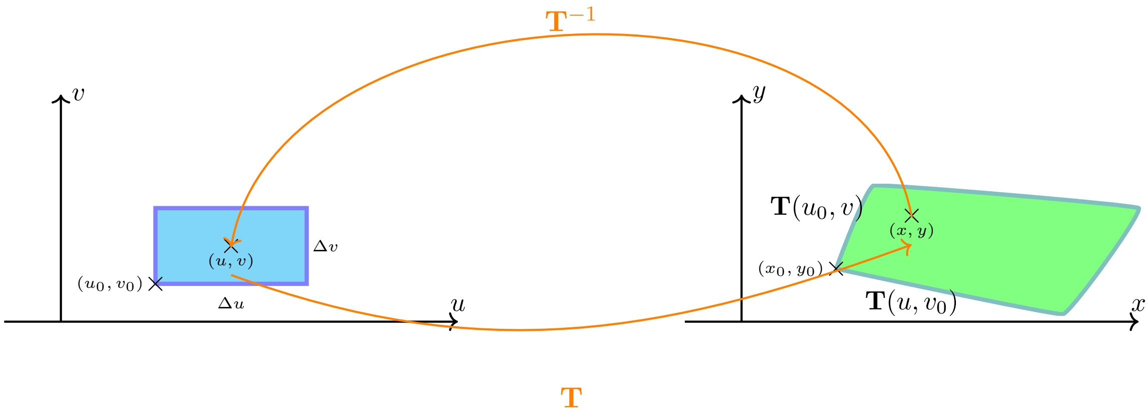
Example 16.12. *Let* $\mathbf{T}$ *be defined by* $x = u^2 - v^2$ *and* $y = 2uv$*. Find the image of the square* $[0,1]\times[0,1]$ *in the* uv*-plane.*

Now let us construct what happens to the Riemann integral under a change of variables. What we need to know is how the Riemann sum changes, i.e. how does the following formula look in u, v variables:

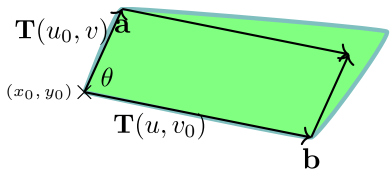
$$\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} f(\check{x}_{ij}, \check{y}_{ij})\Delta x \Delta y. \tag{651}$$

In theory, we can substitute $(\breve{x}_{ij}, \breve{y}_{ij}) = \mathbf{T}(\breve{u}_{ij}, v_{ij})$ in f, we don't yet know how $\Delta x \Delta y$ changes.

To do this consider a very small rectangle R in the uv-plane whose sides are Δu and Δv with lower left corner at (u_0, v_0). Under $\mathbf{T}$, R is mapped to $\mathbf{T}(R)$ and (u_0, v_0) is mapped to $(x_0, y_0) = \mathbf{T}(u_0, v_0)$. We claim that the boundary points are mapped to boundary point (this is not obvious, see a topology course). So, (x_0, y_0), $\mathbf{T}(u_0, v)$ and $\mathbf{T}(u, v_0)$ sit on the boundary of $T(R)$. This is shown in the following picture:



We now construct an approximation of the area in green on the right hand side of the diagram. The claim (that one has to justify in a proper proof) is that we can approximate the green region with a parallelogram with sides given by the vectors $\mathbf{a}$ and $\mathbf{b}$ with an angle θ between then as shown below:



How does one justify this? Well, it shouldn't be too hard to convince yourself that for a linear transformation this is true as such a transformation will give a parallelogram identically (as in the above example)! The key point is that for a C^1-transformation the partial derivatives of $x(u, v)$ and $y(u, v)$ are continuous and therefore, the linear approximation is valid close to some point (u_0, v_0) in R, i.e.

$$x(u, v) \approx x(u_0, v_0) + \partial_u x(u_0, v_0)(u - u_0) + \partial_v x(u_0, v_0)(v - v_0),$$
$$y(u, v) \approx y(u_0, v_0) + \partial_u y(u_0, v_0)(u - u_0) + \partial_v y(u_0, v_0)(v - v_0).$$

Therefore, we approximately have a linear transform! The area A of a parallelogram is its height $h = ||\mathbf{a}|| \sin\theta$ multiplied by its base length $b = ||\mathbf{b}||$:

$$A = bh = ||\mathbf{a}||||\mathbf{b}|| \sin\theta$$

If we extend $\mathbf{a}$ and $\mathbf{b}$ to vectors in $\mathbb{R}^3$, as

$$\mathbf{a} = (a_1, a_2, 0), \qquad \mathbf{b} = (b_1, b_2, 0),$$

we can write

$$A = ||\mathbf{a} \times \mathbf{b}|| = \left\| \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & 0 \\ b_2 & b_2 & 0 \end{pmatrix} \right\| = \left| \det \begin{pmatrix} a_1 & a_2 \\ b_2 & b_2 \end{pmatrix} \right| ||\mathbf{k}|| = \left| \det \begin{pmatrix} a_1 & a_2 \\ b_2 & b_2 \end{pmatrix} \right|.$$

Now, $\mathbf{a}$ is the point $\mathbf{T}(u_0, v_0 + \Delta v)$ along the vector valued function $\mathbf{T}(u_0, v)$ from $\mathbf{T}(u_0, v_0)$. So,

$$\mathbf{a} = \mathbf{T}(u_0, v_0 + \Delta v) - \mathbf{T}(u_0, v_0)$$

and, similarly,

$$\mathbf{b} = \mathbf{T}(u_0 + \Delta u, v_0) - \mathbf{T}(u_0, v_0).$$

However, note that the partial derivatives of $\mathbf{T}$ are defined as

$$\begin{aligned}\partial_u \mathbf{T}(u_0, v_0) &= \lim_{\Delta u \to 0} \frac{\mathbf{T}(u_0 + \Delta u, v_0) - \mathbf{T}(u_0, v_0)}{\Delta u} \\ \partial_v \mathbf{T}(u_0, v_0) &= \lim_{\Delta v \to 0} \frac{\mathbf{T}(u_0, v_0 + \Delta v) - \mathbf{T}(u_0, v_0)}{\Delta v}.\end{aligned}$$

So, one has the approximation

$$\begin{aligned}\mathbf{b} &= \mathbf{T}(u_0 + \Delta u, v_0) - \mathbf{T}(u_0, v_0) \approx \partial_u \mathbf{T}(u_0, v_0)\Delta u \\ \mathbf{a} &= \mathbf{T}(u_0, v_0 + \Delta v) - \mathbf{T}(u_0, v_0) \approx \partial_v \mathbf{T}(u_0, v_0)\Delta v.\end{aligned}$$

So,

$$A = \left| \det \begin{pmatrix} \partial_u T_1 & \partial_u T_2 \\ \partial_v T_1 & \partial_v T_2 \end{pmatrix} \right| = \left| \det \begin{pmatrix} \partial_u x & \partial_u y \\ \partial_v x & \partial_v y \end{pmatrix} \right|$$

Therefore, the approximate change in area for the Riemann sum formula is

$$\Delta x \Delta y \approx \left| \det \begin{pmatrix} \partial_u x & \partial_u y \\ \partial_v x & \partial_v y \end{pmatrix} \right| \Delta u \Delta v.$$

If we do this for our partitioning of our domain of integration for each rectangle we will conclude that, for f Riemann integrable,

$$\int_D f(x,y)dxdy = \int_{\mathbf{T}(D)} f(\mathbf{T}(u,v))|\det[J(\mathbf{T}(u,v))]|dudv.$$

The matrix

$$J(\mathbf{T}(u,v)) = \begin{pmatrix} \partial_u x & \partial_u y \\ \partial_v x & \partial_v y \end{pmatrix}$$

is called the **Jacobian** matrix and is defined as the matrix of all first partial derivatives of a (vector-valued) multivariable function. In general, the **Jacobian** matrix for a function $\mathbf{f} = (f_1, ..., f_n)$ of m-variables $(x_1, ..., x_m)$ is

$$J(\mathbf{f}(\mathbf{x})) = \begin{pmatrix} \partial_{x_1} f_1 & \dots & \partial_{x_1} f_n \\ \vdots & \ddots & \\ \partial_{x_m} f_1 & \dots & \partial_{x_m} f_n \end{pmatrix}, \tag{652}$$

which is the generalisation of the gradient!

What the above discussion motivates (but does not prove, if you're interested check Spivak) is the following theorem for change of variables

Theorem 16.3. *Suppose f is a continuous function of n-variables $(x_1, ..., x_n)$ on $D \subseteq \mathbb{R}^n$. Suppose that $\mathbf{T}$ is a C^1 transformation from D to $\mathbf{T}(D)$ with non-vanishing Jacobian, which is one-to-one except possibly on the boundary of D. Then,*

- *for $n = 2$, where $(x_1, x_2) = (x, y)$,*

$$\int_D f(x,y)dxdy = \int_{\mathbf{T}(D)} f(\mathbf{T}(u,v)) \left| \det \begin{pmatrix} \partial_u x & \partial_u y \\ \partial_v x & \partial_v y \end{pmatrix} \right| dudv.$$

- *for $n = 3$, where $(x_1, x_2, x_3) = (x, y, z)$,*

$$\int_D f(x,y,z)dxdydz = \int_{\mathbf{T}(D)} f(\mathbf{T}(u,v,w))\Big|\det\begin{pmatrix} \partial_u x & \partial_u y & \partial_u z \\ \partial_v x & \partial_v y & \partial_v z \\ \partial_w x & \partial_w y & \partial_w z \end{pmatrix}\Big| dudvdw.$$

- *for arbitrary for n,*

$$\int_D f(\mathbf{x})\mathrm{dvol}(\mathbf{x}) = \int_{\mathbf{T}(D)} f(\mathbf{T}(\mathbf{u}))\big|\det[J(\mathbf{T}(\mathbf{u}))]\big|\mathrm{dvol}(\mathbf{u}).$$

16.5.1 Integrals in Polar Coordinates

Lets start by doing the change of variables from Cartesian coordinates to polar coordinates on $\mathbb{R}^2$. Our C^1-transformation is given by

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

Therefore, the Jacobian is

$$J = \begin{pmatrix} \partial_r x & \partial_r y \\ \partial_\theta x & \partial_\theta y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}.$$

The determinant is then

$$\det(J) = r\cos^2\theta - (-r\sin^2\theta) = r. \tag{653}$$

So, to integrate of a domain D is polar coordinates one would compute

$$\int_D f(x,y)dxdy = \int_{\mathbf{T}(D)} f(r,\theta)rdrd\theta.$$

This is particularly useful if one wishes to integrate of radially symmetric region. For example:

- a disk of radius R, $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = R\}$. Note that this is the union of two normal domains in (x,y) coordinates,

$$D_1 = \{(x,y) : -R \le x \le R, 0 \le y \le \sqrt{R - x^2}\} \tag{654}$$
$$D_2 = \{(x,y) : -R \le x \le R, -\sqrt{R - x^2} \le y \le 0\}. \tag{655}$$

However, in (r,θ)-plane this is a 'rectangle' since

$$\mathbf{T}(D) = \{(r,\theta) \in \mathbb{R}^2 : 0 \le r \le R, 0 \le \theta \le 2\pi\}.$$

Therefore, integrals of continuous functions (or conditions so that Fubini applies) can be very simple on disks:

$$\int_D f(x,y)dxdy = \int_0^{2\pi}\int_0^R f(r,\theta)rdrd\theta.$$

For example, if $f(x,y) = 1 = f(r,\theta)$ would be

$$\int_D f(x,y)dxdy = \int_0^{2\pi}\int_0^R rdrd\theta = \pi R^2$$

which is the area of a disk.

- an annulus is a region between two circles of radius $R_1 < R_2$, i.e. our annulus for integration is

$$D = \{(x, y) \in \mathbb{R}^2 : R_1 \le x^2 + y^2 \le R_2\}.$$

Again, you can split this up into normal domains in the (x, y) plane. However, if you write this in polar coordinates things simplify and, once again we have a 'rectangle' in the (r, θ) plane:

$$\mathbf{T}(D) = \{(r, \theta) \in \mathbb{R}^2 : R_1 \le r \le R_2, , 0 \le \theta \le 2\pi\}.$$

So,

$$\int_D f(x, y) dx dy = \int_0^{2\pi} \int_{R_1}^{R_2} f(r, \theta) r dr d\theta.$$

If $f = 1$ again, we get the area of the annulus:

$$\int_D dx dy = \int_0^{2\pi} \int_{R_1}^{R_2} r dr d\theta = \pi(R_2^2 - R_1^2).$$

- a flower. Consider the domain of integration:

$$D = \{(r, \theta) : -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}, 0 \le r \le \cos(2\theta)\}.$$

Note that this is a normal domain. Suppose we wanted to find its area then

$$A(D) = \int_D 1 r dr d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\cos(2\theta)} r dr d\theta = \frac{\pi}{8}.$$

16.5.2 Integrals in Cylindrical Coordinates

Cylindrical coordinates in $\mathbb{R}^3$ are defined by

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z$$

which is a C^1-transformation. The Jacobian of this transformation is

$$J = \begin{pmatrix} \partial_r x & \partial_r y & \partial_r z \\ \partial_\theta x & \partial_\theta y & \partial_\theta z \\ \partial_z x & \partial_z y & \partial_z z \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore,

$$\det(J) = r$$

and

$$\int_D f(x, y) \mathrm{dvol} = \int_{T(D)} f(r, \theta) r dr d\theta dz.$$

Example 16.13. *The mass m of an object is the integral of its density $\rho(x, y, z)$. Find the mass of a solid that lies between the cylinders $x^2 + y^2 = R_1^2$ $x^2 + y^2 = R_2^2$ with $R_1 < R_2$, the xy-plane and the $\{z = 1\}$ plane with a density*

$$\rho(x, y, z) = (x^2 + y^2)^2.$$

So the mass

$$m = \int_D \rho \mathrm{dvol} = \int_{T(D)} r^4 r dr d\theta d\varphi$$

where

$$D = \{\mathbf{x} \in \mathbb{R}^3 : R_1 \le \sqrt{x^2 + y^2} \le R_2, z \in [0, 1]\}$$
$$T(D) = \{(r, \theta, z) \in \mathbb{R}^3 : R_1 \le r \le R_2, \theta \in [0, 2\pi), z \in [0, 1]\}.$$

So,

$$m = \int_{R_1}^{R_2} \int_0^{2\pi} \int_0^1 r^5 dr d\theta d\varphi = 2\pi \int_{R_1}^{R_2} r^5 dr = \frac{\pi}{3}\left(R_2^6 - R_1^6\right).$$

16.5.3 Integrals in Spherical Coordinates

Spherical coordinates (r, θ, φ) in $\mathbb{R}^3$ are defined by

$$x = r\sin\theta\cos\varphi, \qquad y = r\sin\theta\sin\varphi, \qquad z = r\cos\theta$$

which, again is a C^1-transformation. The Jacobian of this transformation is

$$J = \begin{pmatrix} \partial_r x & \partial_r y & \partial_r z \\ \partial_\theta x & \partial_\theta y & \partial_\theta z \\ \partial_\varphi x & \partial_\varphi y & \partial_\varphi z \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ r\cos\theta\cos\varphi & r\cos\theta\sin\varphi & -r\sin\theta \\ -r\sin\theta\sin\varphi & r\sin\theta\cos\varphi & 0 \end{pmatrix},$$

which has determinant

$$\det(J) = r^2\sin\theta.$$

Therefore,

$$\int_D f(x,y)\mathrm{dvol} = \int_{T(D)} f(r,\theta)r^2\sin\theta drd\theta d\varphi.$$

Example 16.14. *Let's compute the volume between two surfaces:*

- *the cone* $z^2 = x^2 + y^2$
- *the unit sphere* $x^2 + y^2 + (z-1)^2 = 1$.

$$\mathrm{Vol}(D) = \int_D 1dxdydz = \int_{T(D)} r^2\sin\theta drd\theta d\varphi.$$

We can express our sphere in spherical coordinates as

$$r^2 = z = r\cos\theta \Longleftrightarrow r = \cos\theta.$$

Our cone has equation

$$r^2\sin^2\theta = r^2\cos^2\theta \Longleftrightarrow \sin\theta = \cos\theta \implies \theta = \frac{\pi}{4}.$$

$$T(D) = \{(r,\theta,\varphi) : 0 \le \varphi \le 2\pi, 0 \le \theta \le \frac{\pi}{4}, 0 \le r \le \cos\theta\}.$$

So,

$$\mathrm{Vol}(D) = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\cos\theta} r^2\sin\theta drd\theta d\varphi.$$

