CORRIGENDUM TO "ENVELOPING ALGEBRAS WITH JUST INFINITE GELFAND-KIRILLOV DIMENSION"

SUSAN J. SIERRA, BASED ON WORK BY NATALIA K. IYUDU AND SUSAN J. SIERRA

In this corrigendum we correct an error in [IS]. In particular, [IS, Proposition 4.8] is incorrect and thus there is an error in the proof of [IS, Theorem 4.1]. We explain the error and then give a corrected proof of [IS, Theorem 4.1], proving a new version of [IS, Proposition 4.8] in the process.

We adopt the notation of [IS]; in particular, fix a ground field $\mathbb{K}$ of characteristic zero, and let W be the *Witt algebra*, with $\mathbb{K}$-basis $\{e_n : n \in \mathbb{Z}\}$ and Lie bracket

$$[e_n, e_m] = (m-n)e_{n+m}. \tag{1}$$

Both W and the universal enveloping algebra $U(W)$ are graded by setting $\deg(e_n) = n$. The subalgebra W_+, the *positive Witt algebra*, is spanned by $\{e_n : n \geqslant 1\}$.

To fix notation, we restate [IS, Theorem 4.1] here.

Theorem 2. ([IS, Theorem 4.1].) *Let $I = (g)$ be a two-sided ideal in $U(W)$ generated by one nonzero element $g \in U(W)$. Then $A = U(W)/I$ has polynomial growth.*

The main purpose of the current work is to prove Theorem 2.

We fix the meanings of g, I, and A as in the statement of Theorem 2. Let $\pi : U(W) \to A$ be the natural map.

The technical result [IS, Proposition 4.8] is concerned with how to rewrite elements of $U(W)$ modulo (g). Recall that a *standard momomial* is a word $e_{i_1} \dots e_{i_N}$ with $i_1 \leqslant \dots \leqslant i_N$. Let $k = |g|$ (that is, the length of the longest monomial in g) and let n be the positive integer defined in [IS, Lemma 4.4]. By [IS, Lemma 4.4], A is spanned by the set $NS = NS(k,n)$ of standard monomials m which admit a factorisation

$$m = aub, \tag{3}$$

where a is a standard monomial of length $< k$ in e_{-n} and smaller letters, u is a standard monomial in $e_{1-n}, \dots e_0, e_1, e_2, \dots, e_{n-1}$, and b is a standard monomial of length $< k$ in e_n and bigger letters. We refer to elements of NS as *normal words*.

The proof of [IS, Proposition 4.8] considers two normal words

$$m_1 = a_1u_1b_1 \text{ and } m_2 = a_2u_2b_2,$$

with the same constraints on the a_i, b_i, and u_i as in (3). It is claimed in the proof of [IS, Proposition 4.8] that, when we use the commutation relations (1) to rewrite m_1m_2 as a linear combination of standard monomials, all of the obtained standard monomials have $\leqslant 2k-2$ letters with indices $\geqslant n$ (and symmetrically $\leqslant 2k-2$ letters with indices $\leqslant -n$). However, this is false; for example, if r is any positive integer, then the monomial e_n^r occurs when expressing $e_{n-1}^r e_1^r$ as a linear combination of standard monomials; note that e_{n-1}^r and e_1^r are normal words.

Let us correct the statement of [IS, Proposition 4.8] and prove the new result. For any $C, C' \in \mathbb{N}$, define a function

$$\delta_{C,C'} : \{ \text{ standard monomials in } U(W) \} \to \mathbb{N}$$

by $\delta_{C,C'}(e_{i_1} \dots e_{i_N}) = |i_1| + \dots + |i_N| + NC + C'$. Let $\delta = \delta_{0,0}$.

Date: July 26, 2022.

Proposition 4. (Compare [IS, Proposition 4.8].) *Let $\ell = \max\{|i| : e_i \text{ features in } g\}$, and let $C = 2k\ell$ and $C' = 4k^2\ell$. Then for any two normal words $m_1, m_2 \in NS$, we can write*

$$\pi(m_1 m_2) = \sum c_i \pi(w_i),$$

where the $c_i \in \mathbb{K}$ and the w_i are normal words so that

$$\delta_{C,C'}(w_i) \leqslant \delta_{C,C'}(m_1) + \delta_{C,C'}(m_2) \text{ for all } i. \tag{5}$$

Before proving Proposition 4 we must consider how the process of reducing a standard monomial m modulo (g) affects $\delta(m)$. We restate and refine the key reduction lemma [IS, Lemma 3.1].

