SELF-SIMILARITY AND RECURSION

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1. LOOSE EXAMPLES OF SELF-SIMILARITY / RECURSION

- Mutually recursive types, e.g.

$$T = L + T^2$$

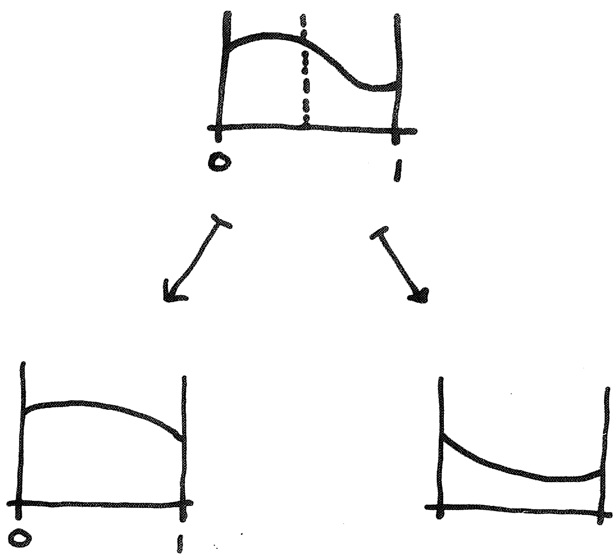
$$L = 1 + T.L$$

- Integration:

$$\int_0^1 : \{\text{integrable functions on } [0,1]\} \xrightarrow{\text{linear}} \mathbb{R}$$

is more-or-less characterized by

$$\int_0^1 f(x)\,dx = \frac{1}{2}\left\{ \int_0^1 f\left(\frac{x}{2}\right) dx + \int_0^1 f\left(\frac{x+1}{2}\right) dx \right\}.$$



19. Julia set

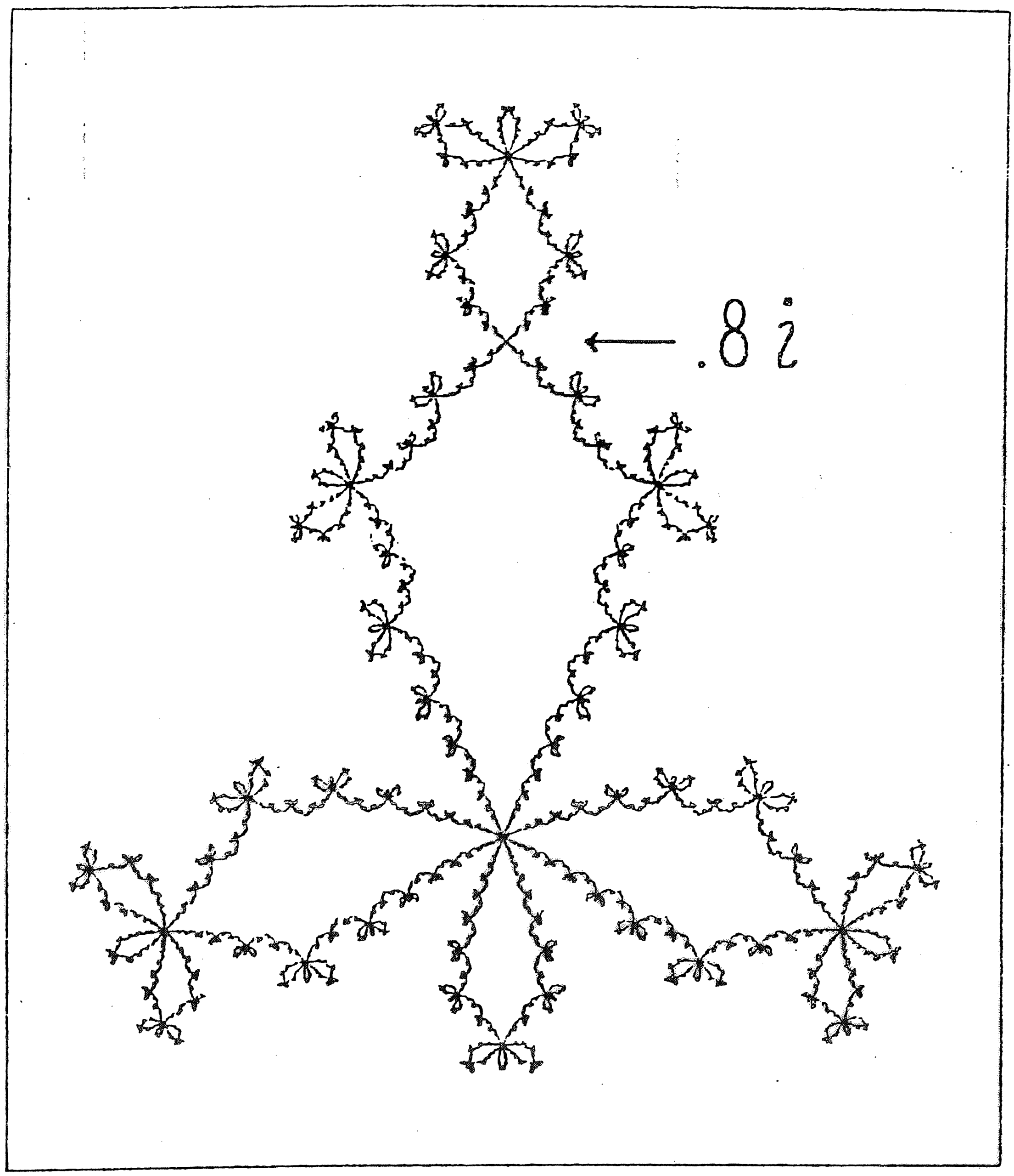


Figure 4. *Julia set for* $f(z) = z^3 + \frac{12}{25}z + \frac{116}{125}i$.