Therefore,

$$\mathrm{Vol}(D) = 2\pi\int_0^{\frac{\pi}{4}} \frac{1}{3}\cos^3\theta\sin\theta d\theta = \frac{2\pi}{12}(-\cos^4\theta)\Big|_0^{\frac{\pi}{4}} = \frac{\pi}{8}.$$

17 Vector Calculus

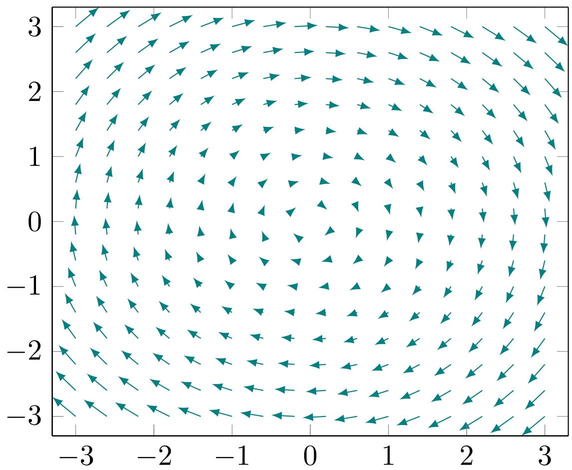
17.1 Vector Fields

For the purposes of this course the following definition of vector field suffices:

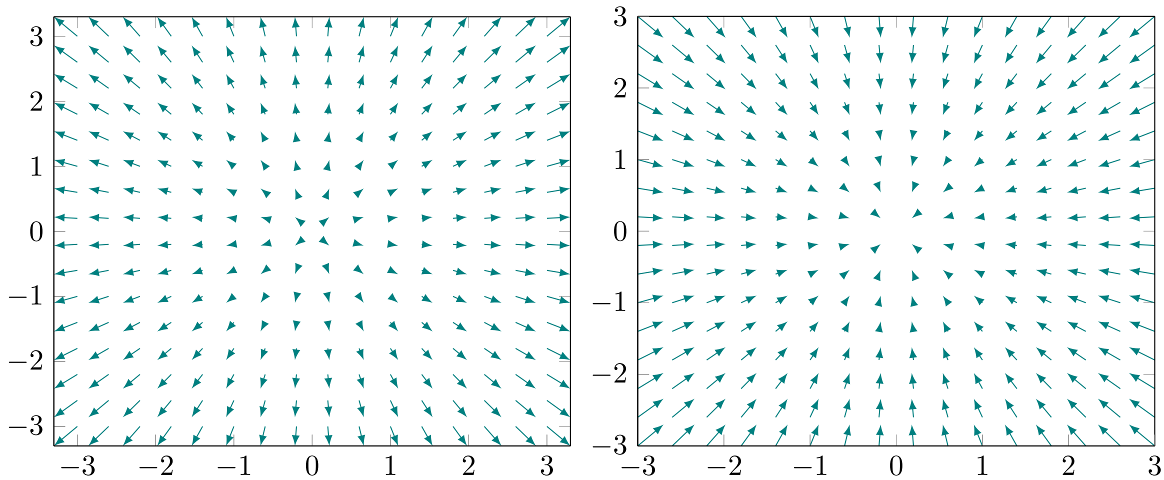
Definition 17.1. *Let D be a subset of $\mathbb{R}^n$. A vector field on D is an assignment of a vector in $\mathbb{R}^n$ to each point $\mathbf{x} \in D$, i.e. it is a vector-valued multivariable function $\mathbf{f}$ on D which maps $\mathbf{x} \in D \subset \mathbb{R}^n$ to $\mathbf{f}(\mathbf{x}) \in \mathbb{R}^n$.*

The picture you should have in mind is an arrow at each $\mathbf{x}$ representing $\mathbf{f}(x)$ as plotted blow:

Example 17.1. *Let's draw the vector field $\mathbf{f}(\mathbf{x}) = y\mathbf{i} - x\mathbf{j}$ in the plane. Notice that on a circle of radius R $||\mathbf{f}(\mathbf{x})|| = x^2 + y^2 = R^2$ therefore the magnitude increases radially and is constant on each circle. In the first quadrant it points positively in the x-direction and negatively in the y-direction. You can deduce the other quadrants. You'll find that it looks something like this.*



Example 17.2. *Let's draw the vector field $\mathbf{f}(\mathbf{x}) = x\mathbf{i} + y\mathbf{j}$ and $\mathbf{f}(\mathbf{x}) = -x\mathbf{i} - y\mathbf{j}$ and in the plane.*



Remark 17.1. *As with vector-valued functions of a single variable, a vector field is continuous if its components*

$$\mathbf{f}(\mathbf{x}) = f_1(\mathbf{x})\mathbf{i} + f_2(\mathbf{x})\mathbf{j} + f_3(\mathbf{x})\mathbf{k}$$

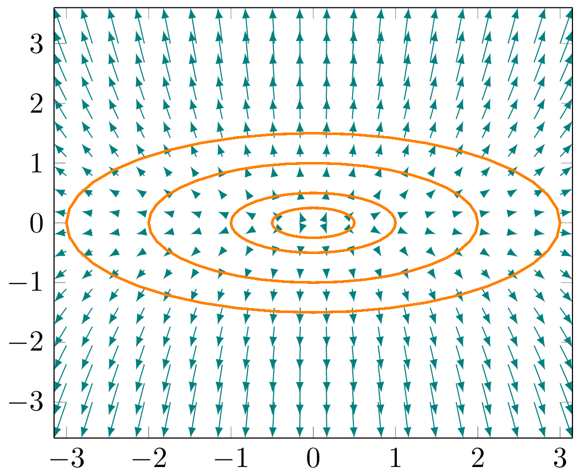
are continuous. Similarly, a vector field is differentiable if its components are differentiable.

Let's think about the gradient again briefly. Let f be a scalar function of two variables $(x, y) \in D \subseteq \mathbb{R}^2$ then

$$\nabla f = \partial_x f \mathbf{i} + \partial_y f \mathbf{j}.$$

This is a vector-valued multivariable function! So the proper interpretation of ∇f is that of a vector field.

Example 17.3. *Consider the level sets of the (elliptic parabaloid) function $\frac{x^2}{4} + y^2$. The gradient vector field $\nabla f = \frac{1}{2}x\mathbf{i} + 2y\mathbf{j}$. We can plot the level curves and the gradient vector field together.*



As we saw before, the gradient vector field ∇f is orthogonal to the level curves of f.

Definition 17.2. *A vector field $\mathbf{f}$ is called conservative if it can be written as the gradient of a scalar function of many variables, i.e.*

$$\mathbf{f} = \nabla f$$

for some f.

The reason for this name will become apparent with line integrals. We can interpret line intergals as measure work done (energy used) by a vector field moving a particle between points. Effectively, conservative vector fields have zero line integral around a closed loop, so no energy used.

Remark 17.2. *In Newtonian gravity, the gravitational force between to masses M and m is defined to be conservative:*

$$\mathbf{f} = \nabla \Phi, \qquad \Phi = \frac{GMm}{||\mathbf{x}||} \implies \mathbf{f} = -m\frac{GM}{||\mathbf{x}||^2}\hat{x}.$$

By Newton's second law:

$$\mathbf{f} = m\mathbf{a} \implies \mathbf{a} = -\frac{GM}{||\mathbf{x}||^2}\hat{\mathbf{x}}.$$

If you compute $\frac{GM}{||\mathbf{x}||^2}$ using $G \approx 6 \times 10^{-11}$, $M = M_E \approx 6 \times 10^{24}$ and $||\mathbf{x}|| = r_E = 6 \times 10^6$, you find $10m/s^2$ as the magnitude of the acceleration close to the surface of the Earth. Note we have treated the Earth as an idealised point mass. The direction of $\mathbf{f}$ is $\hat{\mathbf{x}}$. This points radially inwards as one might expect.

We define two useful operations on vector fields:

Definition 17.3. *The divergence of a vector field $\mathbf{f}$ in $\mathbb{R}^n$ is*

$$\mathrm{div}\mathbf{f} = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i},$$

if the partial derivatives on the RHS exist.

Note that if we think of ∇ as the vector

$$\nabla = (\partial_{x_1}, \ldots, \partial_{x_n})$$

then one can write

$$\mathrm{div}\mathbf{f} = \nabla \cdot \mathbf{f} = \langle \nabla, \mathbf{f} \rangle.$$

Definition 17.4. *The curl of a vector field $\mathbf{f} = (f_1, f_2)$ in $\mathbb{R}^2$ is a scalar multivariable function*

$$\mathrm{curl}\mathbf{f} = \partial_x f_2 - \partial_y f_1.$$

The curl of a vector field $\mathbf{f} = (f_1, f_2, f_3)$ in $\mathbb{R}^3$ is a vector field

$$\mathrm{curl}\mathbf{f} = \Big(\partial_y f_3 - \partial_z f_2\Big)\mathbf{i} - \Big(\partial_x f_3 - \partial_z f_1\Big)\mathbf{j} + \Big(\partial_x f_2 - \partial_y f_1\Big)\mathbf{k}.$$

if the partial derivatives on the RHS exist.

Remark 17.3. *You may wonder how the* curl *generalises to $\mathbb{R}^n$. The most natural generalisation of the curl differential operator is the 'exterior derivative' d of vector field $\mathbf{f} = (f_1, ..., f_n)$ which is a matrix, denoted $d\mathbf{f}$, with components:*

$$(d\mathbf{f})_{ij} = \partial_{x_i} f_j - \partial_{x_j} f_i.$$

Note this matrix is antisymmetric.

Let's see how we get the curl back from this. So in $2D$ we have the matrix:

$$d\mathbf{f} = \begin{pmatrix} 0 & \partial_{x_1} f_2 - \partial_{x_2} f_1 \\ -\partial_{x_1} f_2 + \partial_{x_2} f_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{curl}\mathbf{f} \\ -\mathrm{curl}\mathbf{f} & 0 \end{pmatrix} = \mathrm{curl}\mathbf{f} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

As a repesentation of the information in it, this matrix has redundancy. Typically then one out identify $d\mathbf{f}$ with the $\mathrm{curl}\mathbf{f}$*.*

In $3D$ we find

$$d\mathbf{f} = \begin{pmatrix} 0 & \partial_{x_1} f_2 - \partial_{x_2} f_1 & \partial_{x_1} f_3 - \partial_{x_3} f_1 \\ \partial_{x_2} f_1 - \partial_{x_1} f_2 & 0 & \partial_{x_2} f_3 - \partial_{x_3} f_2 \\ \partial_{x_3} f_1 - \partial_{x_1} f_3 & \partial_{x_3} f_2 - \partial_{x_2} f_3 & 0 \end{pmatrix}.$$

Note that, again, this matrix is a representation of the vector

$$\mathbf{v} = \begin{pmatrix} \partial_{x_2} f_3 - \partial_{x_3} f_2 \\ \partial_{x_3} f_1 - \partial_{x_1} f_3 \\ \partial_{x_1} f_2 - \partial_{x_2} f_1 \end{pmatrix} = \mathrm{curl}\mathbf{f}.$$

In $\mathbb{R}^3$, if we think of ∇ as the vector

$$\nabla = (\partial_x, \partial_y, \partial_z)$$

then one can write

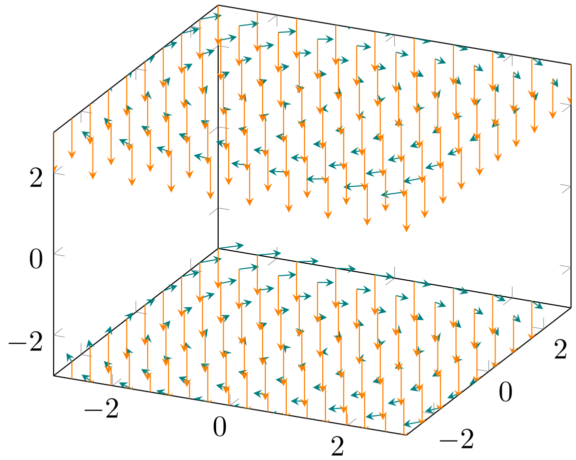
$$\nabla \times \mathbf{f} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ f_1 & f_2 & f_3 \end{pmatrix}$$

How should you think of these curl and div operations? The divergence of a vector field is a measure at each point of how much a vector field is 'outgoing' or 'ingoing'. The curl is a vector field representing how much a vector field would rotate an object sitting in its flow (thinking of the vector field as a fluid) and, in $\mathbb{R}^3$ it gives the axis of that rotation (with the right-hand rule convention). The following examples illustrate this:

Example 17.4. *Let's promote our first example above to a vector field in* $\mathbb{R}^3$ *by trivially extending in* z*. Let* $\mathbf{f}(x, y, z) = y\mathbf{i} - x\mathbf{j}$*. Now the lets compute curl and divergence of this vector field:*

$$\mathrm{curl}\mathbf{f} = -2\mathbf{k} \qquad \mathrm{div}\mathbf{f} = 0.$$

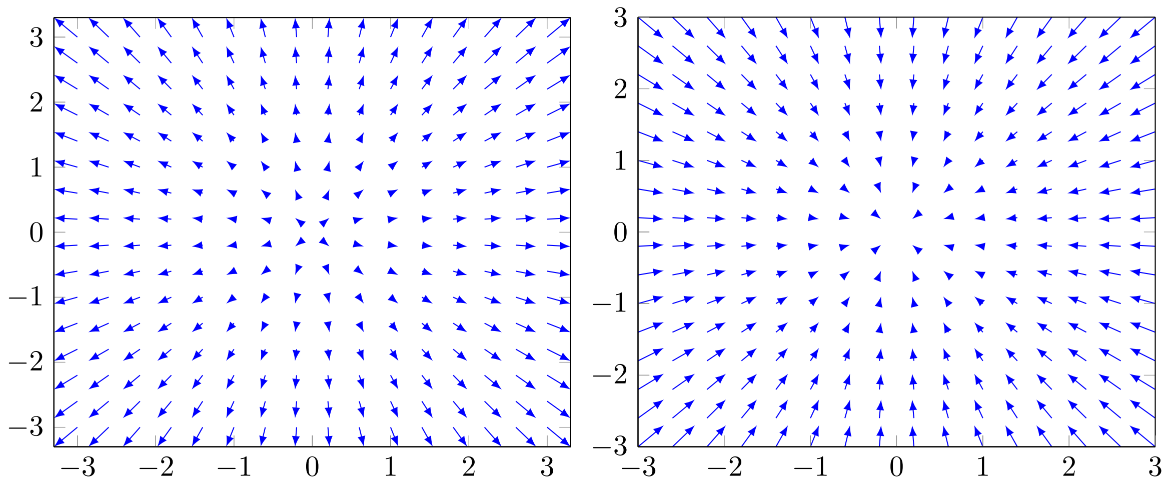
The vector field is plotted in green below with its curl *in orange.*



Example 17.5. *Let's compute the divergence of the vector fields* $\mathbf{f}_1(\mathbf{x}) = x\mathbf{i} + y\mathbf{j}$ *and* $\mathbf{f}_2(\mathbf{x}) = -x\mathbf{i} - y\mathbf{j}$*:*

$$\mathrm{div}\mathbf{f}_1 = 2, \qquad \mathrm{div}\mathbf{f}_2 = -2.$$

Note that its curl *is* 0*. Compare to the figures below* $\mathbf{f_1}$ *is on the left and* $\mathbf{f_2}$ *is on the right. We see that* $\mathrm{div}\mathbf{f}_1 > 0$ *and its flow is outward,* $\mathrm{div}\mathbf{f}_2 < 0$ *and its flow is inward.*



Proposition 17.1. *Let* f *be a function of two or three variables that has continuous second partial derivatives, then*

$$\mathrm{curl}(\nabla f) = 0.$$

In other words, if a vector field $\mathbf{f}$ *is conservative, then* $\mathrm{curl}\mathbf{f} = 0$*.*

Proof. In $2D$ this goes as follows:

$$\nabla f = (\partial_x f, \partial_y f) \implies \mathrm{curl}(\nabla f) = \partial_x \partial_y f - \partial_y \partial_x f$$

which vanishes when second partial derivatives are continuous since in this case the partial derivatives commute.

In $3D$ this goes as follows:

$$\operatorname{curl}(\nabla f) = (\partial_y \partial_z f - \partial_z \partial_y f)\mathbf{i} - (\partial_x \partial_z f - \partial_z \partial_x f)\mathbf{j} + (\partial_x \partial_y f - \partial_y \partial_x f)\mathbf{k}.$$

Since second partials commute when the partial derivatives are continuous all terms vanish and we are left with the zero vector. □

This theorem says if we show $\operatorname{curl}\mathbf{f} \neq 0$ then we know $\mathbf{f}$ is not conservative. In general, the converse of the theorem is **not** true, unless the vector field is well-defined on all of $\mathbb{R}^3$.

