Ideals of enveloping algebras of loop algebras

w/ Petukhov

§1 Enveloping algebras of $\left\{ \begin{array}{c} \text{loop} \\ \text{affine} \end{array} \right\}$ Lie algebras

§2 The question and the partial answer

§3 The proof

§1 Enveloping algebras of $\left\{\begin{array}{c}\text{loop}\\ \text{affine}\end{array}\right\}$ Lie algebras

$\mathfrak{g}$ = f. dim simple Lie algebra / $\mathbb{C}$

$L\mathfrak{g}$ = loop algebra of $\mathfrak{g}$ = $\mathrm{Map}(\mathbb{C}^\times, \mathfrak{g})$

$= \mathfrak{g}[s, s^{-1}] \qquad [xs^i, ys^j] = [x,y]s^{i+j}$

$\hat{\mathfrak{g}}$ = affine Lie algebra $=_{vsp} L\mathfrak{g} \oplus \mathbb{C}c$

= ! nontrivial central extension of $L\mathfrak{g}$

$$[xs^i + \alpha c, ys^j + \beta c] = [x,y]s^{i+j} + \delta_{i,-j}\, i\, K(x,y)c$$

§ 1 Enveloping algebras of $\left\{ \begin{array}{l} \text{loop} \\ \text{affine} \end{array} \right\}$ Lie algebras

$U(\hat{\mathfrak{g}})$ = universal enveloping algebra of $\hat{\mathfrak{g}}$

$$= T(\hat{\mathfrak{g}}) \Big/ \big(XY - YX = [X, Y] \;\big|\; X, Y \in \hat{\mathfrak{g}} \big)$$

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Not L or R noetherian: e.g. $\underbrace{\sum_{i \in \mathbb{N}} x^i \, U(\hat{\mathfrak{g}}) \text{ where } x \in \mathfrak{g}}$

right ideal of $U(\hat{\mathfrak{g}})$ that cannot be finitely generated

Thm (Biswal–S. '21)

(1) If $\lambda \in \mathbb{C}^\times$ then $\mathcal{U}(\hat{\mathfrak{g}})(c-\lambda)$ is a simple ring

(2) Ideals in $\mathcal{U}(L\mathfrak{g})$ are big

2-sided

Thm (Biswal–S. '21)

(1) If $\lambda \in \mathbb{C}^\times$ then $U(\hat{\mathfrak{g}}) / (c-\lambda)$ is a simple ring

(2) Ideals in $U(L\mathfrak{g})$ are big ...!

If $0 \neq I \lhd U(L\mathfrak{g}) = \dfrac{U(\hat{\mathfrak{g}})}{(c)}$ then

$U(L\mathfrak{g})/I$ has finite growth:

$\exists\, d : \forall\, V$ fin dim $\subseteq \dfrac{U(L\mathfrak{g})}{I}$, $\dim V^n < n^d$ for $n \gg 0$

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NB: $\mathcal{U}(L\mathfrak{g})$ has ∞ growth, like $\mathbb{C}[x_i \mid i \in \mathbb{Z}]$

(2): $\mathcal{U}(L\mathfrak{g})$ has just-infinite growth

§2 The question and the partial answer

Ideals in $U(\mathcal{L}og)$ are big

Is the ideal lattice of $U(\mathcal{L}og)$ tractable?

Q: What's the structure of 2-sided ideals of $U(\mathcal{L}og)$?

Let $\Phi_1 : U(\mathfrak{g}[s,s^{-1}]) \to U(\mathfrak{g})[t,t^{-1}]$

$$xs^i \longmapsto xt^i$$

<u>Def</u>: For $\ell \geq 1$ define $\Phi_\ell : U(L\mathfrak{g}) \to U(\mathfrak{g}^{\oplus \ell})[t_1^{\pm}, \dots, t_\ell^{\pm}]$

$$xs^i \mapsto x_{[1]} t_1^i + \cdots + x_{[\ell]} t_\ell^i$$

(Here $\mathfrak{g}^{\oplus \ell} = \mathfrak{g}_{[1]} \oplus \cdots \oplus \mathfrak{g}_{[\ell]}$, $x_{[j]} \in \mathfrak{g}_{[j]}$)

