Overrings of Sklyanin algebras

Sue Sierra

University of Edinburgh

29 September 2017

This is joint work with Dan Rogalski and Toby Stafford.

Let $a, b, c \in \mathbb{C}$. The *Sklyanin algebra* $S = S_{a,b,c}$ is:

$$S = \mathbb{C}\langle x, y, z\rangle/(axy + byx + cz^2, ayz + bzy + cx^2, azx + bxz + cy^2).$$

Theorem (Artin-Tate-Van den Bergh)

For $[a : b : c] \in \mathbb{P}^2 \smallsetminus \{$ known finite set $\}$, then S is a (left and right) noetherian graded domain of global dimension 3 and Hilbert series $1/(1-t)^3$.

Depending on $[a : b : c]$ either $Z(S) = \mathbb{C}[g]$ for some $g \in S_3$, or S satisfies a polynomial identity.

For $[a : b : c]$ (very) general, $Z(S) = \mathbb{C}[g]$.

S is *Artin-Schelter regular*: a good analogue of $\mathbb{C}[x, y, z]$.

For us, $[a : b : c]$ is always general: we assume $Z(S) = \mathbb{C}[g]$.

Question

What are the (nice) rings birational *to S?*

More specifically: What are the

- noetherian domains R
- that are connected graded ($R = \bigoplus_{n \in \mathbb{N}} R_n$, $R_0 = \mathbb{C}$)
- and so that

$$R \subseteq Q_{gr}(S) := S[h^{-1} : h \neq 0 \in S, h \text{ homogeneous }] = Q_{gr}(R)?$$

(Part of general programme to classify (coordinate rings of) *noncommutative projective surfaces*: connected graded noetherian domains of Gelfand-Kirillov dimension 3.)

Theorem (Rogalski-S.-Stafford)

The connected graded subrings *of S that are birational to S and are* maximal orders *are classified. (They are automatically noetherian!)*

(A commutative domain that is a maximal order is integrally closed in its field of fractions.)

Question

What about connected graded overrings *of S?*

There are "cheap" overrings of S:

Example

Let $R = \mathbb{C}\langle S_4 g^{-1} \rangle$.

$R \cong S^{(4)} := \bigoplus_{n\in\mathbb{N}} S_{4n}$ is a noetherian maximal order.

Then $S = \mathbb{C}\langle S_1 g g^{-1} \rangle \subset R \subset Q_{gr}(S)$.

There are "bad" overrings of S:

Example

Let $R' = S\langle xzy^{-1} \rangle$. Then R' is not noetherian. (Not obvious!)

We will show that all connected graded overrings of S are either cheap or bad!

The commutative situation is very different:

Example

Let $f \in \mathbb{C}[x, y, z]_2$ and let $R = \mathbb{C}[x, y, z, fx^{-1}]$.

All such R are noetherian overrings of $\mathbb{C}[x, y, z]$.

Theorem (RSS)

Let $S \subseteq R \subseteq Q_{gr}(S)$, where R is a ring. If R is a connected graded noetherian maximal order, then

$$R = \mathbb{C}\langle S_{3n+1} g^{-n} \rangle$$

for some n. ("R is cheap.")

In particular:

Theorem (RSS)

Let

$$S \subseteq R \subseteq S_{(g)} := S[h^{-1} : h \in S \smallsetminus gS, h \text{ homogeneous }],$$

where R is a ring. If R is connected graded and noetherian, then

$$R = S.$$

We'll prove the second theorem, about subrings of $S_{(g)}$. For technical reasons work with $T = S^{(3)} = \bigoplus S_{3n}$.

We'll prove:

Theorem

Let $T \subseteq R \subseteq T_{(g)}$, where R is connected graded and noetherian. Then $R = T$.

(This easily implies the result about overrings of S.)

Assume that $T \subsetneq R$.

Instead of T, look at:

- $B := T/Tg$, (coordinate ring of a) “closed set” in the “projective variety” defined by T.
- $T^\circ := T[g^{-1}]_0$, coordinate ring of an “affine open set.”

Here $B = B(E, \mathcal{L}, \tau)$ is a *twisted homogeneous coordinate ring* of an elliptic curve E. (Don't worry about $\mathcal{L}$ and τ for now.)

On the other hand, T° is the “coordinate ring of the complement of the elliptic curve.”

T° is a simple hereditary noetherian domain of GK-dimension 2 (ATV), the *elliptic Weyl algebra*.

We can move back and forth between (torsionfree) T and T°-modules:

For a graded T-module M define $M^\circ = (M \otimes_T T[g^{-1}])_0$.

For $N \subseteq Q(T^\circ)$ define

$$\widehat{N} = N[g, g^{-1}] \cap T_{(g)}.$$

We have $(\widehat{N})^\circ = N$.

If $M \subseteq T_{(g)}$ then $\widehat{M^\circ} = \{x \in T_{(g)} : xg^n \in M \text{ for some } n \geqslant 0\} \supseteq M$.