Theorem 17.1. *If a vector field $\mathbf{f}$ is defined everywhere on $\mathbb{R}^3$, and $\operatorname{curl}\mathbf{f} = 0$ then $\mathbf{f}$ is conservative, i.e. $\mathbf{f} = \nabla f$.*

Example 17.6. *Let's consider the following vector field*

$$\mathbf{f}(\mathbf{x}) = y^2 z^3 \mathbf{i} + 2xyz^3 \mathbf{j} + 3xy^2 z^2 \mathbf{k}$$

on all of $\mathbb{R}^3$. If one compute the curl of this vector field

$$\operatorname{curl}\mathbf{f} = 0.$$

From the above theorem we know that $\mathbf{f}$ is conservative so we can try to search for a f such that $\mathbf{f} = \nabla f$. So we want

$$\partial_x f = y^2 z^3, \quad \partial_y f = 2xyz^3, \quad \partial_z f = 3xy^2 z^2.$$

So,

$$f(\mathbf{x}) = xy^2 z^3 + g(y, z).$$

Therefore,

$$\partial_y f = 2xyz^3 + \partial_y g \implies g(y, z) = h(z)$$

and

$$\partial_z f = 3xy^2 z^2 + h' \implies h = \text{constant}.$$

In conclusion,

$$f(\mathbf{x}) = xy^2 z^3 + \text{constant}.$$

Proposition 17.2. *Let $\mathbf{f}$ be a vector field on $\mathbb{R}^3$ which has continuous second partials, then*

$$\operatorname{div}\operatorname{curl}\mathbf{f} = 0.$$

Proof. We compute directly that

$$\begin{aligned}\operatorname{div}\operatorname{curl}\mathbf{f} &= \partial_x (\operatorname{curl}\mathbf{f})_x + \partial_y (\operatorname{curl}\mathbf{f})_y + \partial_z (\operatorname{curl}\mathbf{f})_z \\ &= \partial_x(\partial_y f_z - \partial_z f_y) + \partial_y(\partial_z f_x - \partial_x f_z) + \partial_z(\partial_x f_y - \partial_y f_x).\end{aligned}$$

All terms cancel. □

17.2 Line Integrals

A line integral is an integral of a function along a curve $\gamma \in \mathbb{R}^n$. This can be a scalar function, or a vector field.

17.2.1 Definition

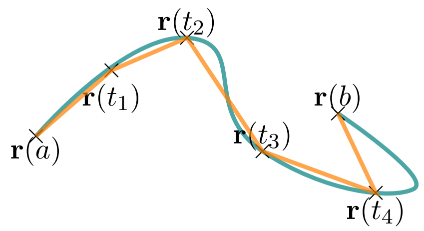
Suppose we have a curve γ in $\mathbb{R}^n$ which we parameterise by $t \in [a, b]$, i.e. γ has the parametric equations

$$x_1 = x_1(t), \ldots, x_n = x_n(t) \qquad \Longleftrightarrow \qquad \mathbf{r}(t) = (x_1(t), \ldots, x_n(t)).$$

We will assume that the curve is C^1 ('continuously differentiable'): the tangent vector $\mathbf{r}'$ is continuous and $\mathbf{r}' \neq \mathbf{0}$ (although this assumption is slightly unusual for a definition of C^1). We divide up out interval $[a, b]$ into n intervals $[t_{i-1}, t_i]$ of length $\Delta t = \frac{b-a}{n}$ with $t_0 = a$ and $t_n = b$. Further, on each subinterval, we sample the curve at the point $\mathbf{r}(t_i)$ (you can pick other points). We then approximate the curve on the interval $[t_{i-1}, t_i]$ with the a straight line between $\mathbf{r}(t_{i-1})$ and $\mathbf{r}(t_i)$. We then denote

$$\Delta s_i = ||\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})||, \qquad i = 1, ..., n.$$

This is the distance between sample points, or in otherwords, the length of the straight line approximating the curve on the interval $[t_{i-1}, t_i]$ (an approximate length of the curve between $\mathbf{r}(t_i)$ and $\mathbf{r}(t_{i-1})$).



If we sum over all these distances we get an approximate length of the curve:

$$\sum_{i=1}^{n} \Delta s_i.$$

Taking the limit of this as $n \to \infty$ and $\Delta s_i \to 0$ gives the **arc length** of the curve:

$$s = \int_\gamma ds.$$

You should think of ds as the infinitesemal arc length.

Now, the mean value theorem[12] tells us that there exists a $t_{i,0} \in [t_{i-1}, t_i]$ such that

$$\frac{d\mathbf{r}(t_{i,0})}{dt} = \frac{\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})}{t_i - t_{i-1}} \implies ||\mathbf{r}'(t_{i,0})|| = \frac{\Delta s_i}{\Delta t_i}.$$

So we can replace Δs_i with (approximately) $||\mathbf{r}'(t_i)||\Delta t_i$ (note the drop of the $t_{i,0}$). So, taking the limit gives

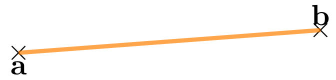
$$s = \int_\gamma ds = \int_a^b ||\mathbf{r}'(t)||dt.$$

Note that gives the length of the curve γ and, in particular, is always positive! In other words, there's no directional information in this integral: it doesn't matter what way you move along the curve:

$$s = \int_\gamma ds = \int_{-\gamma} ds$$

If you are unconvinced consider a simple straight line segment in $\mathbb{R}^n$ connecting $\mathbf{a}$ and $\mathbf{b}$ as shown below:

[12]The mean value theorem says that a continuous function on a closed interval $[a, b]$ which is differentiable on the interior (a, b) then there exists a $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.



We can describe it with the equation

$$\mathbf{r}_1(\lambda_1) = (1-\lambda_1)\mathbf{a} + \lambda\mathbf{b}, \qquad \lambda_1 \in [0,1].$$

Now $-\gamma$ is the curve from $\mathbf{b}$ to $\mathbf{a}$, so we can describe it with the equation

$$\mathbf{r}_2(\lambda_2) = (1-\lambda_2)\mathbf{b} + \lambda_2\mathbf{a} \qquad \lambda_2 \in [0,1].$$

Therefore,

$$\frac{d\mathbf{r}_2}{d\lambda_2} = \mathbf{a} - \mathbf{b} = -\frac{d\mathbf{r}_1'}{d\lambda_1}.$$

Hence,

$$\int_\gamma ds = \int_0^1 \|\mathbf{r}_1'(\lambda_1)\| d\lambda_1 = \int_0^1 \|\mathbf{b}-\mathbf{a}\| d\lambda_1 = \int_1^0 -\|-(\mathbf{b}-\mathbf{a})\| d\lambda_2 = \int_0^1 \|\mathbf{r}_2'(\lambda_2)\| d\lambda_2 = \int_{-\gamma} ds.$$

Since one approximates a curve with straight line segments. You can generalise this argument infinitesemally for any curve. So we have

$$\int_\gamma ds = \int_{-\gamma} ds.$$

Another thing to notice is that the arc length s is a very natural way to parameterise a curve since it is literally how far along the curve you have travelled. Typically we can pick any number of ways to parameterise a curve; whatever is most convenient.

Example 17.7. *Let's consider a unit circle in the plane:*

$$\mathbb{S}^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

We can parameterise this curve via the length of the curve which is just $s = \theta \in [0, 2\pi]$ or we may pick a parameterisation:

$$x = \cos(t) \qquad y = \sin(t)$$

or

$$x = \cos(3t) \qquad y = \sin(3t)$$

or

$$x = \cos(7t+1) \qquad y = \sin(7t+1)$$

etc...

Consider a continuous function f from $\mathbb{R}^n$ to $\mathbb{R}$ which we wish to integrate along the curve, i.e. we want to consider $f(\mathbf{x}(s))$, i.e. the function evaluate at the points on the line parameterised by the arc length. We can return to the above Riemann sum for the arclength and then construct the Riemann sum of the function along the curve:

$$\sum_{i=1}^n f(\mathbf{x}(s))\Delta s_i.$$

Taking the limit of this as $n \to \infty$ and $\Delta s_i \to 0$ gives the line integral of f along γ:

$$\int_\gamma f(\mathbf{x}(s))ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(\mathbf{x}(s_i))\Delta s_i.$$

Recall that above we found that arc length was related to our prefered choice of parameterisation (say $\mathbf{r}(t)$) by $||\mathbf{r}'(t_{i,0})|| = \frac{\Delta s_i}{\Delta t_i}$. So,

$$\int_\gamma f ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(\mathbf{r}(t_i))\Delta s_i = \lim_{n\to\infty} \sum_{i=1}^{n} f(\mathbf{r}(t_i))||\mathbf{r}'(t_i)||\Delta t_i.$$

Therefore,

$$\int_\gamma f ds = \int_a^b f(\mathbf{r}(t))||\mathbf{r}'(t)||dt.$$

Again, the line integral of a function f is **independent of the direction along the curve** we integrate over:

$$\int_\gamma f ds = \int_{-\gamma} f ds.$$

We can relax our assumption of a C^1 curve slightly and require that it is piecewise C^1, i.e. it is a continuous curve made up of C^1 curves $\gamma_1, \ldots, \gamma_n$. Then

$$\int_\gamma f ds = \sum_i \int_{\gamma_i} f ds.$$

This is essentially because integrals don't 'see' points.

Example 17.8. *Let's integrate the function $f(x, y, z) = x^2 y$ along the curve with parametric equations*

$$x = \cos t, \qquad y = \sin t, \qquad z = t \qquad t \in [0, \pi].$$

Note that

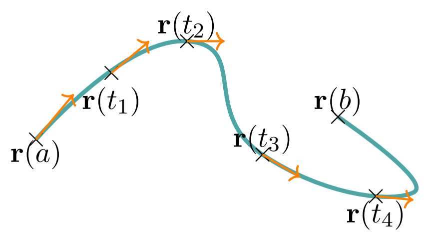
$$||\mathbf{r}'(t)|| = \sqrt{1+1} = \sqrt{2}$$

for $t \in [0, \pi]$.

So,

$$\int_\gamma f ds = \int_0^\pi x(t)^2 y(t)||\mathbf{r}'(t)||dt = \sqrt{2}\int_0^\pi \cos^2(t)\sin(t)dt = \frac{\sqrt{2}}{3}\int_0^\pi \frac{d}{dt}(-\cos^3(t))dt \quad = -\frac{\sqrt{2}}{3}\cos^3 t\Big|_0^\pi = \frac{2\sqrt{2}}{3}.$$

To generalise the above to a vector field $\mathbf{f} : D \subset \mathbb{R}^n \to \mathbb{R}^n$ along a C^1 curve γ in $\mathbb{R}^n$, we have a couple of options. Suppose γ is described by a vector-valued function $\mathbf{r}(t)$ with $t \in [a, b]$. Then γ has tangent vector $\mathbf{r}'$ (shown in orange below):



The tangent vector is unique up to scaling. So let's fix the normalisation by taking the **unit tangent vector**:

$$\hat{\mathbf{t}} = \frac{\mathbf{r}'}{||\mathbf{r}'||}.$$

So we can take the **tangential component** of our vector field by projecting our vector field onto the unit tangent vector $\hat{\mathbf{t}}$, to construct a scalar function $g = \langle \mathbf{f}, \hat{\mathbf{t}} \rangle$. This g we integrate with respect to t:

$$\int_\gamma \langle \mathbf{f}, \hat{\mathbf{t}} \rangle ds = \int_a^b \Big\langle \mathbf{f}(\mathbf{r}(t)), \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||} \Big\rangle ||\mathbf{r}'(t)|| dt = \int_a^b \langle \mathbf{f}(\mathbf{r}(t)), \mathbf{r}'(t) \rangle dt$$

This is what is **defined as the line integral of the vector field $\mathbf{f}$** in $\mathbb{R}^n$ along γ and is often denoted:

$$\int_\gamma \mathbf{f} \cdot d\mathbf{r} = \int_\gamma \langle \mathbf{f}, \hat{\mathbf{t}} \rangle ds$$

In there notes, the latter notation will typically be used, since this makes it clear what we do to integrate a vector field along a curve: we pair it with the unit tangent vector and integrate the scalar function along the curve as above!

One this to notice is that the line integral of a vector field **is direction dependent**. We should think of each curve γ as coming with an orientation, which is the direction along the curve. Equivalently, it can be thought of as the direction of the tangent vector $\hat{\mathbf{t}}$. This is because if integrate along $-\gamma$ instead of γ the tangent vector flips direction. So for **the line integral of a vector field** over a curve γ with tangent $\hat{\mathbf{t}}_\gamma$:

$$\int_\gamma \langle \mathbf{f}, \hat{\mathbf{t}}_\gamma \rangle ds = \int_{-\gamma} \langle \mathbf{f}, \hat{\mathbf{t}}_\gamma \rangle ds = -\int_{-\gamma} \langle \mathbf{f}, -\hat{\mathbf{t}}_\gamma \rangle ds = -\int_{-\gamma} \langle \mathbf{f}, \hat{\mathbf{t}}_{-\gamma} \rangle ds.$$

Example 17.9. *Evaluate the line integral of the vector field* $\mathbf{f}(\mathbf{x}) = \sin(x)\mathbf{i} + \cos(y)\mathbf{j} + xz\mathbf{k}$ *along the curve with parametric equation*

$$\mathbf{r}(t) = (t^3, -t^2, t) \qquad t \in [0, 1].$$

So,

$$\mathbf{r}' = (3t^2, -2t, 1).$$

So,

$$\begin{aligned}\int_\gamma \langle \mathbf{f}, \hat{\mathbf{t}} \rangle ds &= \int_0^1 3t^2 f_1(\mathbf{r}(t)) - 2t f_2(\mathbf{r}(t)) + f_3(\mathbf{r}(t)) dt \\ &= \int_0^1 \Big[3t^2 \sin(t^3) - 2t\cos(-t^2) + t^4 \Big] dt = \frac{6}{5} - \cos(1) - \sin(1).\end{aligned}$$

In $\mathbb{R}^2$, we could look at the normal component of the vector field $\mathbf{f}$ at each point along the curve (this is unique up to scaling).[13] What is the normal to our curve? The normal $\mathbf{n}(t)$ is vector orthogonal to the tangent vector $\mathbf{r}$

$$\langle \mathbf{n}(t), \mathbf{r}'(t) \rangle = 0 \implies n_1(t) r_1'(t) + n_2(t) r_2'(t).$$

and is unique here up to scaling. So $\mathbf{n}(t) \propto (r_2'(t), -r_1'(t))$. To overcome the scaling ambiguity we take the **normal**

$$\mathbf{n}(t) = (r_2'(t), -r_1'(t)).$$

[13] In $\mathbb{R}^n$, we do not have a unique normal direction. For example, in $\mathbb{R}^3$ the normal 'direction' is really a plane.

Additionally, we normalise it to 1:

$$\hat{\mathbf{n}} = \frac{\mathbf{n}}{\|\mathbf{n}\|} = \frac{r_2'(t)}{\|\mathbf{r}'(t)\|}\mathbf{i} - \frac{r_2'(t)}{\|\mathbf{r}'(t)\|}\mathbf{j}.$$

We then contruct a line integal with respect **unit normal**:

$$\int_\gamma \langle \mathbf{f}, \hat{\mathbf{n}}\rangle ds = \int_a^b \left\langle \mathbf{f}(\mathbf{r}(t)), \frac{\mathbf{n}(t)}{||\mathbf{n}(t)||}\right\rangle ||\mathbf{r}'(t)||dt = \int_a^b \langle \mathbf{f}(\mathbf{r}(t)), \mathbf{n}(t)\rangle dt.$$

Remark 17.4. *There is still sign ambiguity here that one has to adopt a convention for; we will return to this later.*

Example 17.10. *Evaluate the line integral normal component of the vector field $\mathbf{f}(\mathbf{x}) = y\mathbf{i} - x\mathbf{j}$ along the curve with parametric equation*

$$\mathbf{r}(t) = (\sin(t), \cos(t)) \qquad t \in [0, 1].$$

So,

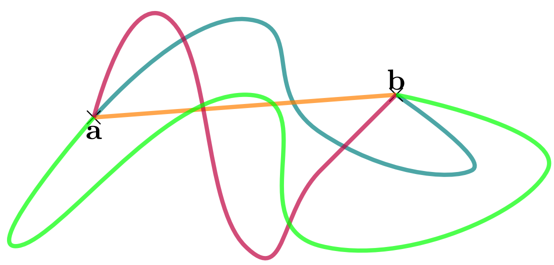
$$\mathbf{r}' = (\cos(t), -\sin(t)) \implies \mathbf{n}(t) = (-\sin(t), -\cos(t)).$$

Therefore,

$$\int_\gamma \langle \mathbf{f}, \hat{\mathbf{n}}\rangle ds = \int_0^1 (-\sin(t)\cos(t) + \cos(t)\sin(t))dt = 0.$$

17.2.2 Path Independence and Conservative Vector Fields

Suppose we consider many paths γ_1 and γ_2 from a point $\mathbf{a}$ to $\mathbf{b}$ in $\mathbb{R}^n$ as shown below:



Suppose γ_1 has unit tangent vector $\hat{\mathbf{t}}_1$ and γ_2 has unit tangent vector $\hat{\mathbf{t}}_2$. In general, the line integral of a vector field $\mathbf{f}$ will depend on the path from $\mathbf{a}$ to $\mathbf{b}$, i.e.

$$\int_{\gamma_1} \langle \mathbf{f}, \hat{\mathbf{t}}_1\rangle ds \neq \int_{\gamma_2} \langle \mathbf{f}, \hat{\mathbf{t}}_2\rangle ds$$

If this is equality then we say that $\mathbf{f}$ is path independent.

