An overview of Artin's conjecture

and some speculations

Sue Sierra

University of Edinburgh

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Outline

Conventions for the talk

- $k = \overline{k}$ = algebraically closed ground field, uncountable, char 0
- all rings are k-algebras, etc.
- A always means a connected graded (cg) k-algebra
  - $\mathbb{N}$-graded: $A = A_0 \oplus A_1 \oplus \cdots$ with $A_i A_j \subseteq A_{i+j}$
  - connected: $A_0 = k$.
  - In addition, we assume A is generated in degree 1.

Gelfand-Kirillov dimension

Let $V \ni 1$ generate a ring R. The GK-dimension of R is

$$\mathrm{GKdim}(R) = \overline{\lim} \log_n \dim V^n.$$

Measures the size (= growth) of R, e.g.
$\mathrm{GKdim}\, k[x_1, \ldots, x_n] = n$.

The conjecture

Let A be a domain with $\operatorname{GKdim} A < \infty$. Can form the graded quotient ring

$$Q_{\mathrm{gr}}(A) := A\langle h^{-1} \mid 0 \neq h \in A \text{ is homogeneous } \rangle.$$

Then $D = Q_{\mathrm{gr}}(A)_0$ is a division ring, the function skewfield of A.

Conjecture (Artin)

If A domain with $\operatorname{GKdim} A = 3$ and A is **suitably nice**, *then D is one of:*

1. **Birationally PI**: *A finite module over its centre K (which must be a field of transcendence degree 2).*
2. **q-ruled**: *A skew extension $K(t; \sigma, \delta)$ where $\operatorname{trdeg} K = 1$.*
3. **q-rational**: *$Q_{\mathrm{gr}}(\mathrm{Skl})_0$, the function skewfield of a 3-dimensional Sklyanin algebra.*

Sklyanin algebras

Example

Let $a, b, c \in k$. Define:

$$\begin{aligned} \mathrm{Skl} = S_{abc} = k\langle x, y, z\rangle/(axy + byx + cz^2, \\ ayz + bzy + cx^2, \\ azx + bxz + cy^2). \end{aligned}$$

Theorem (Artin-Tate-Van den Bergh)

S_{abc} is a **nice** *noetherian domain of GK-dim 3.*
Either S_{abc} is PI or $Z(S_{abc}) \cong k[x]$ and $Z(Q_{\mathrm{gr}}(S_{abc})) = k$.
If a, b, c are generic then the second happens.

S_{abc} is a Sklyanin algebra.

It is thought of as (the coordinate ring of) a "NC $\mathbb{P}^2$" because it shares many properties with $k[x, y, z]$.

Division rings of transcendence degree 2

A finitely generated division ring D has GK-transcendence degree 2 if $D = Q(R)$ where R is a domain of GK-dimension 2.

All D on Artin's list have GK-trdeg 2, and in fact can take R to have quadratic growth: the generators V have $\dim V^n \leq \alpha n^2$.

Question

If A is a domain of GKdim 3, is $Q_{\mathrm{gr}}(A)_0 = Q(\text{quadratic growth})$?

Question

If $D = Q(\text{quadratic growth})$ is $D = Q_{\mathrm{gr}}(A)_0$ for some domain A of GK 3?

Question

Does there exist R with $\mathrm{GKdim}(R) = 2$ but R does not have quadratic growth?

The q-ruled case

Let D be q-ruled, not finite over $Z(D)$. Bell-Rogalski note that D is either

2a $K(t;\sigma)$ where K is a field with trdeg $K = 1$; necessarily $|\sigma| = \infty$.

2b $K(t;\delta)$, K as above.

Thus Artin's conjecture says (morally) that if D has transcendence degree 2 and is not finite over $Z(D)$ then D is one of:

- $k_q(x, y)$, $q^n \neq 1$
- $K(E, \sigma) := k(E)(t;\sigma)$ where E is elliptic, σ an infinite-order translation
- $Q(D(C))$ where C is a (smooth affine) curve, $D(C) =$ differential operators on C
- a Sklyanin function field

Noncommutative projective surfaces

A a domain with $\mathrm{GKdim}(A) = 3$ is called the (coordinate ring of) a noncommutative projective surface, and Artin's conjecture gives a purported "birational classification" of such.

