The Dixmier-Moeglin equivalence for twisted homogeneous coordinate rings

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Goal: work out ideal theory of important class of NC algebras.

- We work over $\mathbb{C}$.
- Let X be a projective variety, let $\sigma \in \mathrm{Aut}(X)$, and let $\mathcal{L}$ be a σ-*ample* invertible sheaf on X.
  - σ-ample means: $\mathcal{L}$ is appropriately positive and σ has reasonable action on homology.
  - Also a definition in terms of cohomology vanishing.
- Let
$$\mathcal{L}_n := \mathcal{L} \otimes \sigma^* \mathcal{L} \otimes \cdots \otimes \sigma^{(n-1)*} \mathcal{L}.$$
- Define the *twisted homogeneous coordinate ring* $B(X, \mathcal{L}, \sigma)$ by
$$B = B(X, \mathcal{L}, \sigma) := \bigoplus_{n \geq 0} H^0(\mathcal{L}_n).$$
- Multiplication given by
$$H^0(\mathcal{L}_n) \otimes H^0(\mathcal{L}_m) \to H^0(\mathcal{L}_n) \otimes H^0(\sigma^{n*} \mathcal{L}_m) \to H^0(\mathcal{L}_{n+m}).$$

Example:

- Let $X = \mathbb{P}^n$. An element $\sigma \in \mathbb{P}GL_{n+1}$ acts on homogeneous forms via pullback. Then

$$B(\mathbb{P}^n, \mathcal{O}(1), \sigma) \cong \mathbb{C}[x_0, \ldots, x_n], \star.$$

- New multiplication $\star$ induced by

$$x_i \star x_j := x_i x_j^\sigma.$$

Some facts about $B(X, \mathcal{L}, \sigma)$.

- B is a graded noetherian domain with $\dim X + 1 \leq \operatorname{GKdim} B < \infty$.
- Can recover $X, \mathcal{L}, \sigma$ from B
- Graded 2-sided ideals of $B \leftrightarrow \sigma$-invariant subschemes of X.

Question

What are the primitive ideals of $B(X, \mathcal{L}, \sigma)$?

Recall: an ideal $I \subset R$ is *primitive* if $I = \mathrm{Ann}_R(M)$ where M_R is simple.

- If (0) is primitive, say R itself is primitive

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$$\mathrm{Spec}\, B = \{\textit{prime ideals of } B\}$$

in the Zariski topology. How are primitive *ideals in* Spec B *distinguished?*

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Primitive ideals in commutative rings

- Primitive = maximal.

In fact, if an affine $\mathbb{C}$-algebra R is commutative then TFAE:

1. $P \in \operatorname{Spec} R$ is primitive
2. P is maximal
3. $\{P\}$ closed in $\operatorname{Spec} R$
4. $Q(R/P) \cong \mathbb{C}$.

Definition

Let R be a noetherian ring. $P \in \operatorname{Spec} R$ is rational *if*

$$Z(Q(R/P)) \cong \mathbb{C}$$

Theorem (Dixmier-Moeglin)

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Let $R = U(\mathfrak{g})$ and let $P \in \operatorname{Spec} R$. TFAE:

1. *P is primitive*
2. *$\{P\}$ is locally closed in $\operatorname{Spec} R$*
3. *P is rational.*

Definition

We say the Dixmier-Moeglin equivalence holds for R *if for all* $P \in \operatorname{Spec} R$ *we have:*
P *rational* $\iff$ P *primitive* $\iff$ $\{P\}$ *locally closed.*

NB:

- In our setting we always have locally closed $\implies$ primitive $\implies$ rational.
- Thus the DM-equivalence is equivalent to showing that rational $\implies$ locally closed.

Main theorem:

Theorem 1

The Dixmier-Moeglin equivalence holds for

- *twisted homogeneous coordinate rings of curves and surfaces;*
- *twisted homogeneous coordinate rings of* $\mathbb{P}^n$.