Lemma 6. *Let $0 \neq g \in U(W)$, and let $I = U(W_+)gU(W_+)$ be the $U(W_+)$-sub-bimodule of $U(W)$ generated by g. Let $k = |g|$. Then there exist integers n and ℓ, with $n \geqslant 1$, so that every standard monomial $m = e_{j_1} \dots e_{j_k}$ with $n \leqslant j_1 \leqslant \dots \leqslant j_k$ satisfies*

$$m = h + \sum c_t m_t, \tag{7}$$

where $h \in I$, the sum is finite, $c_t \in \mathbb{K}$, and the m_t are standard monomials so that for each t, we have $i \geqslant \ell$ for all letters e_i featuring in m_t, and either $|m_t| < k$ or $|m_t| = k$ and $i < n$ for at least one of the letters featuring in m_t. If h is degree-homogeneous, then $\deg(m) = \deg(h) = \deg(m_t)$ for all t.

Finally, for each of the m_t we have

$$\delta(m_t) \leqslant \delta(m) + 2\delta(g). \tag{8}$$

Proof. All statements except for (8) are proved in [IS, Lemma 3.1]. For (8), consider how one iteration of the reduction procedure in [IS, Lemma 3.1] affects $\delta(m)$. Let us call letters with indices $\geqslant n$ *big letters*, so all letters in m are big.

Recall the monomial order $<$ defined in [IS]: if m, m' are standard monomials, then $m < m'$ if and only if:

- $|m| < |m'|$;
- or $|m| = |m'|$ and $\deg m < \deg m'$;
- or $|m| = |m'|$, $\deg m = \deg m'$, and m is less than m' in the left-to-right lexicographic order.

According to the proof of [IS, Lemma 3.1], we first rewrite m as in (7), but here we only assume that each $m_t < m$ and do not assume that m_t contains fewer than k big letters. To do this, we find a product of derivations $D = \partial_{a_1} \dots \partial_{a_k}$, with all the $a_i \geqslant 1$, and some $c \in \mathbb{K}^*$ such that

$$cD(g) = m + \sum c_s m'_s, \tag{9}$$

with the $m'_s < m$ and $c_s \in \mathbb{K}^*$. (Recall here that $\partial_a = [-, e_a]$ as a linear operator on $U(W)$, so $D(g) \in (g)$.) Let m' be some m'_s. Now m' is a standard monomial which falls into one of three cases:

I. $|m'| < |m|$, and m' is obtained from some monomial $\tilde{g}$ in g by applying D and then the commutation relations;

II. $|m'| = |m|$ and m' is obtained from some monomial $\tilde{g}$ in g (which necessarily has length k) by a non-permutational action of D;

III. $|m'| = |m|$ and m' is obtained from some monomial $\tilde{g}$ in g by a permutational action of D.

Here recall that if different ∂_{a_i} act on letters in different places in the monomial $\tilde{g}$ then the action is *permutational*, otherwise it is *non-permutational*.

First, we note what happens to the number of big letters in each case. In case I, since m' is shorter than m, and all letters in m are big, m' contains fewer big letters than m. In case II, since the action is non-permutational, there is a letter in m' which was present in $\tilde{g}$. But monomials of g consist of letters which are not big, by definition of n. Thus the number of big letters in m' is smaller than in m. Thus in cases I and II, we may take $m_t = m'$ and terminate the procedure.

In case III, it may be that the number of big letters in m' is still equal to k, which is the number of (big) letters in m, but certainly there are no more than $k = |m'|$ big letters in m'; and $m' < m$, so we have reduced our monomial modulo (g). However, we may need to repeat this reduction procedure finitely many times to obtain an expansion satisfying (7).

We now consider how δ changes in each of these cases. In case III, as $m' < m$ and m' and m are made of letters $\geqslant e_1$, we have $\delta(m') = \deg(m') \leqslant \deg(m) = \delta(m)$. In cases I and II, $\delta(m')$ may be bigger than $\delta(m)$. Since applying the commutation relations does not increase δ, we may assume that the monomial m' is a (possibly non standard) monomial obtained from applying D to a monomial $\tilde{g}$ of g.

Recall the monomial order $\prec$ defined in [IS], and let $\bar{g}$ be the $\prec$-leading monomial of g. Then recall that m is obtained from the monomial $\bar{g}$ by applying D. Set $\deg(D) = a_1 + \cdots + a_k$, so applying D increases the degrees of homogenous elements by $\deg(D)$. As $\delta(e_{a+b}) \leqslant \delta(e_a) + \delta(e_b)$ and all the a_i are $\geqslant 1$, we have

$$\delta(m') \leqslant \delta(\tilde{g}) + \deg D.$$

So

$$\begin{aligned}
\delta(m') - \delta(m) &\leqslant \delta(\tilde{g}) + \deg D - \delta(m) \\
&= \delta(\tilde{g}) + \deg D - \deg m \qquad \text{as letters in } m \text{ are big} \\
&\leqslant \delta(\tilde{g}) + |\deg m - \deg D| \\
&= \delta(\tilde{g}) + |\deg \bar{g}| \\
&\leqslant 2\delta(g).
\end{aligned}$$