It turns out that if $\mathbf{f}$ is conservative then its line integral is path independent from $\mathbf{a}$ to $\mathbf{b}$, i.e. if $\mathbf{f} = \nabla f$ then the line integral of $\mathbf{f}$ is path independent.

Theorem 17.2. *Let γ be a C^1 curve with unit tangent vector $\hat{\mathbf{t}}$ and initial point $\mathbf{a}$ and end point $\mathbf{b}$. Let f be a differentiable function of many variables whose gradient is continuous on γ. Then,*

$$\int_\gamma \langle \nabla f, \hat{\mathbf{t}}\rangle ds = f(\mathbf{b}) - f(\mathbf{a})$$

Proof. Let γ be given by the vector-valued function $\mathbf{r}(t)$ with parameterisation by $t \in [a,b]$ so that $\mathbf{r}(a) = \mathbf{a}$ and $\mathbf{r}(b) = \mathbf{b}$. Then

$$\hat{\mathbf{t}} = \frac{\mathrm{r}'}{||\mathbf{r}'||}, \qquad \mathbf{r}'(t) = \Big(\frac{dx_1}{dt}, ..., \frac{dx_n}{dt}\Big).$$

We note that

$$\nabla f = (\partial_{x_1} f, ..., \partial_{x_n} f), \implies \langle \nabla f, \mathbf{r}' \rangle = \sum_{i=1}^{n} \partial_{x_i} f \frac{dx_i}{dt} = \frac{df}{dt}$$

using the chain rule. Therefore,

$$\int_\gamma \langle \nabla f, \hat{\mathbf{t}} \rangle ds = \int_a^b \langle \nabla f(\mathbf{r}(t)), \mathbf{r}'(t) \rangle dt = \int_a^b \frac{df}{dt} dt = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

where we've used the fundamental theorem of calculus. □

What can we say about the converse? In particular, is it true that if a vector field is path independent it is conservative? We need two notions for paths before we can state some general theorems addressing this situation.

Definition 17.5. *A curve γ in $\mathbb{R}^n$ is closed if its starting point and end point coincide.*

Definition 17.6. *A curve γ in $\mathbb{R}^n$ is simple if it does not intersect itself between the endpoints.*

Figure 1: Top Left: closed, simple curve, Top Right: not closed, simple curve, Bottom Left: closed, not simple curve, Bottom Right: not closed, not simple curve

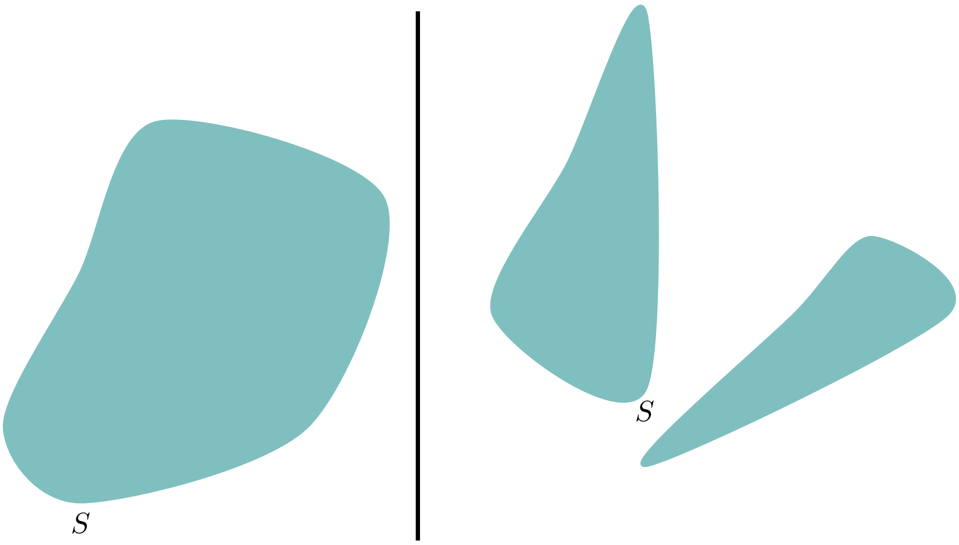
We previously discussed closedness of a set in $\mathbb{R}^2$, here is, the complimentary notion of closedness, openness:

Definition 17.7. *A set S in $\mathbb{R}^n$ is open if for every $\mathbf{x} \in S$ there exists an ϵ such that the ball of radius ϵ at $\mathbf{x}$, $B_\epsilon(\mathbf{x})$, is contained within S.*

We also require a notion of connected set:

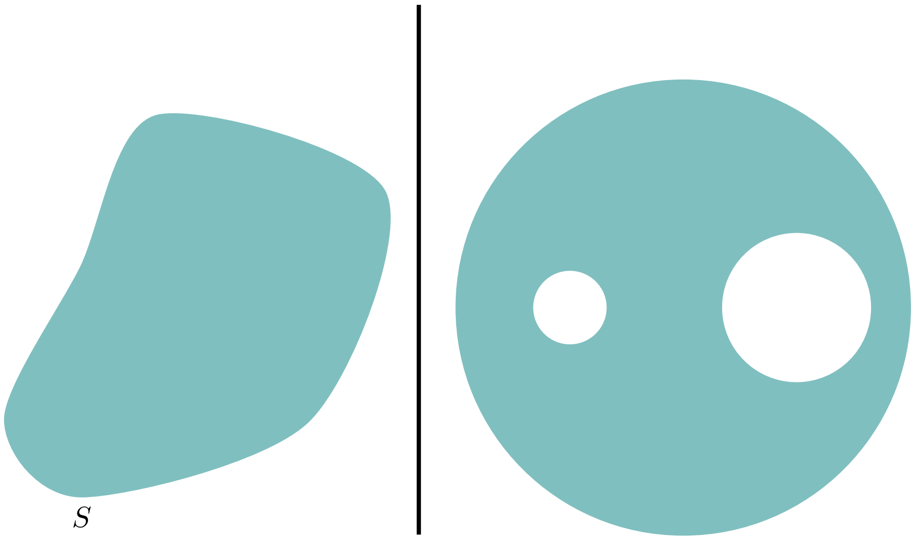
Definition 17.8. *A set S in $\mathbb{R}^n$ is (path) connected if for every pair $\mathbf{a}, \mathbf{b} \in S$ there exists a path within S connecting $\mathbf{a}$ and $\mathbf{b}$.*

On the left is a connected set and on the right is a disconnected set.



Definition 17.9. *A set S in $\mathbb{R}^n$ is simply connected if every closed simple curve can be contracted to a point.*

Effectively this means that the set has no holes. On the left is a simply connected set and on the right is a set that is not simply connected.



We can now state the theorems relating path independence and conservative vector fields.

Theorem 17.3. *Let γ be a path in $D \subset \mathbb{R}^n$ with unit tangent vector $\hat{\mathbf{t}}$. The line integral*

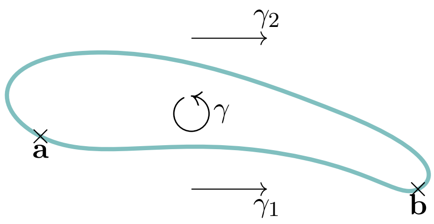
$$\int_\gamma \langle \mathbf{f}, \hat{\mathbf{t}} \rangle ds$$

is independent of the path γ in D if and only if

$$\int_\gamma \langle \mathbf{f}, \hat{\mathbf{t}} \rangle ds = 0$$

for **every closed** *path in D.*

Proof. Consider the following closed curve:



We split γ up into two sections γ_1 and γ_2 so that $\gamma = \gamma_1 - \gamma_2$ and γ_1, γ_2 have the same initial and end points. So, γ_1 has tangent $\hat{\mathbf{t}}_{\gamma_1} = \hat{\mathbf{t}}$ and γ_2 has tangent $\hat{\mathbf{t}}_{\gamma_2} = -\hat{\mathbf{t}}$. Then

$$\int_\gamma \langle \mathbf{f}, \hat{\mathbf{t}} \rangle ds = \int_{\gamma_1} \langle \mathbf{f}, \hat{\mathbf{t}} \rangle ds + \int_{-\gamma_2} \langle \mathbf{f}, \hat{\mathbf{t}} \rangle ds = \int_{\gamma_1} \langle \mathbf{f}, \hat{\mathbf{t}}_{\gamma_1} \rangle ds - \int_{\gamma_2} \langle \mathbf{f}, \hat{\mathbf{t}}_{\gamma_2} \rangle ds.$$

($\Leftarrow$) Suppose then that the $\int_\gamma \langle \mathbf{f}, \hat{\mathbf{t}} \rangle ds = 0$ then

$$\int_{\gamma_1} \langle \mathbf{f}, \hat{\mathbf{t}}_{\gamma_1} \rangle ds = \int_{\gamma_2} \langle \mathbf{f}, \hat{\mathbf{t}}_{\gamma_2} \rangle ds$$

and, therefore, the line integral is path independent.

($\Rightarrow$) Suppose the line integral is path independent, i.e.

$$\int_{\gamma_1} \langle \mathbf{f}, \hat{\mathbf{t}}_{\gamma_1} \rangle ds - \int_{\gamma_2} \langle \mathbf{f}, \hat{\mathbf{t}}_{\gamma_2} \rangle ds = 0$$

for any paths γ_1 and γ_2 linking $\mathbf{a}$ and $\mathbf{b}$. Hence,

$$\int_\gamma \langle \mathbf{f}, \hat{\mathbf{t}} \rangle ds = 0$$

for an arbitrary path. □

Theorem 17.4. *Let $\mathbf{f}$ be a continuous vector field on a open connected region $D \subset \mathbb{R}^n$. If the line integral*

$$\int_\gamma \langle \mathbf{f}, \hat{\mathbf{t}} \rangle ds$$

is path independent in D then $\mathbf{f}$ is conservative vector field, i.e. $\mathbf{f} = \nabla f$ for some f.

Proof. Let's fix a point $\mathbf{a} \in D$ and let $\gamma(\mathbf{a}, \mathbf{y})$ be **some path** from $\mathbf{a}$ to an arbitrary point $\mathbf{y} \in D$. We define a candidate potential function

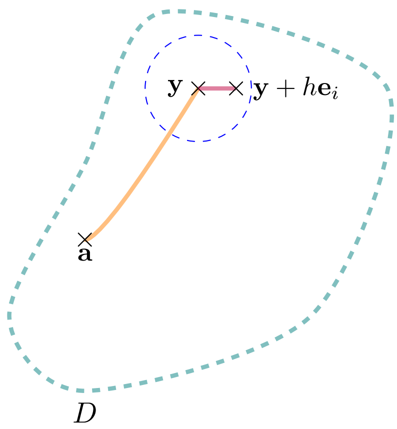
$$f(\mathbf{y}) = \int_{\gamma(\mathbf{a},\mathbf{y})} \langle \mathbf{f}, \hat{\mathbf{t}} \rangle ds.$$

Note that our choice of $\gamma(\mathbf{a}, \mathbf{y})$ does not matter since the line integral is path independent.

Recall that the partial derivative with respect to x_i can be thought of as a directional derivative in the $\mathbf{e}_i$ direction. So, we want to consider

$$\partial_{x_i} f(\mathbf{y}) = (D_{\mathbf{e}_i} f)(\mathbf{y}) = \lim_{h \to 0} \frac{f(\mathbf{y} + h\mathbf{e}_i) - f(\mathbf{y})}{h} = \lim_{h \to 0} \frac{1}{h} \Big[\int_{\gamma(\mathbf{a},\mathbf{y}+h\mathbf{e}_i)} \langle \mathbf{f}, \hat{\mathbf{t}} \rangle ds - \int_{\gamma(\mathbf{a},\mathbf{y})} \langle \mathbf{f}, \hat{\mathbf{t}} \rangle ds \Big]$$

Now why is looking at this limit ok? Well D is open, this means that for any point in $\mathbf{x} \in D$ there is a small open ball of radius ϵ centred at $\mathbf{x}$. So for $h < \epsilon$, $\mathbf{y} + h\mathbf{e}_i \in D$ and, therefore, we are free to look at this limit. The picture you should have in your head is



So, the integral in our derivative above is simple along the purple straight line in the diagram. Let's call this line L_i and it has the equation

$$\mathbf{r}_h(\lambda) = (1-\lambda)\mathbf{y} + \lambda(\mathbf{y} + h\mathbf{e}_i) = \mathbf{y} + \lambda h\mathbf{e}_i, \quad \lambda \in [0,1], \implies \mathbf{r}_h'(\lambda) = h\mathbf{e}_i$$

So,

$$\partial_{x_i} f(\mathbf{y}) = \lim_{h\to 0} \frac{1}{h}\int_{L_i} \langle \mathbf{f}, \hat{\mathbf{t}}\rangle ds = \lim_{h\to 0}\frac{1}{h}\int_0^1 \langle \mathbf{f}(\mathbf{r}_h(\lambda)), h\mathbf{e}_i\rangle d\lambda = \lim_{h\to 0}\int_0^1 f_i(\mathbf{r}_h(\lambda))d\lambda.$$

Now, for h small, $\mathbf{r}_h(\lambda) \approx \mathbf{y}$, so

$$\partial_{x_i} f(\mathbf{y}) = \lim_{h\to 0} f_i(\mathbf{y})\int_0^1 d\lambda = f_i(\mathbf{y}).$$

So, $\mathbf{f} = \nabla f$. □

Theorem 17.5. *Let $\mathbf{f} = f(x,y)\mathbf{i} + g(x,y)\mathbf{j}$ be vector field on a region $D \subseteq \mathbb{R}^2$ with continuous first partial derivatives on D.*

1. *If $\mathbf{f}$ is conservative*

$$\partial_y f = \partial_x g$$

2. *If D is open and simply connected and*

$$\partial_y f = \partial_x g$$

then $\mathbf{f}$ is conservative.

Proof. Proof of 1: If $\mathbf{f}$ is conservative then $\mathbf{f} = \nabla h$. Therefore,

$$f = \partial_x h, \qquad g = \partial_y h.$$

Since, f and g have continuous partial derivatives, h must have continuous second partial derivatives, which means they must commute:

$$\partial_x g = \partial_x\partial_y h = \partial_y\partial_x h = \partial_y f.$$

Proof of 2: follows from Greens' theorem and theorem 17.3.

□

17.2.3 Green's Theorem

Green's theorem states that the integral of the tangential component of a vector field to a curve γ is related to how much the vector field rotates on the interior of the region. We will soon come to its generalisation in $3D$ (in fact there is a generalisation in n-dimensions) where it is called Stokes' theorem.

Definition 17.10. *We will say a simple closed curve is positively orientated if it is traversed* **counter-clockwise**.

Theorem 17.6. *Let γ be a positively oriented, piecewise-C^1 , simple closed curve in $\mathbb{R}^2$ with unit tangent vector $\mathbf{t}$ and let D be the region bounded by γ. If $\mathbf{f} = (f, g)$ is a vector field with continuous first partial derivatives on an open region containing D then*

$$\int_\gamma \langle \mathbf{f}, \hat{\mathbf{t}} \rangle ds = \int_D \mathrm{curl} \mathbf{f} \mathrm{dvol}_2$$

where the path integration is calculated using the positive orientation.