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Loose examples, continued

- Fractal spaces, e.g. Julia sets of self-maps of Riemann surfaces

- Linear equations:

$$
\begin{aligned}
x_1 &= m_{11}x_1 + \cdots + m_{1n}x_n \\
&\;\;\vdots \\
x_n &= m_{n1}x_1 + \cdots + m_{nn}x_n
\end{aligned}
$$

(m_{ij} scalars)

ANALOGIES

Linear equations

- Set $I = \{1, \ldots, n\}$
- $I \times I$ matrix M of scalars
- Vector $\underline{x} = (x_i)_{i \in I}$
- Solutions $\underline{x} = M\underline{x}$,

  i.e. $x_i = \sum_j m_{ij} x_j$

2. DISCRETE SELF-SIMILARITY

A <u>**discrete self-similarity system**</u> is a pair (I, M) where I is a set and $M = (m_{ij})_{i,j \in I} \in \mathbb{N}^{I \times I}$ (subject to a finiteness condition).

Any such induces a functor

$$M \cdot - : \mathbf{Top}^I \longrightarrow \mathbf{Top}^I$$
$$X \longmapsto MX = \Big(\sum_j m_{ij} X_j \Big)_{i \in I}$$

<u>Idea:</u> "Solve the equations

$$X_i \cong \sum_j m_{ij} X_j \quad (i \in I) \text{"},$$

i.e. look for fixed points of $M \cdot -$.

<u>Better idea:</u> Look for the terminal coalgebra!

Review of coalgebras

We have a functor $M\cdot - : \mathrm{Top}^I \longrightarrow \mathrm{Top}^I$.

An <u>M-coalgebra</u> is a pair (X, ξ) where $X \in \mathrm{Top}^I$ and $\xi : X \longrightarrow MX$. A <u>map</u> $(X,\xi) \longrightarrow (X', \xi')$ <u>of M-coalgebras</u> is a map $f : X \to X'$ making

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\xi"'] & X' \arrow[d, "\xi'"] \\ MX \arrow[r, "Mf"'] & MX' \end{tikzcd}$$

commute.

The <u>universal solution</u> for M is the terminal coalgebra, written (U, γ).

<u>Lemma:</u> (Lambek) $\gamma : U \longrightarrow MU$ is an isomorphism. $\square$

(i.e. the universal solution <u>is</u> a solution!)

ANALOGIES

Linear equations	Discrete self-similarity
• Set $I = \{1, \dots, n\}$	• Set I
• $I \times I$ matrix M of scalars	• $I \times I$ matrix M over $\mathbb{N}$
• Vector $\underline{x} = (x_i)_{i \in I}$	• Family $X = (X_i)_{i \in I}$ of spaces
• Solutions $\underline{x} = M\underline{x}$, i.e. $x_i = \sum_j m_{ij} x_j$	• Coalgs $X \longrightarrow MX$, i.e. $X_i \longrightarrow \sum_j m_{ij} X_j$
	• Universal solution $U \xrightarrow{\cong} MU$

Examples

of discrete self-similarity systems & their universal solutions

- $I = \{1\}$, $m_{11} = 1$ corresponds to the single equation

$$X_1 = X_1 .$$

Universal solution is $1 = \bullet$

- $I = \{1\}$, $m_{11} = 2$ corresponds to the single equation

$$X_1 = X_1 + X_1 .$$

Universal solution is Cantor set 

- $I = \{1,2\}$, $M = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ corresponds to the two equations

$$\begin{cases} X_1 = X_1 \\ X_2 = X_1 + X_2 . \end{cases}$$

Universal solution (U_1, U_2) has $U_1 = \bullet$,

$U_2 = \mathbb{N} \cup \{\infty\} = (\bullet \quad \bullet \quad \bullet \; \bullet \bullet \bullet)$

Sample result

Q. What values can U_i take?

I.e. which spaces are "discretely self-similar"?

