Magnitude and magnitude homology

Tom Leinster

Edinburgh

Plan

1. Magnitude, generally
2. The magnitude of a metric space
3. Magnitude homology, generally
4. The magnitude homology of a metric space

1. Magnitude, generally

Size

For many types of mathematical object, there is a canonical notion of size.

- Sets have cardinality. It satisfies

$$\begin{aligned} |X \cup Y| &= |X| + |Y| - |X \cap Y| \\ |X \times Y| &= |X| \times |Y| . \end{aligned}$$

- Subsets of $\mathbb{R}^n$ have volume. It satisfies

$$\begin{aligned} \mathrm{vol}(X \cup Y) &= \mathrm{vol}(X) + \mathrm{vol}(Y) - \mathrm{vol}(X \cap Y) \\ \mathrm{vol}(X \times Y) &= \mathrm{vol}(X) \times \mathrm{vol}(Y). \end{aligned}$$

- Topological spaces have Euler characteristic. It satisfies

$$\begin{aligned} \chi(X \cup Y) &= \chi(X) + \chi(Y) - \chi(X \cap Y) \quad \text{(under hypotheses)} \\ \chi(X \times Y) &= \chi(X) \times \chi(Y). \end{aligned}$$

Challenge Find a general definition of 'size', including these and other examples.

One answer The magnitude of an enriched category.

The magnitude of an enriched category

Let $\mathscr{V} = (\mathscr{V}, \otimes, I)$ be a monoidal category equipped with a 'notion of the size of its objects'.

Formally, suppose we have a function

$$|\cdot| : \mathrm{ob}\,\mathscr{V} \to k$$

taking values in some semiring k, such that

$$U \cong V \Rightarrow |U| = |V|, \qquad |U \otimes V| = |U|\,|V|, \qquad |I| = 1.$$

Given a $\mathscr{V}$-enriched category $\mathbf{X}$ with finitely many objects, define a matrix

$$Z_{\mathbf{X}} = \big(|\mathbf{X}(x, y)|\big)_{x,y\in\mathbf{X}}.$$

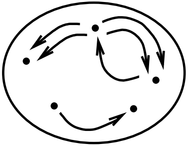
If $Z_{\mathbf{X}}$ is invertible, the magnitude of $\mathbf{X}$ is

$$|\mathbf{X}| = \sum_{x,y\in\mathbf{X}} Z_{\mathbf{X}}^{-1}(x, y) \in k.$$

It can be understood as a notion of the size of $\mathbf{X}$.

The magnitude of an ordinary category

Take $\mathscr{V} = \textbf{FinSet}$ with usual cardinality $|\cdot|$.
This gives a notion of the magnitude $|\mathbf{X}| \in \mathbb{Q}$ of a finite category $\mathbf{X}$.



'Recall': $\mathbf{X}$ gives rise to a topological space $B\mathbf{X}$ (its classifying space), built as follows:

- for each object of $\mathbf{X}$, put a point $\bullet$ into $B\mathbf{X}$;
- for each map $x \to y$ in $\mathbf{X}$, put an interval $\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet$ into $B\mathbf{X}$;
- for each commutative triangle in $\mathbf{X}$, put a 2-simplex $\blacktriangle$ into $B\mathbf{X}$;
- ...

Theorem Let $\mathbf{X}$ be a finite category. Then

$$|\mathbf{X}| = \chi(B\mathbf{X}),$$

under hypotheses ensuring that $\chi(B\mathbf{X})$ is well-defined.

The magnitude of a metric space

Let $\mathscr{V}$ be the category whose objects are the elements of $[0, \infty]$, with one map $u \to v$ when $u \geq v$, and with no maps $u \to v$ otherwise.

It is a monoidal category under addition.

Any metric space gives a $\mathscr{V}$-enriched category $\mathbf{X}$:

- the objects of $\mathbf{X}$ are the points;
- $\mathbf{X}(x, y) = d(x, y) \in [0, \infty]$;
- composition in $\mathbf{X}$ is the triangle inequality.

We have the 'size function'

$$\begin{array}{rccc} |\cdot| : & [0, \infty] & \to & \mathbb{R}, \\ & u & \mapsto & e^{-u}, \end{array}$$

which has the required properties that $e^{-(u+v)} = e^{-u} e^{-v}$, etc.