Proof (Normal Region). Suppose that γ can be described by the parametric equation $\mathbf{r}(t)$. We note that

$$\int_\gamma \langle \mathbf{f}, \hat{\mathbf{t}} \rangle ds = \int_\gamma \left[f\frac{dx}{dt} + g\frac{dy}{dt} \right] dt = \int_\gamma f dx + g dy.$$

Suppose our region can be expressed as

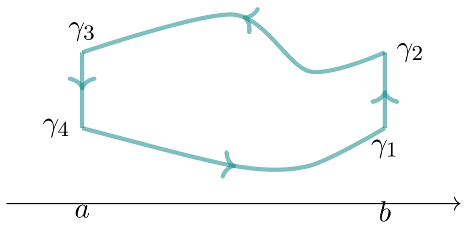
$$D = \{\mathbf{x} \in \mathbb{R}^2 : a \le x \le b, g_1(x) \le y \le g_2(x)\}.$$

A differentiable function (continuous partials imply differentiability) is continuous so we can use Fubini.

$$\int_D \partial_y f \mathrm{dvol} = \int_a^b \int_{g_1(x)}^{g_2(x)} \partial_y f dy dx = \int_a^b \Big[f(x, g_2(x)) - f(x, g_1(x)) \Big] dx$$

from the fundamental theorem of calculus.

We now write $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ as shown below:



Therefore,we can write the integral as

$$\begin{aligned} \int_\gamma f dx &= \int_{\gamma_1} f dx + \int_{\gamma_2} f dx + \int_{\gamma_3} f dx + \int_{\gamma_4} f dx \\ &= \int_a^b f(x, g_1(x)) dx + \int_b^a f(x, g_2(x)) dx \end{aligned}$$

since the integrals on γ_2 and γ_4 are zero. Therefore,

$$\int_\gamma f dx = \int_a^b \Big[f(x, g_1(x)) - f(x, g_2(x))\Big] dx = -\int_D \partial_y f \mathrm{dvol}.$$

Analogously one can show

$$\int_\gamma g dx = \int_D \partial_x g \mathrm{dvol}.$$

So that

$$\int_\gamma g dy + \int_\gamma f dx = \int_D \Big(\partial_x g - \partial_y f\Big) \mathrm{dvol} = \int_D \mathrm{curl}\mathbf{f} \mathrm{dvol}.$$

□

Remark 17.5. *Clearly the proof of this generalises to a finite union of normal domains.*

Green's theorem in $2D$ gives rise to the extremely useful divergence theorem, which tells us that a vector fields outgoingness or ingoingness in a region can be related to the flux of the vector field through the boundary of the region. For this, we need to fix a direction for our normal.

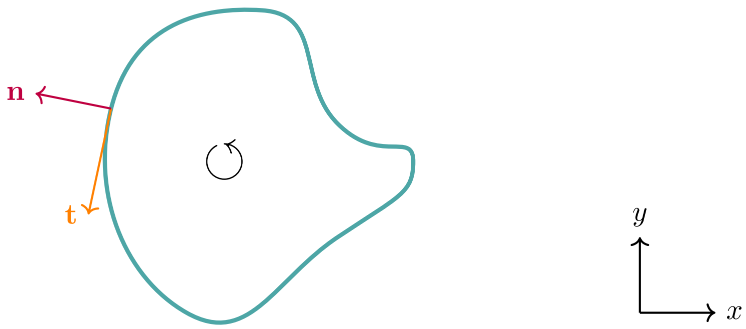
Definition 17.11. *Let γ positively oriented simple closed C^1-curve. Let $\mathbf{r}(t)$ be a parametric equation for the curve. Extend the tangent vector $\mathbf{r}'$ to $\mathbb{R}^3$ by*

$$\mathbf{r}' = (r_1', r_2', 0).$$

We define the outward normal as

$$\mathbf{n} = \mathbf{r}' \times \mathbf{k} = r_2' \mathbf{i} - r_1' \mathbf{j}.$$

You can see that this definition works out using the right-hand rule on the following picture (z, and therefore $\mathbf{k}$, is out of the page):



Corollary 17.1. *($2D$ Divergence Theorem) Let γ be a positively oriented, piecewise-C^1, simple closed curve in $\mathbb{R}^2$ with unit outward normal vector $\hat{\mathbf{n}}$ and let D be the region bounded by γ. If $\mathbf{f} = (f, g)$ is a vector field with continuous first partial derivatives on an open region containing D then*

$$\int_\gamma \langle \mathbf{f}, \hat{\mathbf{n}} \rangle ds = \int_D \mathrm{div}\mathbf{f} \mathrm{dvol}_2$$

Proof. We have

$$\begin{aligned}
\int_\gamma \langle \mathbf{f}, \hat{\mathbf{n}} \rangle ds &= \int_a^b \langle \mathbf{f}, \mathbf{n} \rangle dt \\
&= \int_a^b \Big[f(x(t), y(t))y'(t) - g(x(t), y(r))x'(t)\Big] dt \\
&= \int_D \mathrm{curl}\mathbf{g} \mathrm{dvol}_2
\end{aligned}$$

by Green's theorem for $\mathbf{g} = (f, -g)$ but

$$\mathrm{curl}\mathbf{g} = \mathrm{div}\mathbf{f}.$$

□

Example 17.11. *Let's use Green's theorem to compute*

$$\int_{\mathbb{S}^1_R} \langle \mathbf{f}, \hat{\mathbf{t}} \rangle ds,$$

for $\mathbf{f} = y\mathbf{i} - x\mathbf{i}$.

Our curve γ *is a circle of radius* R*, so its interior is a disk of radius* R*:*

$$D_R = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq R\}.$$

We can compute that

$$\mathrm{curl}\mathbf{f} = -2,$$

so, by Green's theorem,

$$\int_{\mathbb{S}^1_R} \langle \mathbf{f}, \hat{\mathbf{t}} \rangle ds = \int_{D_R} (-2)\mathrm{dvol}_2 = -2(\pi R^2).$$

Let's check that everything works out and compute the line integral explicitly. Let's use the parameterisation

$$\mathbf{r}(t) = (x(t), y(t)) = R(\cos t, \sin t) \implies \hat{\mathbf{t}} = \frac{1}{R}(-y, x).$$

So,

$$\int_{\mathbb{S}^1_R} \langle \mathbf{f}, \hat{\mathbf{t}} \rangle ds = \int_{\mathbb{S}^1_R} \frac{1}{R}\Big[-x^2 - y^2 \Big] \rangle ds = -\int_{\mathbb{S}^1_R} R ds = -2\pi R^2.$$

Example 17.12. *Let's use the divergence theorem to compute*

$$\int_{\mathbb{S}^1_R} \langle \mathbf{f}, \hat{\mathbf{n}} \rangle ds$$

for $\mathbf{f} = x\mathbf{i} + y\mathbf{j}$.

We can compute that

$$\mathrm{div}\mathbf{f} = 2,$$

so, by the divergence theorem,

$$\int_{\mathbb{S}^1_R} \langle \mathbf{f}, \hat{\mathbf{n}} \rangle ds = \int_{D_R} (2)\mathrm{dvol}_2 = 2\pi R^2.$$

Example 17.13. *The work done,* W*, by a force* $\mathbf{f}$ *moving a particle along a curve* γ *is given by*

$$W = \int_\gamma \langle \mathbf{f}, \hat{\mathbf{t}} \rangle ds.$$

Let's use Green's theorem to compute the work done by the force

$$\mathbf{f}(x, y) = x(x + y)\mathbf{i} + xy^2\mathbf{j}$$

in moving a particle from $(0,0)$ to $(0,1)$ to $(0,1)$ and then back to the origin along straight lines.

We note that the curl of $\mathbf{f}$ is

$$\text{curl}\mathbf{f} = y^2 - x$$

We can write our region as the set

$$D = \{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq 1-x\}.$$

So by Green's theorem,

$$W = \int_D \left(y^2 - x\right)\text{dvol}_2 = \int_0^1 \int_0^{1-x} (y^2 - x)dydx = -\frac{1}{12}x(x^3 - 8x^2 + 12x - 4)\Big|_0^1 = -\frac{1}{12}.$$

Remark 17.6. *The divergence theorem is* **extremely useful**. *It can give rise to conservation laws! For example suppose $\Phi(t,x)$ satisfies the wave equation*

$$\Box\Phi = 0.$$

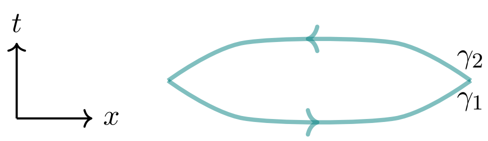
One can compute that

$$\mathbf{f} = \left((\partial_t\Phi)^2 + (\partial_x\Phi)^2, 2\partial_t\Phi\partial_x\Phi\right)$$

is divergence free. This means that

$$\int_\gamma \langle \mathbf{f}, \hat{\mathbf{t}}\rangle ds = 0!$$

So if one has a curve $\gamma = \gamma_1 \cup \gamma_2$ such the following



This gives the convervation law (up to some technicalities with signs)

$$\int_{\gamma_1} \langle \mathbf{f}, \hat{\mathbf{t}}\rangle ds = \int_{\gamma_2} \langle \mathbf{f}, \hat{\mathbf{t}}\rangle ds.$$

For appropriately chosen γ_1 and γ_2 (and some extra theory which we do not develop here) one can use this to construct the conservation law

$$\int_{-\infty}^{\infty} \left[(\partial_t\Phi)^2 + (\partial_x\Phi)^2\right] dx\Big|_{t_0} = \int_{-\infty}^{\infty} \left[(\partial_t\Phi)^2 + (\partial_x\Phi)^2\right] dx\Big|_{t_1}$$

for $t_1 \geq t_0$. One can interprate these integrals as energies for solutions to the wave equation. So the energy associated to a solution is conserved in time t.

One can show that this implies that the solutions to the wave equation on $\mathbb{R}^2$ are bounded, i.e. $|\Phi(t,x)| \leq C$ for a finite constant C and for all $t, x \in \mathbb{R}^2$. The use of the divergence theorem and conservation laws arising from it are important tools in the field of PDE; they allow one to infer many properties of general solutions without actually being able to write down a general formula!

17.3 Surfaces in $\mathbb{R}^3$

So far we have talked about the graph of a function f of two variables (x, y):

$$\text{Graph}(f) = \{(x, y, z) \in \mathbb{R}^3 : z = f(x, y)\},$$

which we viewed as a surface S in $\mathbb{R}^3$.

We also discussed, the level surface S of a function h of three variables (x, y, z):

$$\{(x, y, z) \in \mathbb{R}^3 : h(x, y, z) = \text{constant}\}.$$

Additionally, we introduced curves in space as vector-valued functions of one variable. We can generalise this to **parameteric surfaces** in $\mathbb{R}^3$. These are given by an vector-valued function depend taking two parameters (u, v), i.e.

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}.$$

Remark 17.7. *Note that given a surface specified by a graph $z = f(x, y)$, it has a parametric representation*

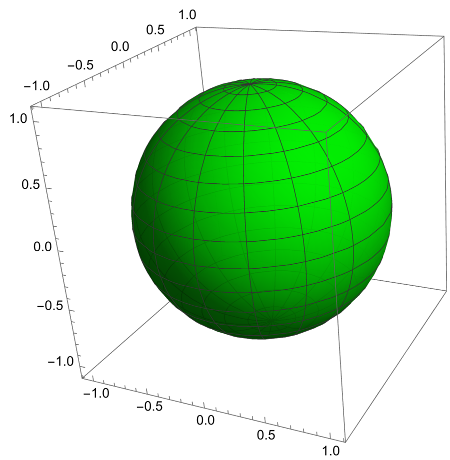
$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + f(u, v)\mathbf{k}$$

Example 17.14. *One can use the spherical coordinate relations*

$$x = r\cos\varphi\sin\theta, \quad y = r\sin\varphi\sin\theta, \quad z = r\cos\theta$$

to write down a parametric equation of a sphere of radius R.

$$r(\theta, \varphi) = R\Big[\cos\varphi\sin\theta\mathbf{i} + \sin\varphi\sin\theta\mathbf{j} + \cos\theta\mathbf{k}\Big]$$

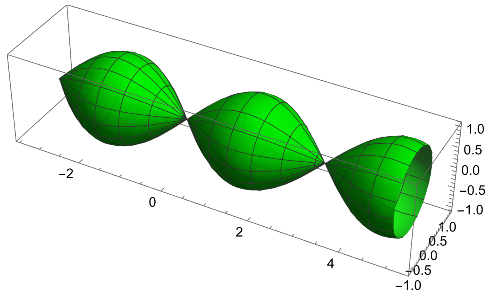


Example 17.15. *One can view surfaces of revolution as surfaces with a parametric representation. For example, we can consider the surface of revolution arising from rotating the curve $\sin(x)$ around the x-axis. The parameteric equations are*

$$x = x, \quad y = \sin x\cos\theta, \quad z = \sin x\sin\theta$$

or

$$r(\theta, \varphi) = x\mathbf{i} + \cos\theta\sin x\mathbf{j} + \sin\theta\sin x\mathbf{k}$$



We can generalise our notions of tangent planes and normals to such surfaces. In particular, suppose we want to construct the tangent plane at (u_0, v_0). If we hold $v = v_0$ fixed then we have a curve that lies in our surface. We can, therefore, construct its tangent line

$$\mathbf{x}_u(\lambda) = (u_0, v_0) + \lambda \partial_u \mathbf{r}(u_0, v_0)$$

Similarly, if we hold $u = u_0$ fixed then we have another curve that lies in our surface. We can, therefore, construct its tangent line

$$\mathbf{x}_v(\lambda) = (u_0, v_0) + \lambda \partial_v \mathbf{r}(u_0, v_0)$$

Assuming our surface behaves nicely ($\mathbf{r}$ has continous first partial derivatives and $\partial_u \mathbf{r} \times \partial_v \mathbf{r} \neq \mathbf{0}$), the plane that contains these two lines is then our tangent plane. This means the tangent plane contains the two vectors

$$\partial_u \mathbf{r} = \partial_u x(u, v)\mathbf{i} + \partial_u y(u, v)\mathbf{j} + \partial_u z(u, v)\mathbf{k},$$
$$\partial_v \mathbf{r} = \partial_v x(u, v)\mathbf{i} + \partial_v y(u, v)\mathbf{j} + \partial_v z(u, v)\mathbf{k}.$$

Therefore, a normal to the tangent plane is

$$\mathbf{n} = \partial_u \mathbf{r} \times \partial_v \mathbf{r}.$$

If we let u and v vary we get a normal in each tangent plane, which we call the **normal to our surface** (this is a vector field in $\mathbb{R}^3$). If $\mathbf{r}$ has continous first partial derivatives and $\partial_u \mathbf{r} \times \partial_v \mathbf{r} \neq \mathbf{0}$ then we will call our surface C^1.

17.3.1 Orientation

The normal vector of a surface can be thought of as being associated to a notion of 'direction' for a surface. Some surfaces have a well defined notion of direction, others do not. As we shall see this is important for defining surface integrals of vector fields. This notion of direction is called an orientation and those surfaces with a well-defined orientation are called oriented.