Let V a rep of $\mathfrak{g}$, let $a \in \mathbb{C}$

$\rightsquigarrow$ evaluation rep V_a of $L\mathfrak{g}$ $\qquad$ $L\mathfrak{g} \xrightarrow{ev_a} \mathfrak{g} \curvearrowright V$

"$s \longmapsto a$"

- Action of $U(L\mathfrak{g})$ on V_a factors through $\overline{\Phi}_1$

- Given $V^1, \ldots, V^\ell, a_1, \ldots, a_\ell$, action of $U(L\mathfrak{g})$ on $V^1_{a_1} \otimes \cdots \otimes V^\ell_{a_\ell}$ factors through $\overline{\Phi}_\ell$

. (Chari – Pressley) any f.d. irrep of $L\mathfrak{g}$ is of

form $V^1_{a_1} \otimes \cdots \otimes V^\ell_{a_\ell}$

. $\operatorname{Ker} \overline{\Phi}_\ell = \bigcap_{\substack{V^1 \ldots V^\ell \\ a_1, \ldots, a_\ell}} \operatorname{Ann}_{U(L\mathfrak{g})} \left(V^1_{a_1} \otimes \cdots \otimes V^\ell_{a_\ell} \right)$

. $\operatorname{Ker} \overline{\Phi}_\ell \triangleleft U(L\mathfrak{g})$

§2 The question and the partial answer

Is the ideal lattice of $U(Lg)$ tractable?

???

<u>Thm</u> (Petukhov – S.)

(1) $U(Lg)$ does not satisfy ACC on 2-sided ideals

(2) $\mathrm{Im}\,\Phi_\ell$ does not satisfy ACC on 2-sided ideals

(3) If $0 \neq J \triangleleft U(Lg)$ then

$\exists\, \ell$ s.t. $\quad J \supseteq \ker \Phi_\ell$

§3 The proof $\mathfrak{g} = \mathfrak{sl}_2 = \mathbb{C}\cdot(e, h, f)$

§3.1 Pf (1, 2) $l = 1$

$$Z(U(\mathfrak{sl}_2)[t, t^{-1}]) = \mathbb{C}[\Omega, t, t^{-1}]$$

$$\Omega = ef - h^2 - h \in Z(U(\mathfrak{sl}_2))$$ Casimir

$$Z(\mathrm{Im}\,\Phi_1) = \mathrm{Im}\,\Phi_1 \cap \mathbb{C}[\Omega, t, t^{-1}]$$

$$= \mathbb{C} \oplus \Omega\,\mathbb{C}[\Omega, t, t^{-1}]$$

not noetherian $\square$

§ 3.2 H-ideals of S(E)

Let $E = \text{"}Le\text{"} = e \cdot \mathbb{C}[s, s^{-1}]$ $H = \text{"}Lh\text{"} = h \cdot \mathbb{C}[s, s^{-1}]$

$$E \subseteq Lsl_2 \Rightarrow \underset{\underset{S(E)}{\shortparallel}}{U(E)} = \mathbb{C}[\underset{\underset{es^i}{\shortparallel}}{e(i)} \mid i \in \mathbb{Z}] \subseteq U(Lsl_2)$$

If $0 \neq J \trianglelefteq U(Lsl_2)$ then

- $J \cap S(E) \neq 0$ b/c $S(E) \not\hookrightarrow \dfrac{U(Lsl_2)}{J}$ ← finite growth
- $\theta \in J \Rightarrow h(i)\theta - \theta h(i) \in J$

  $\Rightarrow J \cap S(E)$ is preserved by adjoint action of H

Def We say $J \cap S(E)$ is an <u>H-ideal</u> of $S(E)$

Lemma If I is an H-ideal of $S(E)$ then I is graded and each I_k is an H-subrep

Pf: $h(i)\cdot e(j) = 2e(i+j)$

$h(0)\cdot(m = e(i_1)\cdots e(i_k)) = 2km$ $\square$

§ 3.3 H-subreps of $S^k(E) = S(E)_k$

Notation

$$\begin{array}{ccc} S^k(E) & \longleftrightarrow & \mathrm{Sym}^k(E) = (E^{\otimes k})^{\mathfrak{S}_k} \\ \uparrow & & \cap \\ E^{\otimes k} & & E^{\otimes k} \end{array}$$

