The magnitude of a presheaf

Tom Leinster

University of Edinburgh and the Maxwell Institute

These slides: on my web page

Background

Magnitude is a numerical invariant of enriched categories.

The magnitude of an enriched category is a measure of its size.

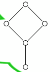
Magnitude homology is a homology theory of enriched categories, categorifying magnitude.

A taste:

	Category $\mathbf{A}$	Metric space A
Magnitude	Euler characteristic of classifying space $B\mathbf{A}$	$(\mathrm{Mag}(tA))_{t\in\mathbb{R}^+}$ determines dimension, volume, surface area, ... of A
Magnitude homology	Homology of $B\mathbf{A}$	Detects non-uniqueness of geodesics, size of holes, ...

Some history

Magnitude arose from ideas such as:



Möbius inversion for posets

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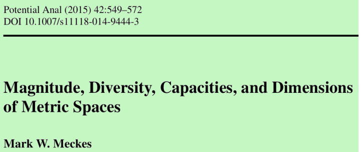
Formulas for cardinalities of colimits

Euler char is analogue of cardinality

It has developed thanks to some serious expertise in. . .

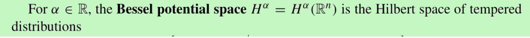
Analysis



Potential Anal (2015) 42:549–572
DOI 10.1007/s11118-014-9444-3

Magnitude, Diversity, Capacities, and Dimensions of Metric Spaces

Mark W. Meckes



For $\alpha \in \mathbb{R}$, the **Bessel potential space** $H^{\alpha} = H^{\alpha}(\mathbb{R}^n)$ is the Hilbert space of tempered distributions

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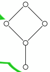

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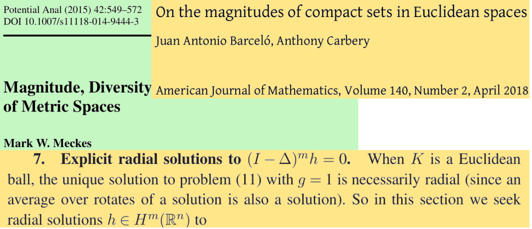

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Magnitude, Diversity of Metric Spaces

Mark W. Meckes

On the magnitudes of compact sets in Euclidean spaces

Juan Antonio Barceló, Anthony Carbery

American Journal of Mathematics, Volume 140, Number 2, April 2018

7. Explicit radial solutions to $(I-\Delta)^m h = 0$. When K is a Euclidean ball, the unique solution to problem (11) with $g = 1$ is necessarily radial (since an average over rotates of a solution is also a solution). So in this section we seek radial solutions $h \in H^m(\mathbb{R}^n)$ to

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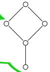

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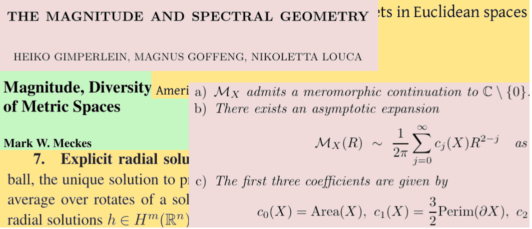

THE MAGNITUDE AND SPECTRAL GEOMETRY

HEIKO GIMPERLEIN, MAGNUS GOFFENG, NIKOLETTA LOUCA

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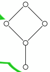

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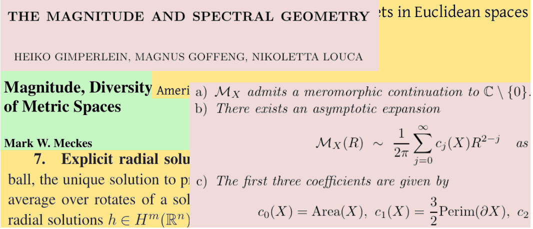

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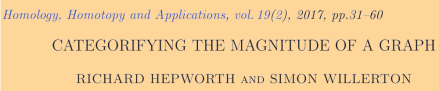
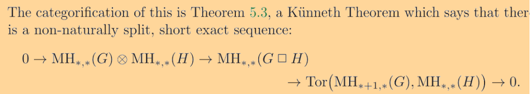


Homology, Homotopy and Applications, vol. 19(2), 2017, pp.31–60

CATEGORIFYING THE MAGNITUDE OF A GRAPH

RICHARD HEPWORTH AND SIMON WILLERTON



The categorification of this is Theorem 5.3, a Künneth Theorem which says that ther is a non-naturally split, short exact sequence:

$$0 \to \mathrm{MH}_{*,*}(G) \otimes \mathrm{MH}_{*,*}(H) \to \mathrm{MH}_{*,*}(G \,\square\, H) \\ \to \mathrm{Tor}(\mathrm{MH}_{*+1,*}(G), \mathrm{MH}_{*,*}(H)) \to 0.$$