The story so far:

- A great deal of progress on understanding rings within various birational classes;
- Much less progress on proving (or disproving) the conjecture.

Noncommutative projective curves

Recall:

Theorem (Artin-Stafford)

Let A be a graded domain of quadratic growth. Then $Q_{\mathrm{gr}}(A)_0$ is a (fin. gen.) field with $\operatorname{trdeg} K = 1$, *and (up to finite dimension)*

$$A = B(C, \mathcal{L}, \sigma),$$

is the twisted homogeneous coordinate ring of a projective curve C with $k(C) = K$. A is birationally commutative.

Definition

Let $\sigma \in \mathrm{Aut}(C)$, $\mathcal{L}$ an ample invertible sheaf on C. The twisted homogeneous coordinate ring of C is

$$B(C, \mathcal{L}, \sigma) = \bigoplus_{n \geq 0} H^0(C, \mathcal{L} \otimes \sigma^* \mathcal{L} \otimes \cdots \otimes (\sigma^{n-1})^* \mathcal{L}).$$

Birationally commutative surfaces: case (1a)

Birationally commutative (i.e. D is commutative) NC projective surfaces are classified:

Theorem (Rogalski-Stafford)

Let A be a domain of GKdim 3 with $Q_{\mathrm{gr}}(A)_0 = K$, a field of trdeg 2. Then A determines and is determined by geometric data:

1. *a projective surface X*
2. $\sigma \in \mathrm{Aut}(X)$
3. *an (appropriately ample) invertible sheaf $\mathcal{L}$ on X*
4. *some other data*

In particular, $A \subseteq B(X, \mathcal{L}, \sigma)$ and is "close to" B.

Point modules

Definition

A point module over A is a cyclic graded module M so that

$$\dim_k M_n = \begin{cases} 1 & \textit{if } n \geq 0 \\ 0 & \textit{otherwise.} \end{cases}$$

That is M has the Hilbert series $1/(1-s) = 1 + s + s^2 + \cdots$ *of a point in* $\mathbb{P}^n$.

(If we replace 1 by m in the definition above we have an m-point.)

Note that if M is a point module so is $M[1] := \bigoplus_n M_{n+1}$.

Criterion for birational commutativity

Proposition

A domain A of GKdim 3 is birationally commutative iff A has a "surface of points."

Let A be arbitrary. Point modules over A are parameterised by a (pro)scheme $\widetilde{X}$. The map $M \mapsto M[1]$ induces an automorphism σ of $\widetilde{X}$, and there is an induced map (Artin-Tate-Van den Bergh) $A \to k(\widetilde{X})[t; \sigma]$.

Rogalski-Stafford point out that if A is noetherian then $\widetilde{X}$ is well-behaved, in particular has finitely many components, and each component is (birationally) of finite type and fixed by σ.

So if $\widetilde{X}$ (contains/is) a surface then $K = k(\widetilde{X})$ is finitely generated of transcendence degree 2, get $A \to K[t; \sigma]$ which must be injective if A is a domain with $\mathrm{GKdim}(A) = 3$.

Conversely if $\mathrm{GKdim}(A) = 3$ and A is birationally commutative then Rogalski-Stafford's classification shows that A has a surface worth of points.

Birationally PI surfaces: case (1b)

If R is in case (1) but not case (1a) then $D(R)$ is a finite module over its centre or, equivalently, satisfies a polynomial identity. We say R is birationally PI.

Example

(D. Chan) Let X be a surface and $\mathcal{A}$ be an order on X (a sheaf of NC algebras finite over X). Let $\mathcal{L}$ be an invertible $\mathcal{A}$-bimodule. The twisted ring

$$B = B(\mathcal{A}, \mathcal{L}) = \bigoplus_{n \geq 0} H^0(X, \mathcal{L}^{\otimes n})$$

is a NC projective surface and is birationally PI. We have

$$D(B) = \mathcal{A} \otimes k(X),$$

a division ring finite over $k(X)$.