To summarize the discussion above: in cases I and II, we remove at least one big letter and increase δ by no more than $2\delta(g)$; and the procedure then terminates as (7) is satisfied. In case III, we do not reduce the number of big letters and do not increase δ; here we may need to iterate our procedure. Thus, iterating as necessary, when we obtain the expansion satisfying (7), we have increased δ by no more than $2\delta(g)$. □

Symmetrically to Lemma 6, we have:

Lemma 10. (See [IS, Lemma 4.2].) *There exist positive integers k and n and an integer ℓ such that every standard monomial $m = e_{j_1} \dots e_{j_k}$ with $j_1 \leqslant \dots \leqslant j_k \leqslant -n$ satisfies*

$$m = h + \sum c_t m_t,$$

where $h \in I$, the sum is finite, $c_t \in \mathbb{K}$, and the m_t are standard monomials so that for each t, we have $i \leqslant \ell$ for all letters e_i featuring in m_t, and either $|m_t| < k$ or $|m_t| = k$ and $i > -n$ for at least one of the letters featuring in m_t. For each of the m_t we have

$$\delta(m_t) \leqslant \delta(m) + 2\delta(g). \tag{11}$$

Proof. The proof of [IS, Lemma 4.2] works here to show the situation is symmetric to that of Lemma 6. □

We now prove Proposition 4.

Proof of Proposition 4. First, apply the commutation relations in $U(W)$ to rewrite $m_1 m_2$ as a linear combination of standard monomials m_3. Write

$$m_3 = a_3 u_3 b_3,$$

where a_3 involves only letters $\leqslant -n$, u_3 involves only letters strictly between $-n$ and n, and b_3 involves only letters $\geqslant n$.

Let us call letters e_i with $|i| \geqslant n$ *big letters*. Let the length of a_3 be r and the length of b_3 be s. If $s > 2k - 2$ then the "extra" big letters in b_3 must have come from applying the commutation

relations (1) to small letters. After each such transposition we get at most one more big letter, and the length of the monomial in which it appears drops by 1. A similar analysis applies to a_3. Thus $\delta_{C,0}(m_3) \leqslant \delta_{C,0}(m_1) + \delta_{C,0}(m_2) - (r+s-4k+4)C$, where here we note that the commutation relations do not increase δ.

We apply our reduction procedure to get rid of $s-k+1$ big letters in b_3 and $r-k+1$ big letters in a_3, rewriting

$$\pi(m_3) = \sum c_i \pi(w_i),$$

where the w_i are normal words. By Lemma 6 (for b_3) and Lemma 10 (for a_3) this increases δ (and thus $\delta_{C,0}$) by no more than

$$2(r+s-2k+2)\delta(g) \leqslant (r+s-2k+2)C,$$

as $C \geqslant 2\delta(g)$. In other words, we have

$$\begin{aligned}\delta_{C,0}(w_i) &\leqslant \delta_{C,0}(m_3) + (r+s-2k+2)C \\ &\leqslant \delta_{C,0}(m_1) + \delta_{C,0}(m_2) + (2k-2)C \leqslant \delta_{C,0}(m_1) + \delta_{C,0}(m_2) + C'.\end{aligned}$$

Adding C' to both sides gives (5). □

We also must show that NS has polynomial growth, where we use $\delta_{C,C'}$ to give a notion of degree.

Lemma 12. (Compare [IS, Lemma 4.5].) *For any $N, C, C' \in \mathbb{N}$, let*

$$p_{C,C'}(N) = \#\{w \in NS : \delta_{C,C'}(w) \leqslant N\}.$$

Then for any C, C', the function $p_{C,C'}(N)$ has polynomial growth in N.

Proof. Since $\delta_{C,C'}(m) \geqslant \delta_{0,C+C'}(m)$ for a standard monomial $m \in NS \setminus \mathbb{K}$, we have

$$p_{C,C'}(N) \leqslant p_{0,C+C'}(N) \quad \text{for } N > C'.$$

But $p_{0,C+C'}$ is shown to have polynomial growth in [IS, Lemma 4.5]. □

Theorem 2 is now proved by by substituting Proposition 4 for [IS, Proposition 4.8] and Lemma 12 for [IS, Lemma 4.5] in the proof of [IS, Theorem 4.1].

The proof of [IS, Theorem 5.3] is also based on [IS, Theorem 4.1], now Theorem 2, and the proof of [IS, Theorem 5.6] uses the proof of [IS, Proposition 4.8]. The proofs require only minor modifications to use the filtration defined by $\delta_{C,C'}$ rather than the functions δ_C of [IS]. We leave the details to the reader.

References

[IS] N. Iyudu and S. J. Sierra. *Enveloping algebras with just infinite Gelfand-Kirillov dimension.* Arkiv för Matematik **58** (2020), no. 2, 285-306.

School of Mathematics, The University of Edinburgh, Edinburgh EH9 3FD, United Kingdom
Email address: s.sierra@ed.ac.uk