Further, in these cases B is primitive $\iff$ σ *has a dense orbit.*

- We conjecture this holds for all THCR's.

An example on $\mathbb{P}^1$:

- Let $\tau \in \mathbb{P}GL_2$ be defined by $\tau[x:y] = [x:x+y]$.
- Let
$$B := B(\mathbb{P}^1, \mathcal{O}(1), \tau) \cong \mathbb{C}\{x,y\}/(xy - yx - x^2).$$
- $\operatorname{Spec} B = \{(0),(x)\} \cup \{(x, y+\lambda) \mid \lambda \in \mathbb{C}\}$
- Rational ideals are (0) and $(x, y+\lambda)$.
- The maximal ideals are obviously locally closed.
- Since
$$\operatorname{Spec} B \setminus \{(0)\} = \{P \mid P \supseteq (x)\},$$
(0) is locally closed.
- So B satisfies the DM-equivalence; in particular B is primitive.
- The point $[1:0]$ has a dense orbit, since
$$\tau^n([1:0]) = [1:n].$$

A geometric condition on σ:

Definition

The pair (X, σ) *is* ordinary *if for all* σ*-invariant* $Z \subseteq X$*, the set*

$$\{x \in Z \mid \{\sigma^n(x)\} \text{ is dense in } Z\}$$

is an open subset of Z*.*

- If G is an algebraic group acting on X and $\sigma \in G$, then (X, σ) is ordinary.

Proposition

If (X, σ) is ordinary, then $B(X, \mathcal{L}, \sigma)$ satisfies the DM-equivalence.

Corollary

Suppose that $\sigma \in G$, G an algebraic group acting on X. Then $B(X, \mathcal{L}, \sigma)$ satisfies the DM-equivalence. In particular, if $X = \mathbb{P}^n$ or X is a curve, then $B(X, \mathcal{L}, \sigma)$ satisfies the DM-equivalence.

Proposition

If X is a surface, then $B(X, \mathcal{L}, \sigma)$ satisfies the DM-equivalence.

Proof:

- Key point: the existence of a σ-ample sheaf constrains σ.
- In particular, σ must be *parabolic* = not exponentially expanding.
- Can then use $\mathbb{C}$ dynamics and work of Gizatullin, Diller, Favre to reduce to a case-by-case analysis of automorphisms of surfaces.
- We show that if X is a surface and σ is parabolic, then (X, σ) is ordinary.

Theorem 2

Let S be a commutative ring of dimension 2, and let $\sigma \in \mathrm{Aut}(S)$. Let

$$T := S[t, t^{-1}; \sigma]$$

Then if $\mathrm{GKdim}\, T < \infty$*, then T satisfies the DM-equivalence. If $S = \mathbb{C}[u, v]$, then this is if and only if.*

Example (David Jordan)

Let $S = \mathbb{C}[u^{\pm}, v^{\pm}]$. Let $\sigma(u) = uv$ and $\sigma(v) = u$. Then

$$T := S[t, t^{-1}; \sigma]$$

does not satisfy the DM-equivalence, since (0) *is primitive but not locally closed.*

- Jordan's example has GKdim $= \infty$.

Question

Does finite GKdim imply the DM-equivalence?

- Recall that THCR's have finite GKdim.

Theorem 3

If GKdim $B(X, \mathcal{L}, \sigma) = \dim X + 1$*, then the DM-equivalence holds for B.*

Proof:

- σ acts on the homology of X to give an integer matrix $[\sigma]$, invariant with respect to the intersection form.
- Keeler: GKdim $B = \dim X + 1 \implies [\sigma]^n = 1$.
- De-Qi Zhang: $\{\sigma \mid [\sigma]^n = 1\}$ is an algebraic group.
- So (X, σ) is ordinary and the DM-equivalence holds for B.

- Jordan's example has GKdim $= \infty$.

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