Definition 17.12. *An orientation for a surface S is a continuous choice of normal $\mathbf{n}$ for the surface, if such a choice exists we say S is orientable. A surface S with an orientation $\mathbf{n}$ is called an oriented surface.*

If the surface is orientable then there are two possible orientations (a surface has two sides) and one must make a choice! For a C^1 orientable surface S given by the parametric equation $\mathbf{r}(u, v)$ the two choices are

$$\mathbf{n} = \partial_u \mathbf{r} \times \partial_v \mathbf{r}, \qquad \mathbf{n} = -\partial_u \mathbf{r} \times \partial_v \mathbf{r}.$$

If we is have a surface given by the graph of a function $z = f(x, y)$ we can think of this as a specific 0 level surface of

$$g(x, y, z) = z - f(x, y),$$

i.e. out surface S is

$$\{\mathbf{x} \in \mathbb{R}^3 : g(x, y, z) = 0\}.$$

Recall that the gradient of g is orthogonal to the surface. So, we have a normal

$$\mathbf{n} = \nabla_3 g = (1, \nabla_2 f) = (1, -\partial_x f, -\partial_y f).$$

Example 17.16. *A sphere of radius R is an example of an orientable surface. Recall the parametric equations of a sphere are*

$$\mathbf{r}(\theta,\varphi) = R\cos\varphi\sin\theta\mathbf{i} + R\sin\varphi\sin\theta\mathbf{j} + R\cos\theta\mathbf{k}$$

with $D = \{(\theta,\varphi) : \theta \in (0,\pi), \varphi \in [0,2\pi)\}$. The normal is then

$$\mathbf{n} = \partial_\theta \mathbf{r} \times \partial_\theta \mathbf{r}.$$

We have

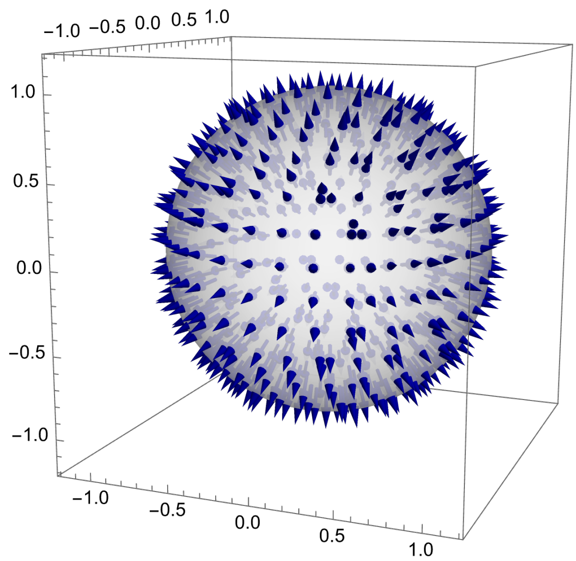
$$\partial_\theta \mathbf{r} = R\cos\varphi\cos\theta\mathbf{i} + R\sin\varphi\cos\theta\mathbf{j} - R\sin\theta\mathbf{k},$$
$$\partial_\varphi \mathbf{r} = -R\sin\varphi\sin\theta\mathbf{i} + R\cos\varphi\sin\theta\mathbf{j}.$$

So,

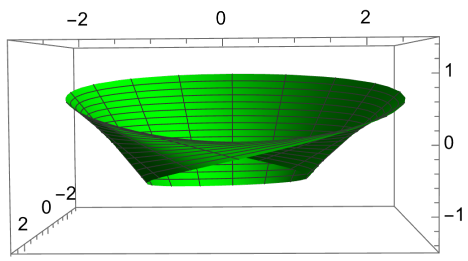
$$\mathbf{n} = R^2\sin^2\theta\cos\varphi\mathbf{i} + R^2\sin^2\theta\sin\varphi\mathbf{j} + R^2\sin\theta\cos\theta\mathbf{k}.$$

We can view S_R^2 as the 0 level surface of

$$g(x,y,z) = x^2 + y^2 + z^2 - 1 \implies \mathbf{n}_\pm = (2x, 2y, \pm 2\sqrt{1 - x^2 - y^2})$$



Example 17.17. *An example of a non-orientable surface is the Möbius strip. Take a strip of paper, make one twist and join the ends. This is plotted below*



The following set of equation describe it parametrically

$$\mathbf{r}(u,v) = \left[2\cos v + u\cos\left(\frac{v}{2}\right)\right]\mathbf{i} + \left[2\sin v + u\cos\left(\frac{v}{2}\right)\right]\mathbf{j} + u\sin\left(\frac{v}{2}\right)\mathbf{k},$$

for $u \in [-\frac{1}{2}, \frac{1}{2}]$ and $v \in [0, 2\pi)$. Let's work out the normal

$$\partial_u \mathbf{r} = \cos\left(\frac{v}{2}\right)(\mathbf{i}+\mathbf{j}) + \sin\left(\frac{v}{2}\right)\mathbf{k}$$
$$\partial_v \mathbf{r} = -\left[2\sin v + \frac{u}{2}\sin\left(\frac{v}{2}\right)\right]\mathbf{i} + \left[2\cos v - \frac{u}{2}\sin\left(\frac{v}{2}\right)\right]\mathbf{j} + \frac{u}{2}\cos\left(\frac{v}{2}\right)\mathbf{k}.$$

$$\mathbf{n} = \partial_u\mathbf{r} \times \partial_v\mathbf{r} = \left[\frac{u}{2} + \sin\left(\frac{v}{2}\right) - \sin\left(\frac{3v}{2}\right)\right]\mathbf{i} + \left[-\frac{u}{2} - \cos\left(\frac{v}{2}\right) + \cos\left(\frac{3v}{2}\right)\right]\mathbf{j} + 2\cos\left(\frac{v}{2}\right)(\cos v + \sin v)\mathbf{k}.$$

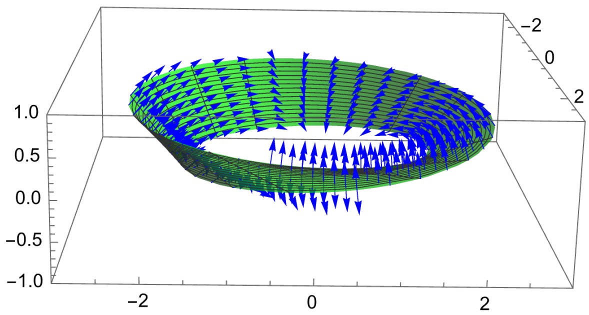
So at $v = 0$ we have

$$\mathbf{n} = \frac{u}{2}\mathbf{i} - \frac{u}{2}\mathbf{j} + 2\mathbf{k},$$

and at $v = 2\pi$ we have

$$\mathbf{n} = \frac{u}{2}\mathbf{i} - \frac{u}{2}\mathbf{j} - 2\mathbf{k}.$$

So, whilst $(0,0)$ and $(0,2\pi)$ describe the same point on the Möbius strip we have $\mathbf{n}(2\pi) = -\mathbf{n}(0)$ which means out normal on the surface is discontinuous, as is shown in the following figure



Effectively what is happening is that the Möbius strip is one-sided, if you traverse around the strip you can reach all points. This is unlike a sphere where you cannot reach the inside from the outside. There is a result say that any surface is non-orientable if and only if it has a Möbius strip as a subset.

17.3.2 Surface Integrals

Suppose you want to **integrate a function** f of three variables over parametric surface S with parametric equation

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k},$$

where $(u,v) \in D \subseteq \mathbb{R}^2$. We define the **surface integral** of f over S as

$$\int_S f \mathrm{dvol}_S = \int_D f(\mathbf{r}(u,v))\|\partial_u\mathbf{r} \times \partial_v\mathbf{r}\| \mathrm{dvol}_{\mathbb{R}^2}$$

where dvol_S should be interpreted as the infinitesmal volume/area element of the surface and $\mathrm{dvol}_{\mathbb{R}^2} = dudv$ is the usual volume element on $\mathbb{R}^2$.

How does this formula arise? In a very similar manner to the change of variables formula. Suppose for simplicity our domain D for (u,v) is a rectangle. We we take D and cover it with small rectangles of side Δu by Δv. We consider the image of these small rectangles under $\mathbf{r}$ which for small Δv and Δu will be a warped shape on S. If S is a C^1-surface then the linear approximation is good and, therefore, one can approximate this shape with a parallelogram (which would be the shape produced by a linear $\mathbf{r}$). The linear approximation gives around a point (u_0, v_0)

$$\mathbf{r}(u,v) = \mathbf{r}(u_0,v_0) + \partial_u\mathbf{r}(u_0,v_0)(u-u_0) + \partial_v\mathbf{r}(u_0,v_0)(v-v_0).$$

Therefore, the vectors of which define the parallelograms sides are

$$\mathbf{a} = \mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0) = \partial_u \mathbf{r}(u_0, v_0)\Delta u,$$
$$\mathbf{b} = \mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0) = \partial_v \mathbf{r}(u_0, v_0)\Delta v.$$

Hence,

$$\Delta \mathrm{vol}_S = ||\mathbf{a} \times \mathbf{b}|| = ||\partial_u \mathbf{r} \times \partial_v \mathbf{r}||\Delta u \Delta v.$$

Therefore, if we construct a Riemann sum and then pass to the limit to define integration we pick find

$$\mathrm{dvol}_S = ||\partial_u \mathbf{r} \times \partial_v \mathbf{r}|| du dv.$$

Example 17.18. *Let's compute the surface integral of y^2 over the surface*

$$S = \{\mathbf{x} \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1, z \geq \sqrt{x^2 + y^2}\}$$

Let's parameterise our sphere with spherical coordinates:

$$x = \cos\varphi \sin\theta, \quad y = \sin\varphi \sin\theta, \quad z = \cos\theta$$

where the parameter ranges are restricted to

$$\theta \in [0, \frac{\pi}{4}], \quad \varphi \in [0, 2\pi]$$

So, we have

$$\partial_\theta \mathbf{r} = \cos\varphi \cos\theta \mathbf{i} + \sin\varphi \cos\theta \mathbf{j} - \sin\theta \mathbf{k},$$
$$\partial_\varphi \mathbf{r} = -\sin\varphi \sin\theta \mathbf{i} + \cos\varphi \sin\theta \mathbf{j},$$
$$\partial_\theta \mathbf{r} \times \partial_\varphi \mathbf{r} = \cos\varphi \sin^2\theta \mathbf{i} + \sin^2\theta \sin\varphi \mathbf{j} + \cos\theta \sin\theta \mathbf{k}.$$

So, $\|\partial_\theta \mathbf{r} \times \partial_\varphi \mathbf{r}\| = \sin\theta$

$$\int_S f \mathrm{dvol}_S = \int_0^{\frac{\pi}{4}} \int_0^{2\pi} \sin^2\varphi \sin^3\theta d\theta d\varphi = \frac{1}{12}(8 - 5\sqrt{2})\pi$$

where one uses trignometric identities or expands in exponentials.

Let's generalise to **integrating a vector field** $\mathbf{f}$ on $\mathbb{R}^3$ over S. We define the **surface integral** of $\mathbf{f}$ over S as

$$\int_S \langle \mathbf{f}, \hat{\mathbf{n}} \rangle \mathrm{dvol}_S$$

where $\hat{\mathbf{n}}$ is the **unit normal**:

$$\hat{\mathbf{n}} = \frac{\partial_u \mathbf{r} \times \partial_v \mathbf{r}}{||\partial_u \mathbf{r} \times \partial_v \mathbf{r}||}.$$

Therefore, parsing the notation gives,

$$\int_S \langle \mathbf{f}, \hat{\mathbf{n}} \rangle \mathrm{dvol}_S = \int_D \langle \mathbf{f}, \partial_u \mathbf{r} \times \partial_v \mathbf{r} \rangle du dv.$$

Example 17.19. *Let's compute the surface integral of $\mathbf{f}(\mathbf{x}) = z\mathbf{i} + y\mathbf{j} + x\mathbf{k}$ over the surface*

$$S = \{\mathbf{x}(u, v) \in \mathbb{R}^3 : \mathbf{x}(u, v) = u \cos v \mathbf{i} + u \sin v \mathbf{j} + v\mathbf{k}, (u, v) \in [0, 1] \times [0, \pi]\}$$

with the normal in the positive z-direction.

We have

$$\partial_u \mathbf{x} = \cos v \mathbf{i} + \sin v \mathbf{j}$$
$$\partial_v \mathbf{x} = -u \sin v \mathbf{i} + u \cos v \mathbf{j} + \mathbf{k}$$

So,

$$\mathbf{n} = \sin v \mathbf{i} - \cos v \mathbf{j} + u \mathbf{k}$$

which points upwards for $u > 0$. So,

$$\int_S \langle \mathbf{f}, \hat{\mathbf{n}} \rangle \mathrm{dvol}_S = \int_D \langle \mathbf{f}, \partial_u \mathbf{r} \times \partial_v \mathbf{r} \rangle du dv,$$

and we compute

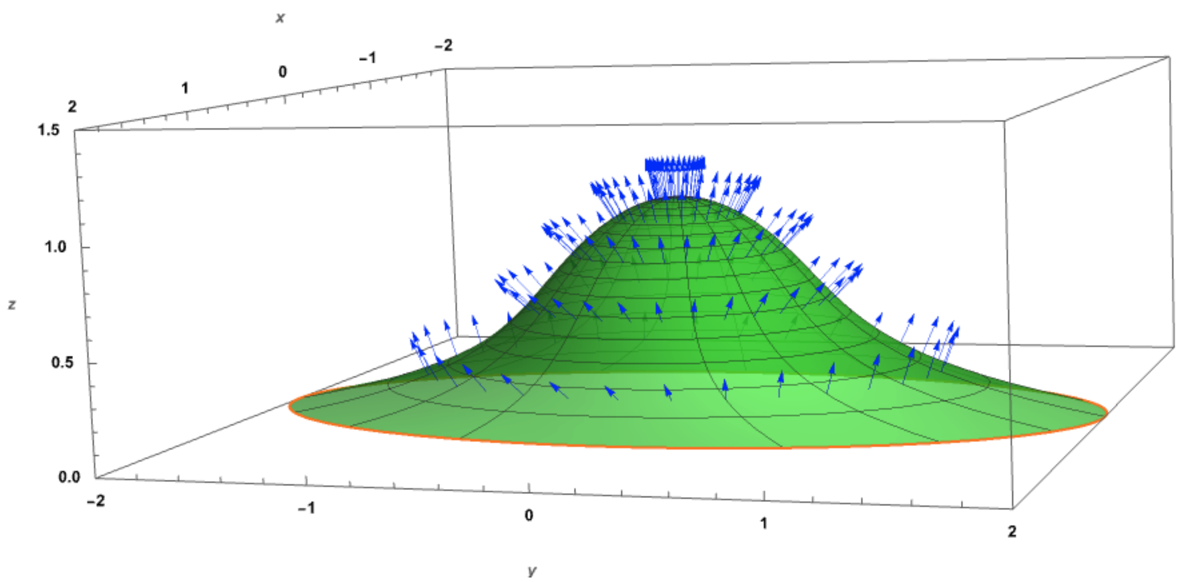
$$\langle \mathbf{f}, \mathbf{n} \rangle = z \sin v - y \cos v + xu = v \sin v - u \sin v \cos v + u^2 \cos v$$

which is continous so we can do the partial integrals:

$$\begin{aligned} \int_S \langle \mathbf{f}, \hat{\mathbf{n}} \rangle \mathrm{dvol}_S &= \int_0^\pi \int_0^1 \Big[v \sin v - u \sin v \cos v + u^2 \cos v \Big] du dv \\ &= \int_0^\pi \Big[v \sin v - \frac{1}{2} \sin v \cos v + \frac{1}{2} \cos v \Big] dv = \pi \end{aligned}$$

17.3.3 Stokes' Theorem

Stokes' Theorem is a generalisation of Green's theorem in the plane to a general surface S bounded by a curve γ sitting in $\mathbb{R}^3$ (note that a subset in $\mathbb{R}^2$ can be thought of as a surface sitting in $\mathbb{R}^3$, i.e. the xy-plane). It says that the line integral tangential component of the vector field $\mathbf{f}$ along γ is equal to the surface integral of the normal component of $\mathrm{curl}\mathbf{f}$. The picture you should have in mind is the following



Note there is some orientation fixing to be done. The rule is the following: suppose S has an orientation with a normal $\mathbf{n}$, this **induces a positive orientation** γ given by the right-hand rule, i.e. point your thumb in the direction of the normal, if you then curl your fingers inwards then this is the orientation of the curve. So in the above diagram the orientation of the curve for the line integral would be the usual counter-clockwise orientation in the xy-plane

Theorem 17.7. *Stokes' Theorem Let S be a oriented piecewise-C^1 surface that is bounded by a simple, closed, piecewise-C^1 curve γ with positive orientation. Let $\hat{\mathbf{n}}$ be the unit normal vector to the surface S and $\hat{\mathbf{t}}$ be the unit tangent vector to the curve γ. Let $\mathbf{f}$ be a vector field with continuous first partial derivative on an open region of $\mathbb{R}^3$ that contains S. Then*

$$\int_\gamma \langle \mathbf{f}, \hat{\mathbf{t}} \rangle ds = \int_S \langle \text{curl}\mathbf{f}, \hat{\mathbf{n}} \rangle \text{dvol}_S.$$

Example 17.20. *Let's verify Stokes' theorem for $\mathbf{f} = -2yz\mathbf{i} + y\mathbf{j} + 3x\mathbf{k}$ for the surface*

$$S = \{\mathbf{x} : z = 5 - x^2 - y^2, z \geq 1\}.$$

We need to parameterise S and the boundary curve γ of S with a consistent choice or orientation. Note that γ is the curve given by

$$x^2 + y^2 = 4.$$

For S let's parameterise as: $x = u\cos v$, $y = u\sin v$, $z = 5 - u^2$, i.e.

$$\mathbf{r}_S(u, v) = (u\cos v, u\sin v, 5 - u^2)$$

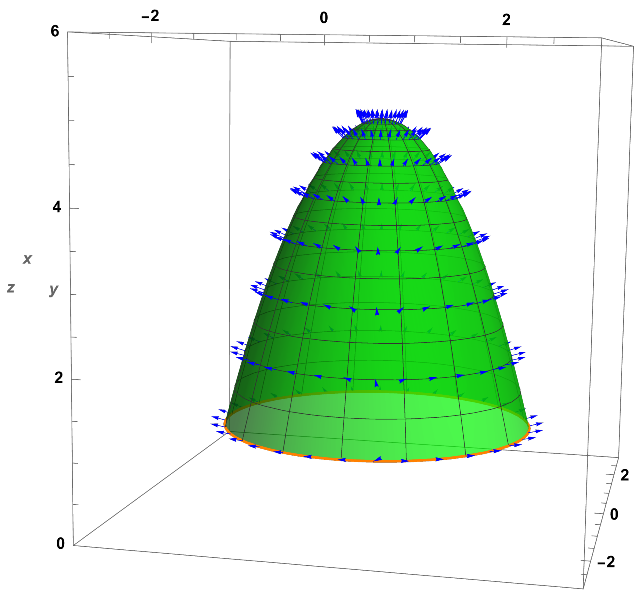
with $u \in [0, 2]$ and $v \in [0, 2\pi]$. We also need the normal to talk about orientation. Let's orientate with $\mathbf{n} = \partial_u\mathbf{r} \times \partial_v\mathbf{r}$. We can compute

$$\partial_u\mathbf{r} = \cos v\mathbf{i} + \sin v\mathbf{j} - 2u\mathbf{k}$$
$$\partial_v\mathbf{r} = -u\sin v\mathbf{i} + u\cos v\mathbf{j}$$

and therefore,

$$\mathbf{n} = 2u^2\cos v\mathbf{i} + 2u^2\sin v\mathbf{j} + u\mathbf{k}$$

Note that $\mathbf{n}$ points upwards in $\mathbf{k}$ and γ lies in the $z = 1$ plane so we want to traverse γ counterclockwise by the right-hand rule as shown in the following diagram:



This leads us to the γ parameterisation: $x = 2\cos u$ $y = 2\sin u$, $z = 1$, i.e.

$$\mathbf{r}_\gamma(u) = (2\cos u, 2\sin u, 1)$$

with $u \in [0, 2\pi]$.