So we get a new invariant: the 'magnitude of a metric space'.

2. The magnitude of a metric space

— done explicitly —

The magnitude of a finite metric space

Let X be a finite metric space.

Write Z_X for the $X \times X$ matrix with entries

$$Z_X(x, y) = e^{-d(x,y)}$$

$(x, y \in X)$.

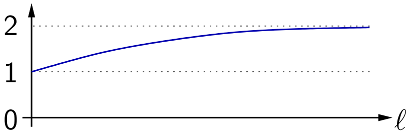
If Z_X is invertible (which it is if $X \subseteq \mathbb{R}^n$), the magnitude of X is

$$|X| = \sum_{x,y \in X} Z_X^{-1}(x, y) \ \in \mathbb{R}$$

—the sum of all the entries of the inverse matrix of Z_X.

First examples

- $|\emptyset| = 0$.
- $|\bullet| = 1$.
- $\left|\bullet \overset{\leftarrow\ \ell\ \rightarrow}{} \bullet\right| = \text{sum of entries of } \begin{pmatrix} e^{-0} & e^{-\ell} \\ e^{-\ell} & e^{-0} \end{pmatrix}^{-1} = \dfrac{2}{1+e^{-\ell}}$

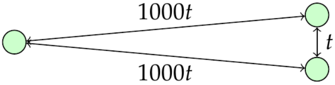


- If $d(x, y) = \infty$ for all $x \neq y$ then $|X| = \text{cardinality}(X)$.

Slogan: Magnitude is the 'effective number of points'.

Example: a 3-point space (Simon Willerton)

Take the 3-point space

$X =$ 

- When t is small, X looks like a 1-point space.

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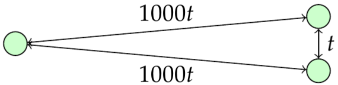
$X =$ 

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- When t is moderate, X looks like a 2-point space.

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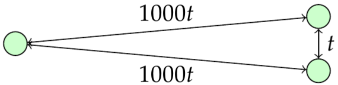
$X =$ 

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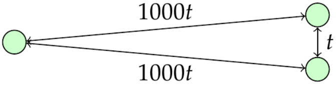
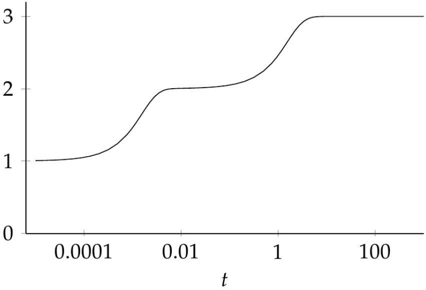
$X =$ 

- When t is small, X looks like a 1-point space.
- When t is moderate, X looks like a 2-point space.
- When t is large, X looks like a 3-point space.

Indeed, the magnitude of X as a function of t is:



Magnitude functions

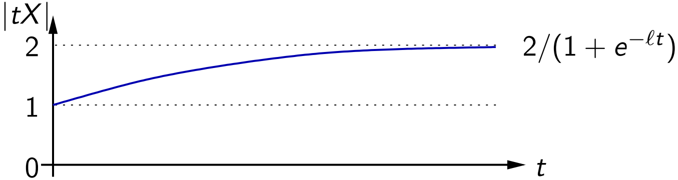
Magnitude assigns to each metric space not just a *number*, but a *function*.

For $t > 0$, write tX for X scaled up by a factor of t.

The magnitude function of a metric space X is the partially-defined function

$$\begin{array}{ccc} (0,\infty) & \to & \mathbb{R} \\ t & \mapsto & |tX|. \end{array}$$

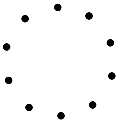
E.g.: the magnitude function of $X = (\bullet \xleftrightarrow{\;\ell\;} \bullet)$ is



A magnitude function has only finitely many singularities (none if $X \subseteq \mathbb{R}^n$).

It is increasing for $t \gg 0$, and $\lim\limits_{t\to\infty} |tX| = \text{cardinality}(X)$.

Dimension at different scales

Let $X =$  , with subspace metric from $\mathbb{R}^2$.

