The magnitude of metric spaces I

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Parts joint with

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Simon Willerton (Sheffield)

These slides are available on my web page

Background

Cardinality-like invariants

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Metric spaces	have	magnitude

. . . and provide evidence that it subsumes many invariants of integral geometry.

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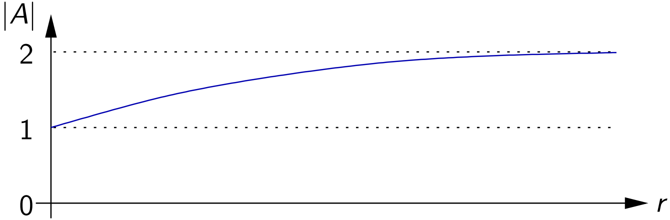

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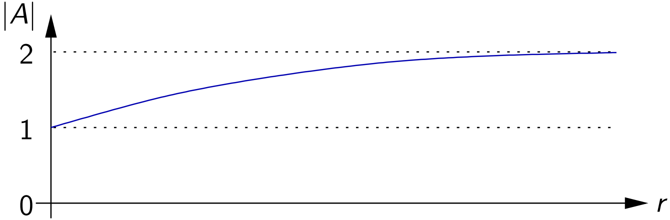

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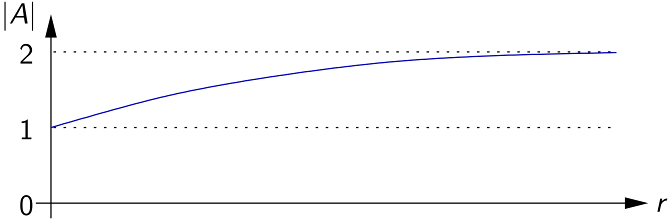

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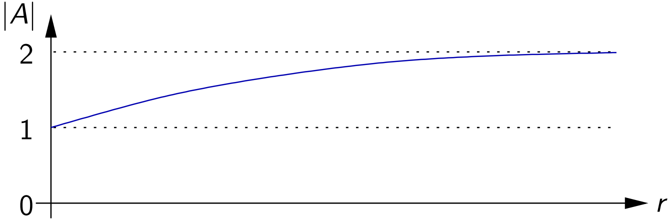

- If $d(a, b) = \infty$ for all $a \neq b$ then $|A| = \#A$: magnitude = cardinality.

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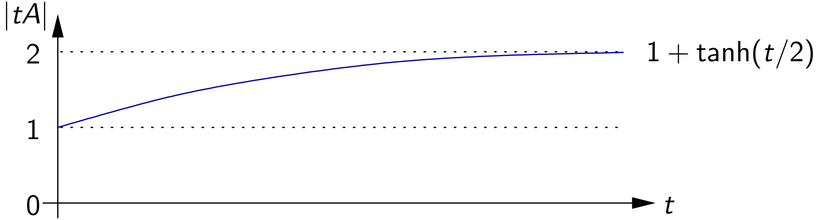

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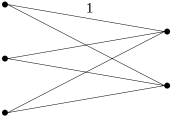

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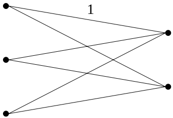

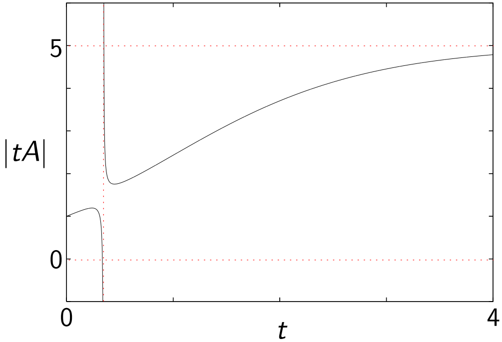

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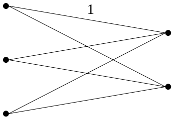

