The magnitude of a graph

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These slides: on my web page

Plan

1. Broad context: cardinality-like invariants

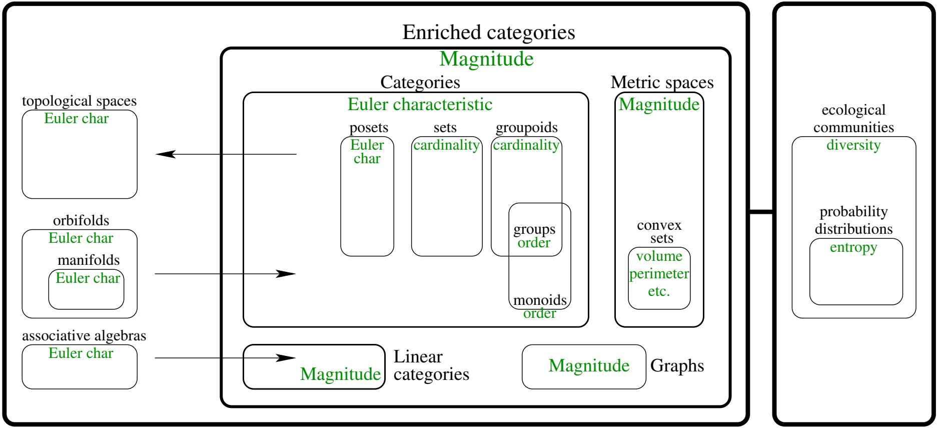
2. The magnitude of a graph

3. What next?

1. Broad context:

Cardinality-like invariants

The big picture (to be explained. . .)



The magnitude of a category

Let $\mathbf{A}$ be a finite category (finitely many objects and arrows).

There is a matrix $Z_{\mathbf{A}} = (|\mathrm{Hom}_{\mathbf{A}}(a,b)|)_{a,b\in\mathbf{A}}$ over $\mathbb{Q}$.

If $Z_{\mathbf{A}}$ is invertible, the magnitude or Euler characteristic of $\mathbf{A}$ is

$$|\mathbf{A}| = \sum_{a,b\in\mathbf{A}} (Z_{\mathbf{A}}^{-1})_{a,b} \in \mathbb{Q}.$$

Theorem

Let $\mathbf{A}$ *be a finite category containing no nontrivial endomorphisms or isomorphisms. Denote by* $B\mathbf{A}$ *its classifying space (geometric realization). Then*

$$|\mathbf{A}| = \chi(B\mathbf{A}).$$

Remark From a notion of the size $|X|$ of a finite set X, we obtained a notion of the size $|\mathbf{A}|$ of a finite category $\mathbf{A}$.

(And from $|X \times Y| = |X| \times |Y|$, we can deduce $|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}| \times |\mathbf{B}|$.)

The magnitude of a linear category

Let **A** be a linear category (the hom-sets are vector spaces and composition is bilinear), with finitely many objects and finite-dimensional hom-spaces.

There is a matrix $Z_{\mathbf{A}} = (\dim \mathrm{Hom}_{\mathbf{A}}(a, b))_{a,b\in\mathbf{A}}$ over $\mathbb{Q}$.

(Background thought: $\dim(X \otimes Y) = \dim(X) \times \dim(Y)$.)

If $Z_{\mathbf{A}}$ is invertible, the magnitude or Euler characteristic of **A** is

$$|\mathbf{A}| = \sum_{a,b\in\mathbf{A}} (Z_{\mathbf{A}}^{-1})_{a,b} \in \mathbb{Q}.$$

Theorem (with Joe Chuang and Alastair King)

Let A be an associative algebra of finite dimension and global dimension, over an algebraically closed field. Denote by **A** *the linear category of indecomposable projective A-modules. Then*

$$|\mathbf{A}| = \sum_{n\geq 0} (-1)^n \dim \mathit{Ext}^n_A(S, S)$$

where S is the direct sum of the simple A-modules.

The magnitude of a metric space

Let A be a finite metric space.

There is a matrix $Z_A = (e^{-d(a,b)})_{a,b \in A}$ over $\mathbb{R}$.

(Background thought: $e^{-(x+y)} = e^{-x} \times e^{-y}$.)

If Z_A is invertible, the magnitude of A is

$$|A| = \sum_{a,b \in A} (Z_A^{-1})_{a,b} \in \mathbb{R}.$$

Can extend this definition from finite metric spaces to many compact metric spaces, e.g. compact subsets of $\mathbb{R}^n$ (as shown by Mark Meckes).

Various results connect magnitude to classical geometric invariants...

Conjecture (with Simon Willerton)

Let A be a compact convex subset of $\mathbb{R}^2$. Then for all $t > 0$,

$$|tA| = \chi(A) + \frac{1}{4}\text{perimeter}(A) \cdot t + \frac{1}{2\pi}\text{area}(A) \cdot t^2.$$

The magnitude of a graph: one-slide version

Let G be a graph (finite, undirected, no loops or multiple edges).