- When t is small, tX looks 0-dimensional.

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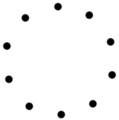
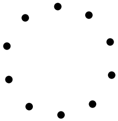
Let $X =$  , with subspace metric from $\mathbb{R}^2$.

- When t is small, tX looks 0-dimensional.
- When t is moderate, tX looks nearly 1-dimensional.



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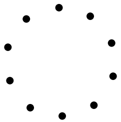
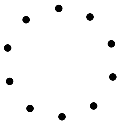
Let $X =$  , with subspace metric from $\mathbb{R}^2$.

- When t is small, tX looks 0-dimensional.
- When t is moderate, tX looks nearly 1-dimensional.
- When t is large, tX looks 0-dimensional again.

Dimension at different scales

Let $X =$  , with subspace metric from $\mathbb{R}^2$.

- When t is small, tX looks 0-dimensional.
- When t is moderate, tX looks nearly 1-dimensional.
- When t is large, tX looks 0-dimensional again.

The magnitude function sees all this!

Here's how. . .

Dimension at different scales (Willerton)

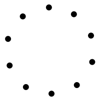
For a function $f\colon (0,\infty) \to \mathbb{R}$, the instantaneous growth of f at $t \in (0,\infty)$ is

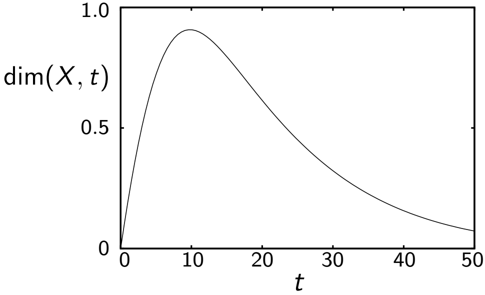
$$\mathrm{growth}(f,t) = \frac{d(\log f(t))}{d(\log t)} = \text{slope of the log-log graph of } f \text{ at } t.$$

E.g.: If $f(t) = Ct^n$ then $\mathrm{growth}(f,t) = n$ for all t.

For a space X, the magnitude dimension of X at scale t is

$$\dim(X,t) = \mathrm{growth}(|tX|\,,t).$$

E.g.: Let $X =$  .



The magnitude of a compact metric space

A metric space M is positive definite if for every finite $Y \subseteq M$, the matrix Z_Y is positive definite.

E.g.: $\mathbb{R}^n$ with Euclidean or taxicab metric; sphere with geodesic metric; hyperbolic space; any ultrametric space.

Theorem (Mark Meckes)
All sensible ways of extending the definition of magnitude from finite metric spaces to compact positive definite spaces are equivalent.

For a compact positive definite space X,

$$|X| = \sup\{|Y| \ : \ \text{finite } Y \subseteq X\}.$$

Example: the magnitude of a rectangle

The straight line $[0, L]$ of length L has magnitude $1 + \frac{1}{2}L$.

So $[0, L]$ has magnitude function

$$t \mapsto |t[0, L]| = |[0, tL]| = \underbrace{\boxed{1}}_{\text{Euler characteristic}} + \tfrac{1}{2}\underbrace{\boxed{L}}_{\text{length}} \cdot t^{\overbrace{\boxed{1}}^{\text{dimension}}}$$

For metric spaces A and B, let $A \times_1 B$ be their 'ℓ^1 product', given by

$$d_{A \times_1 B}\big((a, b), (a', b')\big) = d_A(a, a') + d_B(b, b').$$

Lemma $|A \times_1 B| = |A|\,|B|$.

It follows that the rectangle $[0, L_1] \times_1 [0, L_2]$ has magnitude function

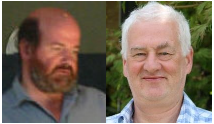
$$t \mapsto \underbrace{\boxed{1}}_{\text{Euler characteristic}} + \tfrac{1}{2}\underbrace{\boxed{(L_1 + L_2)}}_{\text{semiperimeter}}t + \tfrac{1}{4}\underbrace{\boxed{L_1 L_2}}_{\text{area}}t^{\overbrace{\boxed{2}}^{\text{dimension}}}$$

So, the magnitude function of a rectangle knows its Euler characteristic, perimeter, area and dimension!