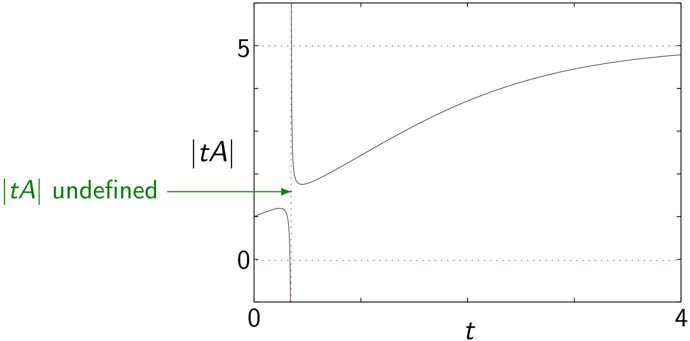

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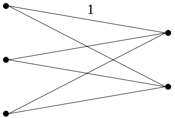

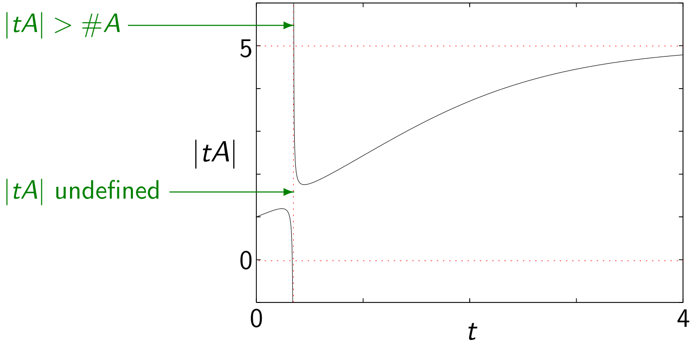

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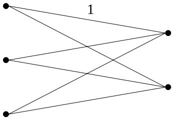

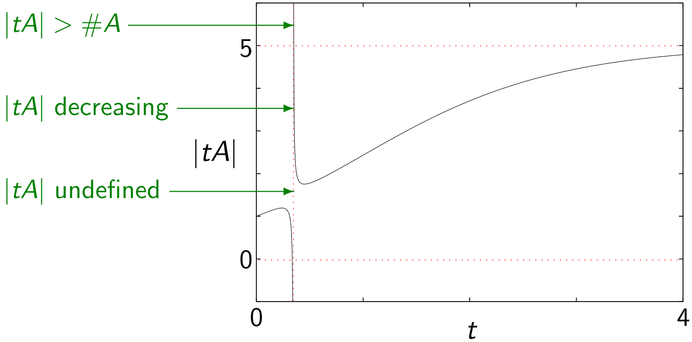

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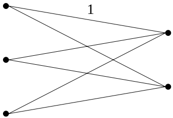

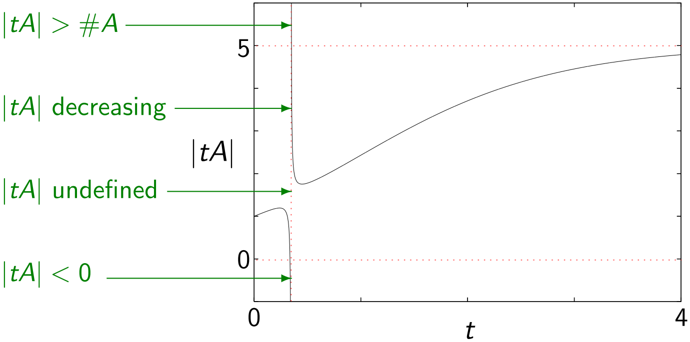

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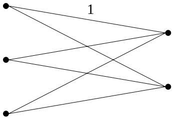

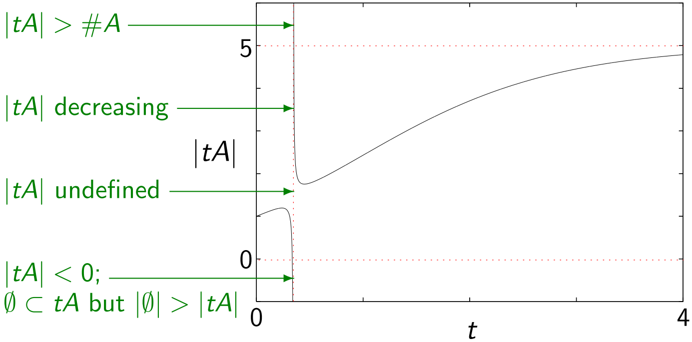

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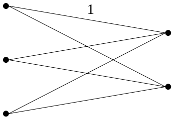