There is a matrix $Z_G = (q^{d(a,b)})_{a,b \in G}$ over $\mathbb{Q}(q)$, where d is geodesic distance in G.

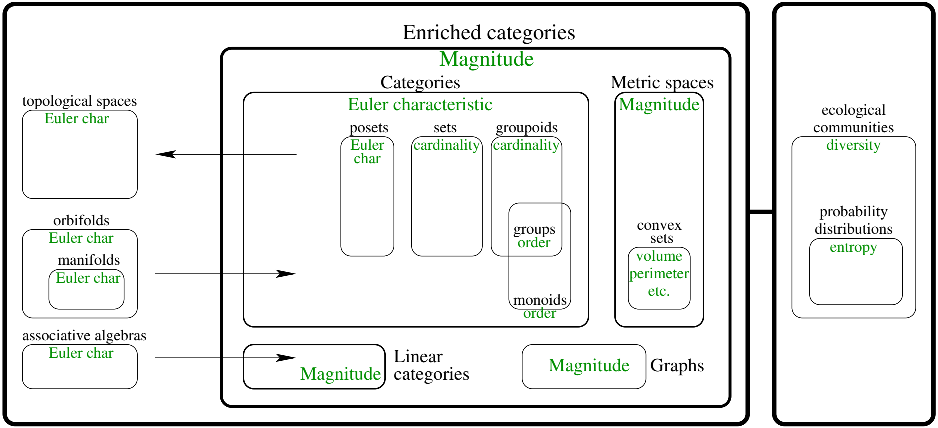
(By convention, $q^\infty = 0$; use this when a and b are in different components.)

The matrix Z_G is always invertible, and the magnitude of G is

$$\#G = \sum_{a,b \in G} (Z_G)^{-1}_{a,b} \in \mathbb{Q}(q).$$

The rest of the talk is about this definition.

The big picture, again



2. The magnitude of a graph

arXiv:1401.4623

The definition

Given a graph G, we defined a matrix

$$Z_G = Z_G(q) = (q^{d(a,b)})_{a,b \in G}$$

over $\mathbb{Q}(q)$.

Then $Z_G(0) = I$, so the polynomial $\det(Z_G(q))$ has constant term 1.

Hence $Z_G(q)$ is invertible in both $\mathbb{Q}(q)$ and $\mathbb{Z}[\![q]\!]$.

The magnitude of G is

$$\#G = \#G(q) = \sum_{a,b \in G} (Z_G^{-1})_{a,b}$$

and can be seen as an element of either $\mathbb{Q}(q)$ or $\mathbb{Z}[\![q]\!]$.

How to compute magnitude: Solve the equations

$$\sum_{b \in G} q^{d(a,b)} w_G(b) = 1 \qquad (a \in G)$$

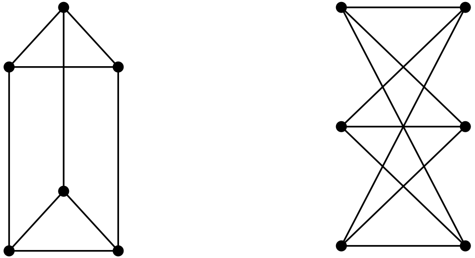
with $w_G(b)$ in $\mathbb{Q}(q)$ or $\mathbb{Z}[\![q]\!]$. Then $\#G = \sum_{a \in G} w_G(a)$.

Basic examples

- The edgeless graph with n vertices has magnitude n.
- Let G be a graph whose automorphism group acts transitively on vertices. Then
$$\#G = \frac{v(G)}{\sum_{a \in G} q^{d(x,a)}}$$
for any $x \in G$, where $v(G)$ is the number of vertices.
- The complete graph K_n on n vertices has magnitude
$$\#K_n = \frac{n}{1 + (n-1)q} = n \sum_{k=0}^{\infty} (1-n)^k q^k.$$
- The cycle graph C_n on n vertices has magnitude
$$\#C_n = \frac{n(q-1)}{q^{\lfloor (n+1)/2 \rfloor} + q^{\lceil (n+1)/2 \rceil} - q - 1}.$$
- The Petersen graph has magnitude
$$\frac{10}{1 + 3q + 6q^2} = 10 - 30q + 30q^2 + 90q^3 - 450q^4 + \cdots .$$

Basic properties

- Disjoint unions: $\#(G \sqcup H) = \#G + \#H$.
- Let $G \,\square\, H$ denote the (graph-theorists') cartesian product of graphs, which has an edge between (a, b) and (a', b') if either $a = a'$ and there is an edge between b and b', or vice versa. Then $\#(G \,\square\, H) = \#G \cdot \#H$.
- These two graphs have the same magnitude but different chromatic numbers:



So magnitude does not know the chromatic number.