NC surfaces with a surface of *m*-points

Theorem (D. Chan)

If A is a NC surface whose m-points are parameterized by a surface (and some technical conditions) then there are $\mathcal{A}, \mathcal{L}$ as above so that $R \subseteq B(\mathcal{A}, \mathcal{L})$, with the same graded quotient ring.

Question

Can algebras $A \subseteq B(\mathcal{A}, \mathcal{L})$ be classified, similarly to the classification of birationally commutative surfaces?

Question

If A is birationally PI, of PI-degree m, does it have a surface of m-points?

A curve of points

All known noetherian graded domains A of GKdim 3 have at least a curve of points.

Question

Must A have a curve Y of points?

Question

Must A have any points?

Question

Is there a module-theoretic criterion for q-ruledness or q-rationality?

An example

Let $B = B(C, \mathcal{L}, \sigma)$ for some curve C. Let

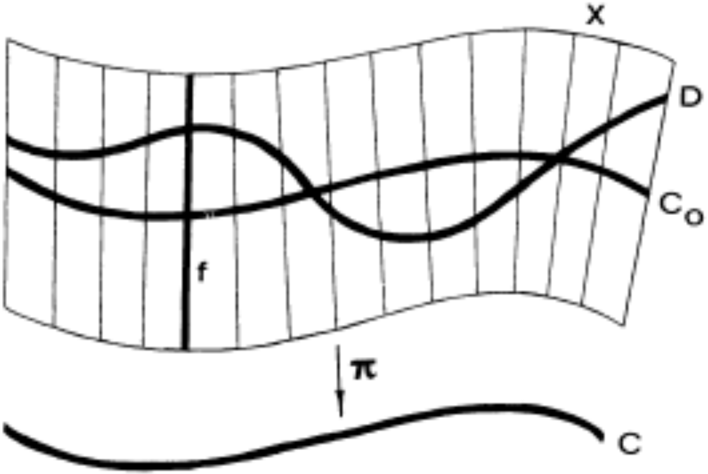
$$A = B[t; \tau],$$

graded so that $A_1 = B_1 + k \cdot t$.

Fact: Right ideals of B whose factor is a point module correspond to points on C.

- Let I be such a right ideal.
- B/I has Hilbert series $1/(1-s)$
- A/IA has Hilbert series $1/(1-s)^2$ and is a "line module."
- There is a component of the “line scheme” of R that is parameterized by C:

Here's a picture



which is of course the classical picture of a ruled surface.

A curve of points

How many points does A have?

- Recall that point modules over B are parameterized by C.
- Since B is a factor of A, all B-modules are A-modules.
- So A has at least a curve worth of points.

That is, there is a section, C_0, of the line scheme, as in the previous picture.

Question

If A has a curve of line modules with a section, is $Q_{\mathrm{gr}}(A)_0$ q-ruled?

Serre's theorem

Theorem (Serre)

Let A be a commutative graded ring generated in degree 1. Let $X = \operatorname{Proj} A$. Then

$$\mathcal{O}_X\text{-mod} \simeq \{\textit{graded } A\textit{-modules}\}/\{\textit{finite dimensional modules}\}.$$

That is: there is a functor

$$\text{gr-}A \to \mathcal{O}_X\text{-mod} \qquad M \mapsto \mathcal{M}$$

We have $\mathcal{M} = \mathcal{M}' \iff M_{\geq n} = M'_{\geq n}$ for $n \gg 0$.

Definition

For any graded A, the category

$$\{\textit{graded } A\textit{-modules}\}/\{\textit{finite dimensional modules}\}$$

makes sense and is called qgr-A *(or A-*qgr*).*

Some functors

Theorem (Chan-Nyman)

If A is a domain of GK-dim 3 so that A and qgr-A *are* **nice** *and so that A has a "well-behaved" family of "rational curve modules" parameterised by a curve C, then there are well-behaved adjoint functors:*

$$\pi^* : \mathcal{O}_C\text{-mod} \leftrightarrows \text{qgr-}A : \pi_*$$

and A is a "NC ruled surface."