A. All totally disconnected compact metrizable spaces.

ANALOGIES

Linear equations	Discrete self-similarity	General self-similarity
• Set $I = \{1, \ldots, n\}$	• Set I	• Category I
• $I \times I$ matrix M of scalars	• $I \times I$ matrix M over $\mathbb{N}$	• Module $M : I \to\!\!\!\!\!+\, I$ (finite)
• Vector $\underline{x} = (x_i)_{i \in I}$	• Family $X = (X_i)_{i \in I}$ of spaces	• Functor $X : I \to \mathbf{Top}$
• Solutions $\underline{x} = M\underline{x}$, i.e. $x_i = \sum_j m_{ij} x_j$	• Coalgs $X \longrightarrow MX$, i.e. $X_i \longrightarrow \sum_j m_{ij} X_j$	• Coalgs $X \longrightarrow M \otimes X$, $X_i \longrightarrow \int^j M(j,i) \times X_j$
	• Universal solution $U \xrightarrow{\cong} MU$	• Universal solution $U \xrightarrow{\cong} M \otimes U$

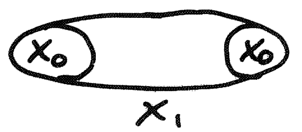
3. GENERAL SELF-SIMILARITY

Motivating example (Freyd)

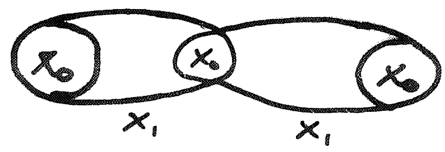
Let $\underline{\mathcal{C}}$ be the category whose objects X are diagrams

$$X_0 \overset{u}{\underset{v}{\rightrightarrows}} X_1$$

where X_0 & X_1 are topological spaces and u&v are continuous closed injections with disjoint images:

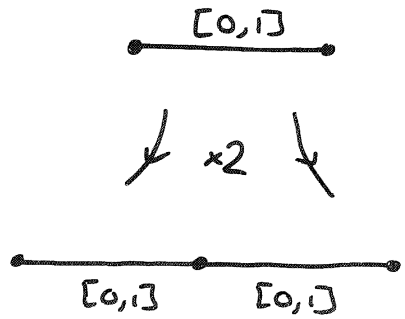
$X =$ 

Define

$M \cdot X =$ 

so that $M \cdot - : \mathcal{C} \longrightarrow \mathcal{C}$.

<u>Thm</u> (Freyd + ε): The terminal coalgebra is



Definitions

A **self-similarity system** is a pair (I, M) where I is a small category and $M : I \to\!\!\!\!\!+\, I$ (i.e. $M : I^{op} \times I \to \mathbf{Set}$) is a finite nondegenerate module.

(E.g. $I = (0 \rightrightarrows 1)$ for Freyd.)

An **M-coalgebra** is a coalgebra for $M \otimes - : (\mathbf{Top}^I)_{\text{nondegen}} \circlearrowleft$.

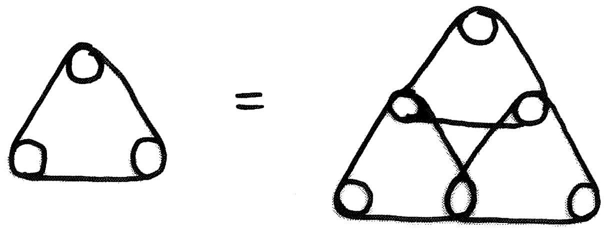
A **universal solution for M** is a terminal M-coalgebra.

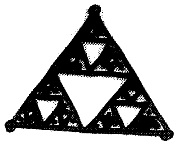
ANALOGIES

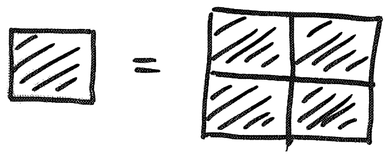
Linear equations	Discrete self-similarity	General self-similarity
• Set $I=\{1,\ldots,n\}$	• Set I	• Category I
• $I\times I$ matrix M of scalars	• $I\times I$ matrix M over $\mathbb{N}$	• Module $M: I \nrightarrow I$ (finite)
• Vector $\underline{x}=(x_i)_{i\in I}$	• Family $X=(X_i)_{i\in I}$ of spaces	• Functor $X: I \longrightarrow \mathbf{Top}$
• Solutions $\underline{x}=M\underline{x}$, i.e. $x_i=\sum_j m_{ij}x_j$	• Coalgs $X \longrightarrow MX$, i.e. $X_i \longrightarrow \sum_j m_{ij} X_j$	• Coalgs $X \longrightarrow M\otimes X$, $X_i \longrightarrow \int^j M(j,i)\times X_j$
	• Universal solution $U \xrightarrow{\cong} MU$	• Universal solution $U \xrightarrow{\cong} M\otimes U$
• $\underline{x}$ nonzero	—	• X nondegenerate
• M nonsingular	—	• M nondegenerate

Examples

- The universal solution of the equation



is Sierpiński's gasket, 

- Many Julia sets
- Squares, cubes, ... : e.g. 
- A space may have several <u>different</u> self-similar structures / recursive decompositions.

Q. Which spaces are "self-similar", i.e. arise as universal solutions?

A. All compact metrizable spaces.

4. POSSIBILITIES

- Handle polynomial equations by linearizing:

e.g.

$$T = 1 + T^2$$

becomes

$$T_0 = T_0$$
$$T_1 = T_0 + T_2$$
$$T_2 = T_1 + T_3$$
$$\vdots$$

whose universal solution has period 6.

- Import concepts from topology to other domains: e.g. connectedness, homotopy, Euler characteristic.