Let's the surface integral. For this we need the curl,

$$\mathrm{curl}\mathbf{f} = 2z\mathbf{k} - (3+2y)\mathbf{j},$$

which if we evaluate on the surface gives

$$\mathrm{curl}\mathbf{f}(\mathbf{r}_S(u,v)) = 2(5-u^2)\mathbf{k} - (3+2u\sin v)\mathbf{j}.$$

Recall that to compute the line integral from a parameterisation we have

$$\int_S \langle \mathrm{curl}\mathbf{f}, \hat{\mathbf{n}}\rangle \mathrm{dvol}_S = \int_D \langle \mathrm{curl}\mathbf{f}(\mathbf{r}_S(u,v)), \partial_u \mathbf{r} \times \partial_v \mathbf{r}\rangle \mathrm{dvol}_{\mathbb{R}^2_{u,v}}$$

where D is the domain for the parameters (u,v). So,

$$\int_S \langle \mathrm{curl}\mathbf{f}, \hat{\mathbf{n}}\rangle \mathrm{dvol}_S = \int_D \Big[10u - 2u^3 - 6u^2\sin v - 4u^3\sin^2 v\Big] \mathrm{dvol}_{\mathbb{R}^2_{u,v}}$$

Our integrand is continuous, so we can apply Fubini to write an iterated integral

$$\int_S \langle \mathrm{curl}\mathbf{f}, \hat{\mathbf{n}}\rangle \mathrm{dvol}_S = \int_0^2 \int_0^{2\pi} \Big[10u - 4u^3 - 6u^2\sin v - 2u^3\cos(2v)\Big] dvdu = \int_0^2 \Big[20\pi u - 8\pi u^3\Big] du = 8\pi.$$

Let's now do the line integral. Recall that to compute the line integral from a parameterisation we have

$$\int_\gamma \langle \mathbf{f}, \hat{\mathbf{t}}\rangle ds = \int_0^{2\pi} \langle \mathbf{f}(\mathbf{r}(u)), \mathbf{r}'_\gamma(u)\rangle du$$

So,

$$\mathbf{f}(\mathbf{r}_\gamma(u)) = -4\sin u\mathbf{i} + 2\sin u\mathbf{j} + 6\cos u\mathbf{k}$$
$$\mathbf{r}'_\gamma(u) = (-2\sin u, 2\cos u, 0).$$

Therefore,

$$\langle \mathbf{f}(\mathbf{r}(u)), \mathbf{r}'_\gamma(u)\rangle = 8\sin^2 u + 4\cos u \sin u = 4 - 4\cos(2u) + 2\sin(2u).$$

Therefore,

$$\int_\gamma \langle \mathbf{f}, \hat{\mathbf{t}}\rangle ds = \int_0^{2\pi} \Big[4 - 4\cos(2u) + 2\sin(2u)\Big] du = 8\pi.$$

Example 17.21. *Let's compute*

$$\int_S \langle \mathrm{curl}\mathbf{f}, \hat{\mathbf{n}}\rangle \mathrm{dvol}_S$$

for $\mathbf{f} = x^2\sin z\mathbf{i} + y^2\mathbf{j} + xy\mathbf{k}$ over

$$S = \{\mathbf{x} : z = 1 - x^2 - y^2, z \geq 0\}.$$

We have that the boundary curve is the unit circle in the xy-plane. So, from Stokes' we have that

$$\int_S \langle \mathrm{curl}\mathbf{f}, \hat{\mathbf{n}}\rangle \mathrm{dvol}_S = \int_\gamma \langle \mathbf{f}, \hat{\mathbf{t}}\rangle ds.$$

Let's orientated the surface upwards so that the curves induced orientation is anti-clockwise in the xy-plane. Moreover, let's parameterise the curve by

$$x = \cos u, \quad y = \sin u, \quad z = 0, \qquad u \in [0, 2\pi)$$

Therefore,

$$\mathbf{r}'(u) = (-\sin u, \cos u, 0)$$

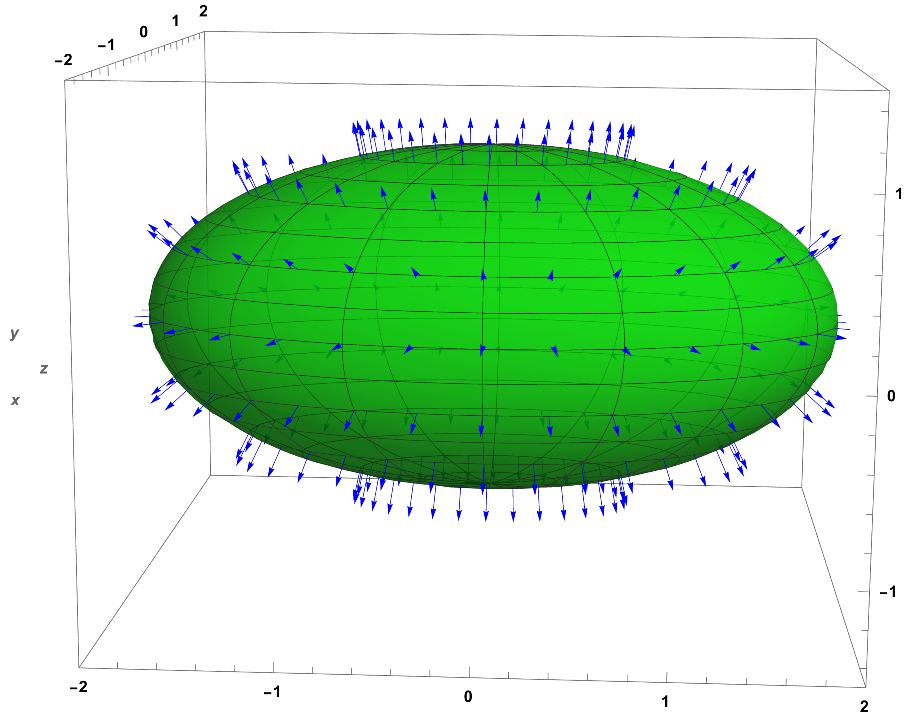
and

$$\langle \mathbf{f}, \mathbf{r}'\rangle = \sin^2 u \cos u = \frac{d}{du}\frac{1}{3}\sin^3 u.$$

So,

$$\int_S \langle \mathrm{curl}\mathbf{f}, \hat{\mathbf{n}}\rangle \mathrm{dvol}_S = \int_0^{2\pi} \frac{d}{du}\frac{1}{3}\sin^3 u du = 0$$

17.3.4 The Divergence Theorem



The generalisation of the divergence theorem to $3D$ tells us that the flux (surface integal of the vector field) of the vector field across the boundary of a region is equal to the integral of the divergence of the vector field. Again, there is orientation issues, we take the outward unit normal.

Theorem 17.8. *Let D be a finite union of normal domains and let S be the boundary surface of D, oriented with the outward unit normal $\hat{\mathbf{n}}$. Suppose $\mathbf{f}$ is a vector field which is continuous on an open region containing D. Then*

$$\int_S \langle \mathbf{f}, \hat{\mathbf{n}} \rangle \mathrm{dvol}_S = \int_D \mathrm{div}\mathbf{f} \mathrm{dvol}_{\mathbb{R}^3}$$

Example 17.22. *Lets compute the surface integal of*

$$\mathbf{f}(x, y, z) = 3xy^2\mathbf{i} + xe^z\mathbf{j} + z^3\mathbf{k}.$$

over S which is the boundary of $D = \{\mathbf{x} : y^2 + z^2 \leq 1, -1 \leq x \leq 2\}$.

We can use the divergence theorem and integrate in $\mathrm{div}\mathbf{f}$ instead. We compute

$$\mathrm{div}\mathbf{f} = 3y^2 + 3z^2$$

So, the divergence theorem gives

$$\int_S \langle \mathbf{f}, \hat{\mathbf{n}} \rangle \mathrm{dvol}_S = 3 \int_D (y^2 + z^2) \mathrm{dvol}_{\mathbb{R}^3}.$$

The integrand is continous so we can compute partial integrals. We can also use cylindricals:

$$y = r\cos\theta, \quad z = r\sin\theta, x = x$$

$$\int_D \mathrm{div}\mathbf{f} \mathrm{dvol}_3 = 3\int_{-1}^{2}\int_0^{2\pi}\int_0^1 r^2 r dr d\theta dx = \frac{9\pi}{2}.$$

Appendices

A Complex Numbers

What x solves the equation

$$x^2 = -1? \tag{656}$$

Clearly, no real number can satisfy this equation. However, we could define a new 'number' $i = \sqrt{-1}$ to be a solution to this equation. This is how imaginary and complex numbers arise.

A.1 Introduction: Definition and Operations

Definition A.1. *The complex numbers, denoted $\mathbb{C}$, is the set or collection of all numbers of the form*

$$z = a + bi \tag{657}$$

where $a, b \in \mathbb{R}$ and i is a specific element, called the imaginary unit, which satisfies $i^2 = 1$.

One can introduce some terminology to discuss complex numbers:

- The **real part** of $z = a + bi$ is a, this is sometimes written $\text{Re}(z) = a$.
- The **imaginary part** of $z = a + bi$ is b, this is sometimes written $\text{Im}(z) = b$.
- The **complex conjugate** of a complex number $z = a + bi$, denoted $\bar{z}$, is given by,

$$\bar{z} = a - bi. \tag{658}$$

- The **absolute value or modulus** of a complex number $z = a + bi$ is denoted $|z|$ and is given by

$$|z| = \sqrt{a^2 + b^2}. \tag{659}$$

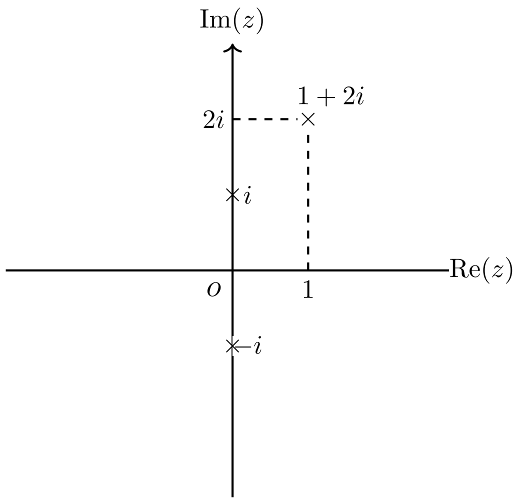
- Two complex numbers $z_1, z_2 \in \mathbb{C}$ are equal if their real parts are equal and their imaginary parts are equal,

$$\text{Re}(z_1) = \text{Re}(z_2), \qquad \text{Im}(z_1) = \text{Im}(z_2). \tag{660}$$

We can identify $\mathbb{C}$ with $\mathbb{R}^2$ via

$$z = a + bi \mapsto (a, b) \in \mathbb{R}^2, \qquad (a, b) \mapsto a + bi \in \mathbb{C}, \tag{661}$$

which means that we can draw $\mathbb{C}$ as we drew $\mathbb{R}^2$. We associate the x-axis with the real part of the complex numbers and the y-axis with the imaginary part of the complex numbers. In this context the plane is called the Argand or complex plane and the x-axis is called the real axis and the y-axis is called the imaginary axis. This is plotted below:



Note that the distance of $z \in \mathbb{C}$ from the origin is given by the absolute value of z, i.e. $|z| = \sqrt{a^2 + b^2}$.

One can define addition and subtraction of imaginary numbers. Suppose

$$z_1 = a + bi \qquad z_2 = c + di. \tag{662}$$

Then,

$$z_1 + z_2 = (a + c) + (b + d)i, \qquad z_1 - z_2 = (a - c) + (b - d)i. \tag{663}$$

Further multiplication is defined with the usual commutative and distributive laws holding:

$$z_1 z_2 = (a + bi)(c + di) = (ac - bd) + (ad + cb)i \tag{664}$$

where we've used $i^2 = -1$.

Complex conjugates satisfy some very nice relations:

Proposition A.1. *For $z, w \in \mathbb{C}$ one has*

$$\overline{z + w} = \bar{z} + \bar{w}, \qquad \overline{zw} = \bar{z}\bar{w}, \qquad \overline{z^n} = \bar{z}^n, \qquad z\bar{z} = |z|^2 \tag{665}$$

Proof. Problem Sheet 10. □

Division of two complex numbers is slightly more tricky. Suppose one has the quotient $\frac{z}{w}$ with $z = a + bi$ and $w = c + di$. One can use the complex conjugate of w and the very useful trick of multiplying by 1 to simplify a quotient:

$$\frac{z}{w} = \frac{z}{w}\frac{\overline{w}}{\overline{w}} = \frac{z\overline{w}}{|w|^2} = \frac{(a + bi)(c - di)}{|w|^2} = \frac{ac + bd}{|w|^2} + \frac{bc - ad}{|w|^2} i \tag{666}$$

A.2 Polar Form, DeMoivre's Theorem, Roots

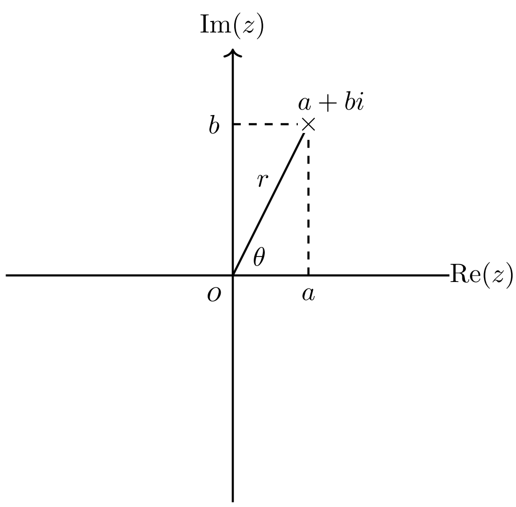
As we've discussed above $\mathbb{C}$ can be identified with the plane $\mathbb{R}^2$. If you recall section 1, we discussed polar coordinates (r, θ). Where we had

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{667}$$

We can simply translate now to write any complex number z as

$$z = a + bi = r(\cos\theta + i\sin\theta). \tag{668}$$

This is the polar form of a complex number. The following picture should help with visualisation:



From our discussion on polar coordinates in section 1, we have that

$$r = |z| = \sqrt{a^2 + b^2}, \qquad \tan\theta = \frac{b}{a}. \tag{669}$$

One often calls the angle θ the **argument or phase** of z, denoted $\theta = \arg(z)$ which is usually restricted to $\theta \in [0, 2\pi)$ (or often $\theta \in (-\pi, \pi]$).