Magnitude encodes geometric information

Theorem (Juan-Antonio Barceló & Tony Carbery)

For compact $X \subseteq \mathbb{R}^n$ (with Euclidean metric),



$$\mathrm{vol}_n(X) = C_n \lim_{t\to\infty} \frac{|tX|}{t^n}$$

where C_n is a known constant.

Theorem (Heiko Gimperlein & Magnus Goffeng)

Assume n is odd. For 'nice' compact $X \subseteq \mathbb{R}^n$
(meaning that ∂X is smooth and $\mathrm{Cl}(\mathrm{Int}(X)) = X$),

$$|tX| = c_n \,\mathrm{vol}_n(X) t^n + c_{n-1} \,\mathrm{vol}_{n-1}(\partial X) t^{n-1} + O(t^{n-2})$$

as $t \to \infty$, where c_n and c_{n-1} are known constants.

The magnitude function knows the volume and the surface area.

Magnitude encodes geometric information

Magnitude satisfies an asymptotic inclusion-exclusion principle:

Theorem (Gimperlein & Goffeng)

Assume n is odd. Let $X, Y \subseteq \mathbb{R}^n$ with X, Y and $X \cap Y$ nice. Then

$$|t(X \cup Y)| + |t(X \cap Y)| - |tX| - |tY| \to 0$$

as $t \to \infty$.

It also encodes Minkowski dimension, one of the most important notions of fractional dimension:

Theorem (Meckes)

The Minkowski dimension of a compact subset of $\mathbb{R}^n$ is equal to the asymptotic growth of its magnitude function.

The magnitude of the Euclidean ball

Not all results on magnitude are asymptotic!

Let B^n denote the unit ball in $\mathbb{R}^n$.

Theorem (Barceló & Carbery; Willerton)
Assume n is odd. Then $\left|tB^n\right|$ is a known rational function of t over $\mathbb{Z}$.

Examples

- $\left|tB^1\right| = |[-t, t]| = 1 + t$
- $\left|tB^3\right| = 1 + 2t + t^2 + \frac{1}{3!}t^3$
- $\left|tB^5\right| = \dfrac{24 + 72t^2 + 35t^3 + 9t^4 + t^5}{8(3+t)} + \dfrac{t^5}{5!}.$

3. Magnitude homology, generally

Two perspectives on Euler characteristic

So far: Euler characteristic has been treated as an analogue of cardinality.

Alternatively: Given any homology theory H_* of any kind of object X, can define

$$\chi(X) = \sum_{n=0}^{\infty} (-1)^n \operatorname{rank} H_n(X).$$

Note:

- $\chi(X)$ is a *number*
- $H_*(X)$ is an *algebraic structure*, and functorial in X.

We say that H_* is a categorification of χ.

So, homology categorifies Euler characteristic.

The homology of an ordinary category

Any *ordinary* category $\mathbf{X}$ gives rise to a chain complex $C_*(\mathbf{X})$:

$$C_n(\mathbf{X}) = \coprod_{x_0,\ldots,x_n \in \mathbf{X}} \mathbb{Z} \cdot \big(\mathbf{X}(x_0, x_1) \times \cdots \times \mathbf{X}(x_{n-1}, x_n)\big)$$

where $\mathbb{Z} \cdot - \colon \mathbf{Set} \to \mathbf{Ab}$ is the free abelian group functor.

The homology $H_*(\mathbf{X})$ of $\mathbf{X}$ is the homology of $C_*(\mathbf{X})$.

Theorem (classical) $H_*(\mathbf{X}) = H_*(B\mathbf{X})$.

Since

$$\chi(B\mathbf{X}) = \sum (-1)^n \operatorname{rank} H_n(B\mathbf{X}),$$

it follows that

$$|\mathbf{X}| = \sum (-1)^n \operatorname{rank} H_n(\mathbf{X})$$

—for categories, homology categorifies magnitude.

The magnitude homology of an enriched category (Michael Shulman)

Let $(\mathscr{V}, \otimes)$ be a monoidal category whose unit object is terminal. (E.g. **Set** or $[0, \infty]$.)