This:

- is the right way to define a "noncommutative morphism"
- happens in the previous example.
- so the term "section" previously makes sense.

BUT:

- It's very hard to say anything about the ring theory of A from these functors.

NC quadric surfaces

Recall that the 3-dimensional Sklyanin algebra S_{abc} is a NC $\mathbb{P}^2$.

Theorem (Smith-Van den Bergh)

There is a NC projective surface A which has the following properties

- *A is a "quadric surface in a NC* $\mathbb{P}^3$*"*
- *A is birational to* Skl.
- *A is a NC* $\mathbb{P}^1 \times \mathbb{P}^1$.
- *A has a* $\mathbb{P}^1$*-worth of line modules and the previous theorem applies.*
- *But there is no section* $\mathbb{P}^1 \to \text{qgr-}A$.

Approaching the conjecture through Hilbert schemes

Question

Is there a module-theoretic criterion $\mathbb{X}$ for q-rationality, possibly involving a curve of lines without a section?

We have:

- Surface of (fat) points $\Rightarrow$ birationally PI
- Hopefully curve of lines + section $\Rightarrow$ q-ruled
- $\mathbb{X} \Rightarrow$ q-rational

All that's left is to show one of these cases always applies.

Valuations and places

Let D be a division ring

Definition

A (discrete) valuation on D is a map $\nu : D \to \mathbb{Z} \cup \infty$ with $\nu(xy) = \nu(x) + \nu(y)$, $\nu(x+y) \geq \min(\nu(x), \nu(y))$, $\nu(x) = \infty \iff x = 0$.

Definition

A place of D is a local ring $R \subset D$ with $D = Q(R)$ so that for $d \in D^*$ either $d \in R$ or $d^{-1} \in R$.

Definition

(Artin) A prime divisor of D is a place R with R/M a field of transcendence degree 1.

Distinguishing division rings of trdeg 2

All the (non-PI) division rings on Artin's list have prime divisors, and these can be used to distinguish the division rings on the list. Can this be used to prove (pieces of) the conjecture?

For example, $Q_{\mathrm{gr}}(\mathrm{Skl})_0$ has a unique prime divisor (Artin).

Question

If D is non-PI with $\mathrm{trdeg}(D) = 2$, and D has a unique prime divisor, is $D = Q(\mathrm{Skl})_0$?

The elliptic algebras of Rogalski-Stafford-S. have a unique valuation on their function fields, whose residue field is $k(E)$ (same proof as for Skl). Not known if elliptic algebras must be birational to Skl, although all known examples are (?).

Do valauations need to exist?

Question

If D is a division ring of transcendence degree 2, must D have a valuation? A prime divisor?

Question

If D has a prime divisor, is it on Artin's list?

Curves and prime divisors

Surely if A is a GK 3 domain with a curve C of points, then $Q(A)_0$ has a prime divisor centred at C:

There is a map $A \to k(C)[t; \sigma]$ as above; let I be the kernel, which is completely prime.

I is surely localisable and surely $(A_{(I)})_0$ is a place with residue field $k(C)$.

Question

Does this work?

Question

Does a prime divisor of $Q(A)_0$ give rise to a curve of A-points?

Counterexample 1: invariants

Let D be one of the division rings on Artin's list, let $G \leq_{\text{finite}} \mathrm{Aut}(D)$. Let $D' = D^G$.

Then $\mathrm{trdeg}(D') = 2$. Is D' on the list?

Example (Van den Bergh)

Let $D = K(E, \sigma) = k(E)(t; \sigma)$ (recall E is elliptic, σ is translation). Define $\tau : D \to D$ such that $\tau(x) = x^{-1}$ and $\tau|_{k(E)}$ is the automorphism induced by $p \mapsto -p$ on E.

Then $D^\tau \cong Q_{\mathrm{gr}}(\mathrm{Skl})$.

Proof: Write $Q_{\mathrm{gr}}(\mathrm{Skl})$ as the function field of a noncommutative $\mathbb{P}^1 \times \mathbb{P}^1$.