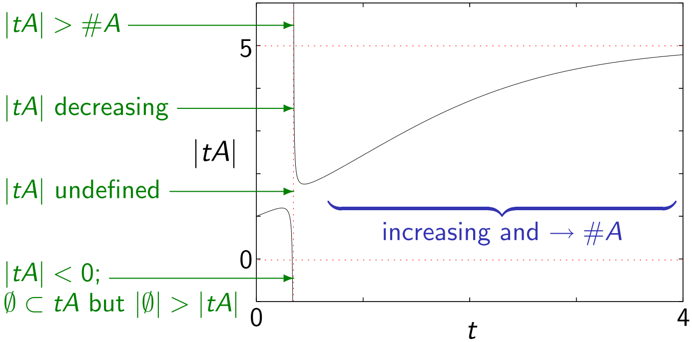

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- $|A| \geq 0$
- *every subspace $B \subseteq A$ is positive definite, and $|B| \leq |A|$*

Positive definite spaces

Roughly, these are the spaces for which 'surprising' behaviour does *not* occur.

Definition

A finite metric space A is positive definite if its matrix Z_A is positive definite.

Positive definite $\Rightarrow$ invertible, so then $|A|$ is defined.

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- *every subspace $B \subseteq A$ is positive definite, and $|B| \leq |A|$*
- $|A| = \sup_{v \in \mathbb{R}^n \setminus \{0\}} \dfrac{(\sum v_i)^2}{v^{\mathrm{t}} Z_A v}$.

Subsets of $\mathbb{R}^N$

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Theorem (Meckes)

Let $p \leq 2$. Then every finite subset of ℓ_p^N is positive definite.

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Magnitude can be understood as something like maximum entropy.

2. Magnitude of infinite metric spaces

From finite to infinite spaces

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(These definitions are consistent with the definitions for finite spaces.)

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But the measure approach currently has some limitations.
So in what follows, we use the finite-approximation definition of magnitude.

Line segments

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Theorem

Let $L \geq 0$. *Let* (A_k) *be a sequence of finite subsets of* $\mathbb{R}$ *such that*

$$\lim_{k \to \infty} A_k = [0, L]$$

in the Hausdorff topology.

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Hence $|[0, L]| = 1 + \frac{1}{2}L$, and $[0, L]$ has magnitude function

$$t \mapsto |t[0, L]| = |[0, tL]| = 1 + \tfrac{1}{2}L \cdot t$$

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Magnitude comes from enriched category theory...

...but produces geometric invariants.

Products

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Let A and B be metric spaces. Write $A \otimes B$ for their 'ℓ^1-product': the set of points is $A \times B$, and

$$d_{A \otimes B}((a, b), (a', b')) = d_A(a, a') + d_B(b, b').$$

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E.g.: a cuboid $[0, L_1] \times \cdots \times [0, L_N] \subset \ell_1^N$, with the subspace metric, is

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Lemma

$|A \otimes B| = |A| \cdot |B|$.

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In general, the magnitude function of the cuboid $A = [0, L_1] \times \cdots \times [0, L_N] \subset \ell_1^N$ is

$$t \mapsto \sum_{i=0}^{N} 2^{-i} \mu_i(A) t^i$$

where μ_i is i-dimensional intrinsic volume.

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Conjectural principle: The same is true for a much larger class of spaces, including convex subsets of $\mathbb{R}^N$ with the Euclidean metric.

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For a metric space A, define

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So, all of the intrinsic volumes of a convex set (as well as the dimension) can be extracted from its magnitude function.

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- Numerical computations support the conjecture.

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Tomorrow:

Magnitude contains *more* than just known invariants