Let $A \colon \mathscr{V} \to \mathbf{Ab}$ be a functor.

Any $\mathscr{V}$-enriched category $\mathbf{X}$ gives rise to a chain complex $C_*(\mathbf{X}, A)$:

$$C_n(\mathbf{X}, A) = \coprod_{x_0, \ldots, x_n \in \mathbf{X}} A\big(\mathbf{X}(x_0, x_1) \otimes \cdots \otimes \mathbf{X}(x_{n-1}, x_n)\big).$$

Definition: The magnitude homology $H_*(\mathbf{X}, A)$ of $\mathbf{X}$ with coefficients in A is the homology of $C_*(\mathbf{X}, A)$.

Example: When $\mathscr{V} = \mathbf{Set}$ and $A = \mathbb{Z} \cdot - \colon \mathbf{Set} \to \mathbf{Ab}$, this is the ordinary homology of a category.

The general definition also gives a homology theory of metric spaces...

4. The magnitude homology of a metric space

Special case of graphs: Hepworth and Willerton (2015)

General case of enriched categories: Shulman and Leinster (2017)

The shape of the definition

For this talk, a persistence module is a functor

$$A\colon ([0,\infty], \geq) \to \mathbf{Ab}.$$

That is: it's a family $(A(\ell))_{\ell \in [0,\infty]}$ of abelian groups, together with a homomorphism $\alpha_{\ell,k}\colon A(\ell) \to A(k)$ whenever $\ell \geq k$, such that $\alpha_{\ell,k} \circ \alpha_{m,\ell} = \alpha_{m,k}$ and $\alpha_{\ell,\ell} = \mathrm{id}$.

The general definition of magnitude homology specializes to give a definition of

$$H_*(X, A),$$

the magnitude homology of a metric space X with coefficients in a persistence module A.

Each $H_n(X, A)$ is an abelian group.

The definition, explicitly

Let X be a metric space and let A be a persistence module.

There is a chain complex $C_*(X, A)$ with

$$C_n(X, A) = \coprod_{x_0, \dots, x_n \in X} A\big(d(x_0, x_1) + \cdots + d(x_{n-1}, x_n)\big).$$

The differential is

$$\partial = \sum_{i=0}^{n} (-1)^i \partial_i \colon C_n(X, A) \to C_{n-1}(X, A)$$

where (e.g.) in the case $n = 2$, the maps $\partial_0, \partial_1, \partial_2$ are given as follows:

> the inequality $d(x_0, x_1) + d(x_1, x_2) \geq d(x_1, x_2)$ induces
> a homomorphism $\partial_0 \colon A\big(d(x_0, x_1) + d(x_1, x_2)\big) \to A\big(d(x_1, x_2)\big)$.

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The magnitude homology $H_*(X, A)$ is the homology of $C_*(X, A)$.

Magnitude homology with coefficients in a point module

For each $\ell \in [0, \infty]$, define a persistence module A_ℓ by

$$A_\ell(k) = \begin{cases} \mathbb{Z} & \text{if } k = \ell \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$C_n(X, A_\ell) = \mathbb{Z} \cdot \left\{ (x_0, \ldots, x_n) : d(x_0, x_1) + \cdots + d(x_{n-1}, x_n) = \ell \right\}.$$

Equivalently, we can replace $C_*(X, A)$ by a normalized version:

$$C_n^\sharp(X, A_\ell) = \mathbb{Z} \cdot \left\{ (x_0, \ldots, x_n) : d(x_0, x_1) + \cdots + d(x_{n-1}, x_n) = \ell,\ x_0 \neq \cdots \neq x_n \right\}.$$

The differential is $\partial = \sum_{i=1}^{n-1} (-1)^i \partial_i$, where

$$\partial_i(x_0, \ldots, x_n) = \begin{cases} (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) & \text{if } x_i \text{ is between } x_{i-1} \text{ and } x_{i+1} \\ 0 & \text{otherwise.} \end{cases}$$

'Between' means that $d(x_{i-1}, x_i) + d(x_i, x_{i+1}) = d(x_{i-1}, x_{i+1})$.

H_1 detects convexity

A metric space X is Menger convex if for all distinct $x, y \in X$, there exists $z \in X$ between x and y with $x \neq z \neq y$.