Example A.1. *Write the following complex numbers in polar form:*

$$z_1 = 1 + i, \qquad z_2 = 1 - 7i, \qquad z_3 = \sqrt{2} - 5i. \tag{670}$$

Let's find the absolute value $|z_1| = \sqrt{2}$, $|z_2| = 5\sqrt{2}$ *and* $|z_3| = 3\sqrt{3}$. *Now*

$$1 + i = \sqrt{2}(\cos\theta + i\sin\theta) \implies \cos\theta = \frac{1}{\sqrt{2}} = \sin\theta \implies \theta = \frac{\pi}{4}. \tag{671}$$

Similarly

$$1 - 7i = 5\sqrt{2}(\cos\theta + i\sin\theta) \implies \cos\theta = \frac{1}{5\sqrt{2}}, \sin\theta = -\frac{7}{5\sqrt{2}} \implies \theta = 2\pi - \arccos(\frac{1}{5\sqrt{2}}), \tag{672}$$

or $\theta \approx 1.545\pi$. *Finally,*

$$\sqrt{2} - 5i = 3\sqrt{3}(\cos\theta + i\sin\theta) \implies \cos\theta = \frac{\sqrt{2}}{3\sqrt{3}}, \sin\theta = -\frac{5}{3\sqrt{3}} \implies \theta = 2\pi - \arccos(\frac{\sqrt{2}}{3\sqrt{3}}), \tag{673}$$

or $\theta \approx 1.588\pi$.

The polar form lets us gain new insight on mulitplication and division. Recall the trigonometic formulas

$$\cos\theta_1 \sin\theta_2 = \frac{1}{2}\Big[\sin(\theta_1 + \theta_2) - \sin(\theta_1 - \theta_2)\Big] \tag{674}$$

$$\cos\theta_1 \cos\theta_2 = \frac{1}{2}\Big[\cos(\theta_1 - \theta_2) + \cos(\theta_1 + \theta_2)\Big] \tag{675}$$

$$\sin\theta_1 \sin\theta_2 = \frac{1}{2}\Big[\cos(\theta_1 - \theta_2) - \cos(\theta_1 + \theta_2)\Big] \tag{676}$$

This means

$$z_1 z_2 = r_1 r_2\Big[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_2\cos\theta_1\Big] \tag{677}$$

$$= r_1 r_2\Big[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\Big] \tag{678}$$

and

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\Big[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\Big], \qquad z_2 \neq 0. \tag{679}$$

We are about to state a theorem about the polar form of z^n. The proof of this is by a method known as induction. Theorems that can be proved in this way depend on a natural number n. The idea is to prove the **base** case $n = 0$ or $n = 1$. Then we **assume** that the statement holds for some $n = k > 1$ and prove that it holds for $n = k + 1$. Since k is arbitrary this then must hold for all n.

Theorem A.1 (De Moivre's Theorem)**.** *If* $z = r(\cos\theta + i\sin\theta)$ *and* $n \in \{1, 2, 3, ...\}$ *then*

$$z^n = (r\cos\theta + ir\sin\theta)^n = r^n(\cos(n\theta) + i\sin(n\theta)). \tag{680}$$

Proof. (Non-examinable). This is clear for $n = 1$. So lets prove the base case $n = 2$

$$z^2 = r^2(\cos^2\theta - \sin^2\theta) + 2r^2 i\cos\theta\sin\theta \tag{681}$$

From our trignometic formulas in equations (674)-(676) one has

$$z^2 = r^2\cos(2\theta) + r^2 i\sin(2\theta). \tag{682}$$

Lets assume this holds for $n-1$ and prove it for n. Under this assumption

$$\begin{aligned} z^n &= r^n(\cos((n-1)\theta) + i\sin((n-1)\theta))(\cos(\theta) + i\sin(\theta)) \qquad (683) \\ &= r^n\Big[\cos((n-1)\theta)\cos\theta - \sin((n-1)\theta)\sin\theta\Big] + ir^n\Big[\cos((n-1)\theta)\sin\theta + \sin((n-1)\theta)\cos\theta\Big]. \end{aligned}$$

Using the trig. identities (equations (674)-(676)) once more gives the result. Therefore, by induction we have the result. □

De Moivre's theorem can be use to find the nth root of a complex number z, i.e. find w such that

$$w^n = z. \tag{684}$$

Writing w and z in polar form as

$$w = s\big(\cos\varphi + i\sin\varphi\big), \qquad z = r\big(\cos\theta + i\sin\theta\big) \tag{685}$$

gives, via DeMoivre's theorem,

$$s^n\big(\cos(n\varphi) + i\sin(n\varphi)\big) = r\big(\cos\theta + i\sin\theta\big). \tag{686}$$

This requires,

$$s = r^{\frac{1}{n}}, \quad \cos(n\varphi) = \cos\theta, \quad \sin(n\varphi) = \sin\theta. \tag{687}$$

This can be simplified to

$$s = r^{\frac{1}{n}}, \qquad \varphi = \frac{\theta + 2k\pi}{n}, \tag{688}$$

where k is an integer. This gives distinct solutions for $k \in \{0, 1, ..., n-1\}$. So we have the following result:

Proposition A.2. *Let $z = r(\cos\theta + i\sin\theta)$ and $n \in \{1, 2, 3, ...\}$. Then z has n distinct roots given by*

$$w_k = r^{\frac{1}{n}}\Big[\cos\Big(\frac{\theta + 2k\pi}{n}\Big) + i\sin\Big(\frac{\theta + 2k\pi}{n}\Big)\Big], \tag{689}$$

for $k = 0, ..., n-1$.

Remark A.1. *All n roots of z have modulus $r^{\frac{1}{n}}$. Hence, they lie on a circle. Moreover, successive roots are spaced by $\frac{2\pi}{n}$, i.e. they are spaced equally on the circle.*

Example A.2. *Find all distinct w such that $w^4 = 1$.*

The above proposition gives us 4 distinct roots:

$$w_k = r^{\frac{1}{4}}\Big[\cos\Big(\frac{\theta + 2k\pi}{4}\Big) + i\sin\Big(\frac{\theta + 2k\pi}{4}\Big)\Big] \tag{690}$$

for $k = 0, 1, 2, 3$. The absolute value of z is

$$r = |z| = \sqrt{1^2} = 1 \tag{691}$$

and $\theta = \arctan(0) = 0$. So,

$$w_k = \Big[\cos\Big(\frac{k\pi}{2}\Big) + i\sin\Big(\frac{k\pi}{2}\Big)\Big]. \tag{692}$$

for $k = 0, 1, 2, 3$. Evaluating gives

$$w_0 = 1, \quad w_1 = i, \quad w_2 = -1, \quad w_3 = -i. \tag{693}$$

A.3 Complex Functions and the Fundamental Theorem of Algebra

So far in this course, we've studied functions that take real numbers as inputs and give out real numbers as outputs. What about functions that take complex numbers as inputs and give out complex numbers as outputs? We would denote this notationally,

$$f : \mathbb{C} \to \mathbb{C}. \tag{694}$$

If $z = x + iy$ then

$$f(z) = u(x, y) + iv(x, y) \tag{695}$$

for $u, v : \mathbb{R}^2 \to \mathbb{R}$.

A.3.1 Polynomials

Let's start with a univariate polynomial of a complex number, z. This is an expression of the form

$$P(z) = a_n z^n + a_{n-1} z^{n-1} ... + a_1 z^1 + a_0, \qquad a_0, ..., a_n \in \mathbb{C}. \tag{696}$$

The **degree** of the polynomial is the largest power appearing in it's equation, i.e. n. The **coefficients** of the polynomial are $a_0, ..., a_n$. A **root** of a polynomial is a complex number z_0 such that $P(z_0) = 0$. A root's **multiplicity** is the number of times that root appears in the factorisation, i.e.

$$P(x) = (x - 1)^2 (x - 7) \tag{697}$$

has roots $x = 1$ with multiplicity 2 and $x = 7$ with multiplicity 1.

For univarite polynomial's one has a very beautiful theorem about the existence of roots called the Fundamental Theorem of Algebra.[14]

Theorem A.2. *Every non-zero, univariate polynomial of degree n with complex coefficients has, counted with multiplicity, exactly n complex roots.*

We do not give the proof here but we will content ourselves with a proof for quadratic equations

$$P(z) = az^2 + bz + c. \tag{698}$$

To study the roots sets $P(z) = 0$. Then

$$az^2 + bz + c = 0. \tag{699}$$

One can rewrite this as

$$a\left(z + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c = 0. \tag{700}$$

So,

$$\left(z + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}. \tag{701}$$

Now when we consider a, b, c as complex numbers and allow for complex roots as above, $\sqrt{\frac{b^2-4ac}{a^2}}$ is well-defined. Therefore,

$$z = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}, \tag{702}$$

which is two roots unless $b^2 - 4ac = 0$, in which case $\frac{b}{2a}$ appears as a root with multiplicity 2.

[14] This theorem is usually attributed to Gauss in 1799. However, as is usual in mathematics there is a complicated history here. If you're interested in the history Wikipedia lays it out well.

A.3.2 Complex Exponentials

The complex exponential is the extension of e^x to the complex plane. We denote this with e^z. How is this defined? Well, there's a couple of ways but one is included here for completeness. The most common is through an infinite series[15]

$$e^z \doteq \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + ... \tag{703}$$

It turns out that the complex exponential has the same property as the real exponential, namely

$$e^{z_1+z_2} = e^{z_1}e^{z_2}. \tag{704}$$

There is a further property in relation to the trignometric functions $\sin$ and $\cos$. Let's do some rough manipulation with this formula.[16] Consider, e^{ix} for $x \in \mathbb{R}$. By the definition via the series one has

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + ... \tag{705}$$

Noting that $i^{2n} = (-1)^n$ for $n = 1, 2, 3, ...$ and $i^{2n+1} = (-1)^n i$ for $n = 1, 2, 3, ...$ one can collect real and imaginary parts of e^{ix} as

$$e^{ix} = \Big(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ...\Big) + i\Big(x - \frac{x^3}{3!} + \frac{x^5}{5!} + ...\Big). \tag{706}$$

The real part is the series expansion of $\cos(x)$ and the imaginary part is the series expansion of $\sin(x)$. We have arrived at Euler's formula

$$e^{ix} = \cos x + i \sin x. \tag{707}$$

Let's compute using this formula $e^{i\pi}$:

$$e^{i\pi} = \cos\pi + i\sin\pi = 1 \implies e^{i\pi} = -1. \tag{708}$$

This is often viewed as one of the most remarkable relations in mathematics. On the left-hand side you have two irrational numbers, e and π and the imaginary unit. Via Euler's formula, these combine to give the integer number -1!

Let's note one last thing about the complex exponential. From Euler's formula one has

$$e^{ix} = \cos(x) + i\sin(x), \qquad e^{-ix} = \cos(-x) + i\sin(-x) = \cos(x) - i\sin(x). \tag{709}$$

Therefore,

$$\cos(x) = \frac{1}{2}\Big(e^{ix} + e^{-ix}\Big), \qquad \sin(x) = \frac{1}{2i}\Big(e^{ix} - e^{-ix}\Big), \tag{710}$$

which are very useful identities to know if you forget your trignometric identities.

A.3.3 A Couple of Interesting Uses/Properties Associated to Complex Numbers

This section is non-examinable.

Definition A.2 (Complex Differentiable/Holomorphic Functions). *A complex-valued function f of a single complex variable z is* **complex differentiable/holomorphic** *at z_0 if the following limit exists*

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}. \tag{711}$$

(Note this is defined in the complex plane the limit has to agree on any path.)

[15] For those of you concerned with convergence of this series one can check via the ratio test that this convergence for all $z \in \mathbb{C}$ and is therefore a well-defined object.

[16] Note: the following manipulation can be made rigorous by noting the absolute convergence of the series. If this doesn't mean anything to you do not worry about it.

Theorem A.3. *Let $z = x + iy$ and let f be a complex-valued function of z, i.e. $f : \mathbb{C} \to \mathbb{C}$ and*

$$f(z) = u(x, y) + iv(x, y). \tag{712}$$

If the function f is **complex-differential/holomorphic** *it satisfies the Cauchy–Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{713}$$

If f satisfies the Cauchy–Riemann equations and is continuous and the first partial derivatives of u and v exist then f is holomorphic.

Example A.3. *The function $f(z) = z^2$ is holomorphic:*

$$f(z) = (x + iy)^2 = x^2 - y^2 + 2ixy, \implies u(x, y) = x^2 - y^2, \qquad v(x, y) = 2xy. \tag{714}$$

$$\partial_x u = 2x, \qquad \partial_y v = 2x \tag{715}$$

$$\partial_y u = -2y, \qquad \partial_x v = 2y. \tag{716}$$

So, the Cauchy–Riemann equations are satisfied.

Why are holomorphic functions useful you may ask? Well they have some very nice properties:

- They are infinitely differentiable.
- At any point z_0 in it's domain you can find a small disk where the function coincides with its Taylor series

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{1}{2} f''(z_0)(z - z_0)^2 + \frac{1}{3} f'''(z_0)(z - z_0)^3 + \ldots. \tag{717}$$

  This is extremely useful for approximating functions beyond the linear approximation.
- You can differentiate this series freely, just like a polynomial.
- The whole behaviour of the function can be reconstructed from knowledge of the function in the neighbourhood of a single point.

Let's end the section on complex numbers with a result that allows us to solve an ordinary differential equation.

Example A.4. *Let $u(x)$ satisfy the ordinary differential equation*

$$\frac{d^2 u}{dx^2} + ku = 0, \tag{718}$$

for k constant. This is the equation modelling simple harmonic motion, which arises everywhere in physics. A simple example is a mass on the end of a spring.

We want to solve for u. You may or may not of seen how to solve this equation before. Here we use a trick: let's write

$$\Big(\frac{d^2}{dx^2} + k\Big) u = 0, \tag{719}$$

which we can manipulate to

$$\Big(\frac{d}{dx} + ik\Big)\Big(\frac{d}{dx} - ik\Big) u = 0. \tag{720}$$

So, this tells us that

$$\Big(\frac{d}{dx} + ik\Big) f = 0 \tag{721}$$

where

$$f = \Big(\frac{d}{dx} - ik\Big)u. \tag{722}$$

Lets assume $f \neq 0$*, then the first equation* (721) *can be rewritten as*

$$\frac{1}{-ikf}\frac{d}{dx}f = 1 \implies \frac{d}{dx}\Big(\frac{i}{k}\log(f)\Big) = 1 \tag{723}$$

where we've used $\frac{1}{i} = -i$ *and* $\frac{d}{dx}\log f = \frac{1}{f}\frac{d}{dx}f$ *by the chain rule. We can now integrate both sides to obtain*

$$\frac{i}{k}\log(f) = x + c_1 \implies f(x) = e^{-ik(c_1+x)} = c_2 e^{-ikx}, \tag{724}$$

where $c_2 = e^{-ikc_1}$*.*

We can return to equation 722 and substitute in f *to find*

$$\Big(\frac{d}{dx} - ik\Big)u = c_2 e^{-ikx}. \tag{725}$$

Multiply both sides by e^{-ikx} *then we find*

$$e^{-ikx}\frac{d}{dx}u - ike^{-ikx}u = c_2 e^{-2ikx}. \tag{726}$$

The left hand side can be written as

$$\frac{d}{dx}(e^{-ikx}u). \tag{727}$$

So, our equation becomes

$$\frac{d}{dx}(e^{-ikx}u) = c_2 e^{-2ikx}. \tag{728}$$

Both sides can be integrated to find

$$e^{-ikx}u = c_2 e^{-2ikx} + c_3. \tag{729}$$

Therefore,

$$u(x) = c_3 e^{ikx} + c_2 e^{-ikx}. \tag{730}$$

You can invert the relations:

$$\cos(kx) = \frac{1}{2}\Big(e^{ikx} + e^{-ikx}\Big), \qquad \sin(kx) = \frac{1}{2i}\Big(e^{ikx} - e^{-ikx}\Big), \tag{731}$$

to find

$$u(x) = c_4\cos(kx) + ic_5\sin(kx). \tag{732}$$

where c_4 *and* c_5 *are related to* c_3, c_4*.*

Appendices

A Abbreviations

- $\therefore$: therefore, or 'it follows that'.
- $\text{Prop}^{\underline{\text{n}}}$: proposition. This is a statement of a result, like a theorem only smaller and less important.
- $\text{f}^{\underline{\text{n}}}$: function
- $\text{def}^{\underline{\text{n}}}$: definition
- $\text{not}^{\underline{\text{n}}}$: notation
- Rem. or Rmk: remark/a comment.
- Cor.: a corollary. a statement of a result that follows from a proposition or theorem above it.
- pt: shorthand for 'point'.
- $\#$: shorthand for 'number'.
- w/: shorthand for 'with'.
- s.t.: shorthand for 'such that'.
- w.r.t: shorthand for 'with respect to'.
- std: shorthand for 'standard'.
- resp.: shorthand for 'respectively'
- //: shorthand for 'parallel'.
- $\perp$: shorthand for 'perpendicular' or orthogonal.
- ($\dagger$), ($\ddagger$), ($\dagger\dagger$) ($\star$), ($\star\star$), ($\circ$): said 'dagger', 'double dagger', 'star', 'double star', 'circle'. Labelling of equations to refer back to in lectures. Does not hold over multiple lectures, only for that lecture.
- RHS: right-hand side
- LHS: right-hand side