Lemma (Finiteness of hats)

Let $T \subseteq R \subseteq T_{(g)}$, where R is a connected graded noetherian algebra. Then $\dim_{\mathbb{C}} \widehat{R^\circ}/R < \infty$.

(For example, $T = \widehat{T^\circ}$.)

Replace R by $\widehat{R^\circ}$ if necessary to assume from now on that we have $T \subsetneqq R \subseteq T_{(g)}$, where R is connected graded, noetherian, and $R = \widehat{R^\circ}$.

The assumption that $R = \widehat{R^\circ}$ means that there is an extremely tight relationship between R-modules and R°-modules.

Let $T \subseteq M \subseteq R$, where $K := M/T$ is a GK 2-critical ("irreducible") T-module. This means K° is simple.

Key fact:

Theorem (Goodearl)

Because T° is hereditary, R° is a categorical localisation *of T°.*

That is, if we have $T^\circ \subseteq \mathcal{M} \subseteq Q(T^\circ)$ with $\mathcal{M}/T^\circ \cong K^\circ$, then

$$\mathcal{M} \subseteq R^\circ.$$

Corollary

If $T \subseteq M' \subseteq T_{(g)}$ with $(M'/T)^\circ \cong K^\circ$, then $M' \subseteq R$.

(Here we use that $R = \widehat{R^\circ}$.)

This suggests that R is likely to be big.

Recall $B = B(E, \mathcal{L}, \tau) \cong T/Tg$ is a twisted homogeneous coordinate ring of the curve E.

We have $Q_{gr}(B) = \mathbb{C}(E)[t, t^{-1}; \tau]$ where $\tau \in \mathrm{Aut}(E)$ is an infinite order (because $Z(S)$ small) translation.

$\mathcal{L}$ is an invertible sheaf on E and $B_1 = H^0(E, \mathcal{L})$.

Fact: The GK 1-critical graded right B-modules are in bijection with points on E:

$$p \leftrightarrow M(p) := B/H^0(E, \mathcal{L}(-p))B$$

These are *point modules*: $\operatorname{hilb} M(p) = 1/(1-t)$.

(In fact, these are all of the GK 1-critical graded T-modules.)

Let $T \subseteq M \subseteq R$ and $K = M/T$ as before, and consider K/Kg, which is a GK-dimension 1 B-module. It has a composition series whose factors are point modules; the number of points is the *multiplicity* of K.

Now we have two key lemmas:

Lemma

Because T is the 3-Veronese of S, all GK 2 T-modules have multiplicity > 1 (in fact at least 3).

Lemma

Because K has multiplicity > 1, there is K' so that:

1. $(K')^{\circ} \cong K^{\circ}$
2. *We have $T \subseteq M' \subseteq T_{(g)}$ with $K' \cong M'/T$*
3. $\exists y \in M'_0 \smallsetminus \mathbb{C}$

(Proof: $\mathrm{Ext}^1_T(K, T)$ is controlled by the composition factors of K/Kg.)

Now we have our contradiction:

- By the corollary to Goodearl's Theorem (using finiteness of hats), we have $M' \subseteq R$.
- So $\mathbb{C}[y] \subseteq R_0$.
- But R is connected graded.

So $R = T$, proving the theorem.

Theorem

Let $T^\circ \subsetneqq R \subseteq Q(T^\circ)$*. Then* $\operatorname{GKdim} R \geqslant 3 > \operatorname{GKdim} T^\circ = 2$*.*

Comes from:

Proposition

Let $S \subsetneqq R \subseteq S_{(g)}$*, where* R *is noetherian. Then* $\operatorname{GKdim} R \geqslant 4$*.*

Proof: R contains T and $\mathbb{C}[y]$. Now work modulo g to do explicit calculations in $\mathbb{C}(E)[t;\tau]$.

Compare:

Theorem (Makar-Limanov)

Let $A_1[x^{-1}] \subsetneqq R \subseteq Q(A_1)$*. Then* $\operatorname{GKdim} R = \infty$*.*

Fun fact:

Theorem (RSS)

All subalgebras of T° are finitely generated and noetherian.

(The comparable statement is false for subalgebras of A_1 or of any commutative ring of Krull dimension 2.)

Q: What about other rings birational to S? or T?

We are starting to understand inclusions among "nice" subrings of $Q_{gr}(S)$ but are far from a general classification.

There are good analogues of "blowing up a point" or "contracting a (-1) curve".

Q: Are all connected graded noetherian rings birational to S produced by (a composition of) such procedures?

Q: What is the "birational classification" of connected graded noetherian domains of GK-dimension 3?

Conjecture (Artin)

If R is a connected graded noetherian domain of GK-dimension 3, then $Q_{gr}(R)_0$ is one of:

- *a division ring finite over a central field of transcendence degree 2*
- $Q(K(t;\sigma))$ *or* $Q(K(t;\delta))$ *where* $\operatorname{trdeg} K = 1$
- $Q_{gr}(S)_0$ *where* $S = S_{abc}$ *is a Sklyanin algebra.*