Theorem Let X be a metric space. Then

$$X \text{ is Menger convex} \iff H_1(X, A_\ell) = 0 \text{ for all } \ell > 0.$$

Corollary Let X be a closed subset of $\mathbb{R}^n$. Then

$$X \text{ is convex} \iff H_1(X, A_\ell) = 0 \text{ for all } \ell > 0.$$

And, for instance, if

$$X = \quad \bullet\ \bullet \quad \bullet\bullet\ \bullet\bullet\ \bullet\ \bullet\ \bullet\ \bullet \quad \subseteq \mathbb{R}$$

with all gaps of length $< \varepsilon$, then $H_1(X, A_\ell) = 0$ for all $\ell \geq \varepsilon$.

Back to Euler characteristic

Let X be a metric space. For any persistence module A, put

$$\chi(X, A) = \sum_{n=0}^{\infty} (-1)^n \operatorname{rank} H_n(X, A)$$

(if defined). In particular, we have an Euler characteristic

$$\chi(X, A_\ell) = \sum_{n=0}^{\infty} (-1)^n \operatorname{rank} H_n(X, A_\ell)$$

for each $\ell \in [0, \infty)$. Not just one Euler characteristic: many!

Make these Euler characteristics into the coefficients of a formal expression:

$$\chi(X) = \sum_{\ell \in [0,\infty)} \chi(X, A_\ell)\, q^\ell.$$

Claim: $\chi(X)$ is formally equal to $|tX|$, where $q = e^{-t}$. So I'm claiming:

Magnitude homology categorifies magnitude

Magnitude homology categorifies magnitude: ‘proof’

$$
\begin{aligned}
\chi(X) &= \sum_{\ell\in[0,\infty)} \sum_{n\in\mathbb{N}} (-1)^n \operatorname{rank} H_n(X, A_\ell)\, q^\ell = \sum_{\ell,n} (-1)^n \operatorname{rank} C_n^\sharp(X, A_\ell)\, q^\ell \\
&= \sum_{n,\ell} (-1)^n \Big| \big\{ (x_0, \ldots, x_n) : d(x_0, x_1) + \cdots + d(x_{n-1}, x_n) = \ell, x_0 \neq \cdots \neq x_n \big\} \Big| q^\ell \\
&= \sum_n (-1)^n \sum_{x_0, \ldots, x_n \colon x_0 \neq \cdots \neq x_n} q^{d(x_0,x_1)+\cdots+d(x_{n-1},x_n)} \\
&= \sum_n (-1)^n \sum_{x_0, \ldots, x_n \in X} \left(q^{d(x_0,x_1)} - \delta_{x_0,x_1} \right) \cdots \left(q^{d(x_{n-1},x_n)} - \delta_{x_{n-1},x_n} \right) \\
&= \sum_n (-1)^n \sum_{x_0, \ldots, x_n \in X} (Z_{tX} - I)_{x_0,x_1} \cdots (Z_{tX} - I)_{x_{n-1},x_n} \\
&= \sum_n (-1)^n \operatorname{sum}\left((Z_{tX} - I)^n \right) \qquad \text{where } \textcolor{red}{\text{sum}} \text{ means the sum of all entries} \\
&= \operatorname{sum}\Bigg(\sum_{n\in\mathbb{N}} (I - Z_{tX})^n \Bigg) = \operatorname{sum}\Bigg(\big(I - (I - Z_{tX}) \big)^{-1} \Bigg) = \operatorname{sum}\big(Z_{tX}^{-1} \big) \\
&= |tX|
\end{aligned}
$$

. . . formally, at least!

Open questions

1. What information does the magnitude homology $H_*(X, A)$ capture when we use other persistence modules A as our coefficients?
2. What is the relationship between magnitude homology and persistent homology?
3. Which theorems about magnitude of metric spaces can be categorified to give theorems about magnitude homology?

   Compare:

   - Many theorems about *topological* Euler characteristic are shadows of theorems about homology.
   - For the special case of graphs, Hepworth and Willerton already proved a Künneth theorem (categorifying the formula for $|X \times Y|$) and a Mayer–Vietoris theorem (categorifying formula for $|X \cup Y|$).

References

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