Let's look at H-subreps of $\mathrm{Sym}^k(E)$

Let $Z_k = \mathbb{C}[x_1^{\pm 1}, \dots, x_k^{\pm 1}]$

$$E^{\otimes k} \underset{\tau_k}{\overset{g_k}{\rightleftarrows}} Z_k \qquad \circlearrowleft H$$

$\exists$ bijections

$$\begin{array}{ccc} e(i_1) \otimes \cdots \otimes e(i_k) & \longmapsto & x_1^{i_1} \cdots x_k^{i_k} \\ \Big\downarrow h(j)\cdot & & \Big\downarrow h(j)\cdot \\ 2e(i_1+j) \otimes e(i_2) \otimes \cdots \otimes e(i_k) + \cdots + 2e(i_1) \otimes \cdots \otimes e(i_k+j) & \longmapsto & x_1^{i_1} \cdots x_k^{i_k} \big(\underbrace{2x_1^j + \cdots + 2x_k^j}_{\in Z_k^{\sigma_k}} \big) \end{array}$$

Prop. H-subreps of $\mathrm{Sym}^k(E) \leftrightarrow$ ideals of $Z_k^{\sigma_k}$

This gives procedure to obtain a $\mho_\infty$-invariant ideal of $Z_\infty = \mathbb{C}[x_i^{\pm}, \dots]$ from an H-ideal I of $S(E)$:

$$I \overset{(H)}{\longmapsto} \left\langle \sum_{k \geq 1} g_k(I_k) \cdot Z_\infty \right\rangle_{\mho_\infty} \trianglelefteq_{\mho_\infty} Z_\infty$$

§ 3.4 Discriminants

Thm (Cohen, 60's) (1) Z_∞ satisfies ACC on $\mho_\infty$-ideals

(2) Let $0 \neq K \trianglelefteq_{\mho_\infty} Z_\infty$.

$\exists\, \ell$ s.t. $P_\ell = \prod_{1 \leq i < j \leq \ell} (x_i - x_j) \in K$

Prop (Petukhov – S.)

(1) (H) is not bijective

(2) $S(E)$ has ACC on H-ideals

(3) Define $\Delta_\ell \in S(E)$ by:

$$
\begin{tikzcd}
P_\ell \arrow[r, mapsto] \arrow[d, phantom, sloped, "\in"] & P_\ell^2 \arrow[r, mapsto] \arrow[d, phantom, sloped, "\in"] \arrow[d, phantom, "\sigma_\ell"'] & D_\ell \arrow[r, mapsto] \arrow[d, phantom, sloped, "\in"] & \Delta_\ell \arrow[d, phantom, sloped, "\in"] \\
Z_\ell & Z_\ell \arrow[r, "\sim", "\tau_\ell"'] & \mathrm{Sym}^\ell(E) \arrow[r, leftrightarrow] & S^\ell(E)
\end{tikzcd}
$$

Any H-ideal of $S(E)$ contains some Δ_ℓ

$\ell = 2$: $\mathbb{Z}_2$ $\qquad P_2 = x_1 - x_2$

$\mathbb{Z}_2^{\sigma_2}$ $\qquad P_2^2 = x_1^2 - 2x_1x_2 + x_2^2$

$\mathrm{Sym}^2(E)$ $\qquad D_2 = e(2) \otimes e(0) - 2e(1) \otimes e(1) + e(0) \otimes e(2)$

$S^2(E)$ $\qquad \Delta_2 = 2\left(e(0)e(2) - e(1)^2\right) \overset{\overline{\Phi}_1}{\longmapsto} 0$

$\overline{\Phi}_1 : e(i)e(j) \mapsto e^2 t^{i+j}$

$\Rightarrow \Delta_2 \in \ker \overline{\Phi}_1$

(in general, $\Delta_{\ell+1} \in \ker \overline{\Phi}_\ell$)

§ 3.5 Pf of Thm (3)

$0 \neq J \trianglelefteq U(Lsl_2)$

By Prop, $J \cap S(E) \ni \Delta_\ell$ for some ℓ

$$J \supseteq (\Delta_\ell) \supseteq \operatorname{Ker} \Phi_{3\ell}$$

$$\trianglelefteq$$
$$U(Lsl_2)$$

$\square$

Cor: W a fin dim rep of Lg.

Then $\exists \ell$ s.t. $\operatorname{Ker} \Phi_\ell \subseteq \operatorname{Ann}_{U(Lg)}(W)$

Thank you!