Fryer's example

Let q not be a root of 1, and let $\phi \in \mathrm{Aut}(k_q(x, y))$ be defined by

$$\phi(x) = (y^{-1} - q^{-1}y)x^{-1}, \quad \phi(y) = -y^{-1}.$$

Note $\phi^2 = \mathrm{Id}$.

Theorem (Fryer, 2013)

$k_q(x, y)^\phi \cong k_q(x, y)$.

NC Cremona groups

Let D be a (non-PI) division ring on Artin's list.

Question

- *What are the finite subgroups of* $\mathrm{Aut}(D)$*?*
- *What is* $\mathrm{Aut}(D)$*?*

Not known for any D. (Except maybe $Q(D(C))$ with $g(C) > 0$?)

Example

$\mathrm{End}(k_q(x,y)) \neq \mathrm{Aut}(k_q(x,y))$ (Fryer).
$\mathrm{End}(k_q(x,y))$ is generated by:

- $x \mapsto x^{-1}$, $y \mapsto y^{-1}$
- Elementary automorphisms sending $x \mapsto x$ or $y \mapsto y$
- Conjugation by $z \in k_q(y)((x))$.

It's not known which z give automorphisms of $k_q(x,y)$.

Counterexample 2: factors of cocycle twists

Let A be a 4-dimensional Sklyanin algebra. Recall that A is an AS-regular noetherian domain with a curve of points parameterised by an elliptic curve E, and has central elements Ω_1, Ω_2 so that $A/(\Omega_1, \Omega_2) \cong B(E, \mathcal{L}, \sigma)$.

The Klein 4-group V acts on A and on $M_2(k)$, and define $A' = (A \otimes_k M_2(k))^V$. (First defined by Odesskii.)

Theorem (Davies; Chirvasitu-Smith)

A' is a noetherian AS-regular domain with of GKdim 4 with only 20 point modules, but A' has a curve of multiplicity 2 fat points parameterised by E^V. Further, Ω_1, Ω_2 survive in A' and $A'/(\Omega_1, \Omega_2)$ is the homogeneous coordinate ring of an order on E^V, and is prime but not a domain.

Stafford's counterexample

Theorem (Stafford)

Suppose there exists $\Omega = \lambda_1\Omega_1 + \lambda_2\Omega_2$ *so that*

$$T = A'/(\Omega)$$

is a domain. Then $D = Q_{\mathrm{gr}}(T)_0$ *is a counterexample to Artin's conjecture.*

Proof.

D has no prime divisors. □

Question

Does such Ω *exist? (Toby thinks yes, I think no.)*

Not all places are prime divisors

D on the previous slide does have a place whose "residue field" is a 2×2 matrix ring.

Question

It would be useful to classify the places, or other NC valuations, of the division rings on Artin's list.

Counterexample 3: a complete counterexample (Bell)

“I’m particularly interested in looking at the “completed" version of Artin’s conjecture—there is an interesting case that appears in the complete case that doesn’t seem to occur in the ordinary case. It’s a bit mysterious and I hope people will have some insights.”

Where's the beef?

Where's the noncommutative geometry?

I can't help noticing that very little noncommutative algebraic geometry in the usual sense has appeared in the talk — I barely used the category qgr-A, for example, or Chan and Nyman's noncommutative Mori contractions.

The birational transformations that Michel and Dan, Toby, and I have worked on also haven't been discussed.

This is because I really don't know how to use these techniques to approach the conjecture. Maybe some of you have better ideas.

Counterexamples must be strange!

A counterexample is likely to be sporadic (not in a family), and not to be a deformation of a commutative surface.

- A family of graded domains of GK 3 is likely to have a commutative member.
- Artin argues that if A is a domain of GK 3 that is a deformation of the (commutative) homogeneous coordinate ring of a surface X, then the function field of A is on the list.

If the previous questions have positive answers, then a counterexample has:

- Fewer points than any known graded domain of GK 3.
- Its function field has no prime divisors.
- Does not deform to any commutative surface.

It would be unlike any known example.

An exotic counterexample

Thank you!