Interlude on graph invariants

Every graph G has a Tutte polynomial $T_G(x, y) \in \mathbb{N}[x, y]$.

This is the king of graph invariants.

Many other invariants are specializations of the Tutte polynomial (e.g. chromatic number).

First question about any new graph invariant: is it a specialization of the Tutte polynomial?

For magnitude, trivially 'no', since the edgeless graph on n vertices has magnitude n but Tutte polynomial 1. But what about for connected graphs?

We'll see that the answer is again 'no'. So:

Magnitude contains information that the Tutte polynomial does not.

An explicit formula for magnitude

When G is viewed as a power series over $\mathbb{Z}$, what are its coefficients?

Can show:

$$\#G = \sum_{k=0}^{\infty} \sum_{a_0 \neq a_1 \neq \cdots \neq a_k} (-1)^k q^{d(a_0,a_1)+\cdots+d(a_{k-1},a_k)}.$$

Equivalently, $\#G = \sum_{n=0}^{\infty} c_n q^n$ where

$$c_n = \sum_{k=0}^{n} (-1)^k |\{(a_0, \ldots, a_k) \mid a_0 \neq a_1 \neq \cdots \neq a_k \text{ and } d(a_0, a_1) + \cdots + d(a_{k-1}, a_k) = n\}|.$$

In particular, $c_0 = v(G)$ and $c_1 = -2e(G)$. So magnitude knows the numbers of vertices and edges.

Inclusion-exclusion: a theorem

Fact There is no graph invariant I that is multiplicative with respect to $\square$ and satisfies the inclusion-exclusion formula

$$I(G \cup H) = I(G) + I(H) - I(G \cap H)$$

whenever G and H are subgraphs of some other graph. However:

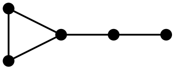
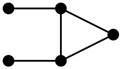
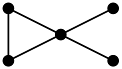
Theorem

Let G and H be subgraphs of a graph X, with $G \cup H = X$. Suppose that certain (not too demanding) conditions are satisfied. Then

$$\#X = \#G + \#H - \#(G \cap H).$$

Inclusion-exclusion: examples

- $\#(G \vee H) = \#G + \#H - 1$, where $\vee$ is one-vertex join.
- Hence all these graphs have the same magnitude:

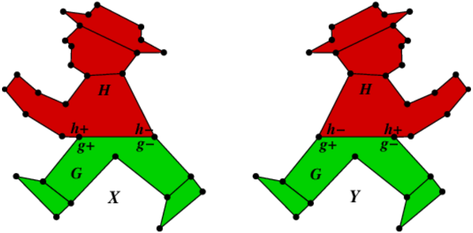
Explicitly, it's

$$\#C_3 + 2 \cdot \#C_2 - 2 = \frac{5 + 5q - 4q^2}{(1+q)(1+2q)}.$$

- All trees with e edges have magnitude $1 + e(1-q)/(1+q)$.
- (Meckes) Whenever G and H are subtrees of some tree, $\#(G \cup H) = \#G + \#H - \#(G \cap H)$.

Whitney twists

Graphs X and Y like this are said to differ by a Whitney twist:



Fact Graphs differing by a Whitney twist have the same Tutte polynomial.

Proposition (Speyer, Willerton) *There exists a pair of connected graphs that differ by a Whitney twist and have different magnitudes.*

Hence magnitude is not a specialization of the Tutte polynomial, even for connected graphs. Nevertheless:

Theorem *Let X and Y be graphs differing by a Whitney twist, such that the two points of identification are adjacent in X (or equivalently in Y). Then $\#X = \#Y$.*

3. What next?

Magnitude homology (Hepworth and Willerton)

Hepworth has categorified the magnitude of a graph.

That is, he has found a homology theory for graphs of which magnitude is the Euler characteristic (cf. Khovanov and the Jones polynomial).

Magnitude	Magnitude homology
Magnitude $\#G(q) \in \mathbb{Z}[\![q]\!]$	Bigraded abelian group $MH_{*,*}(G)$ with $\#G(q) = \sum_{k,n\geq 0} (-1)^k \mathrm{rank}(MH_{k,n}(G)) \cdot q^n$
$\#(G \,\square\, H) = \#G \cdot \#H$	Künneth theorem
$\#(G \cup H) = \#G + \#H - \#(G \cap H)$	Mayer–Vietoris theorem
Whitney twist theorem	?

Open questions

- Exactly what information does magnitude contain that the Tutte polynomial does not?
- What is the significance of the degree of $\#G(q)$, as a rational function of q?
- Can magnitude be extended to infinite graphs in a sensible way?
- What can we do with magnitude (co)homology?

Thanks