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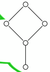

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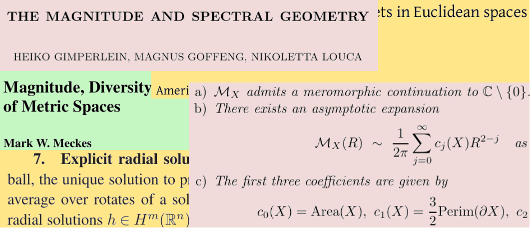

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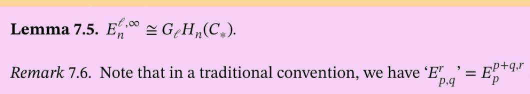
RESEARCH ARTICLE

Bulletin of the London
Mathematical Society

Magnitude homology and path homology

Yasuhiko Asao



Lemma 7.5. $E_n^{\ell,\infty} \cong G_\ell H_n(C_*)$.

Remark 7.6. Note that in a traditional convention, we have '$E^r_{p,q}$' $= E_p^{p+q,r}$

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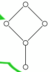

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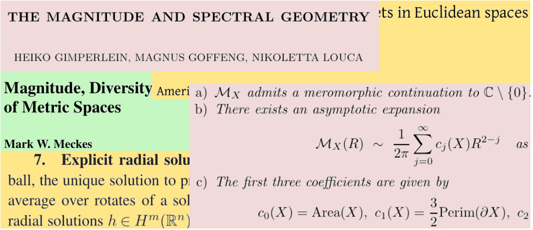

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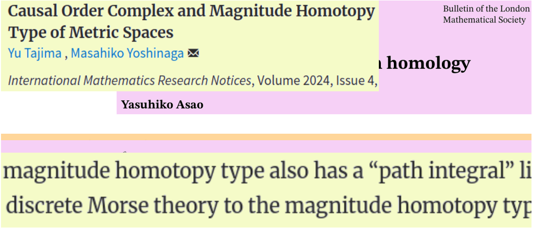

Causal Order Complex and Magnitude Homotopy Type of Metric Spaces

Yu Tajima , Masahiko Yoshinaga ✉

International Mathematics Research Notices, Volume 2024, Issue 4,

Bulletin of the London Mathematical Society

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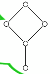

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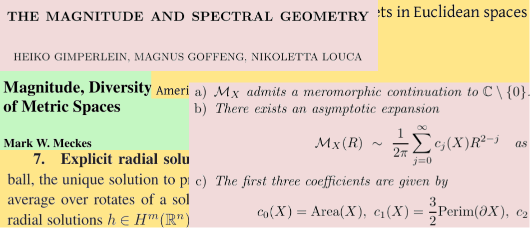

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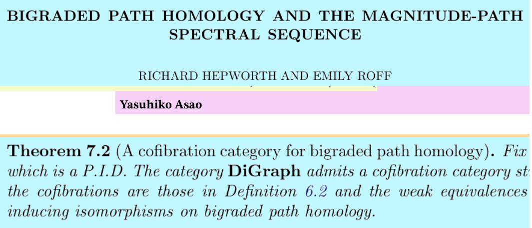

BIGRADED PATH HOMOLOGY AND THE MAGNITUDE-PATH
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RICHARD HEPWORTH AND EMILY ROFF

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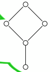

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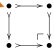

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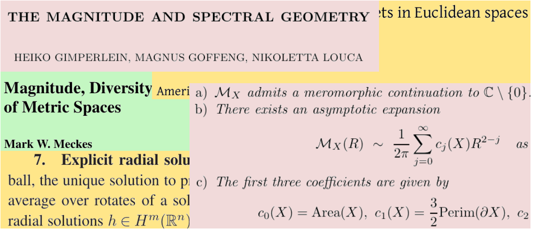

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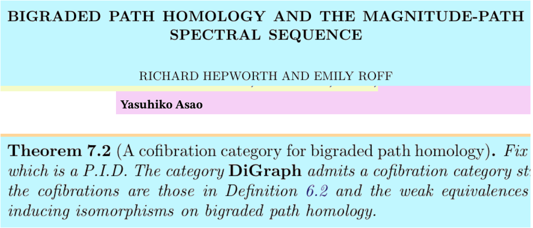

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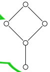

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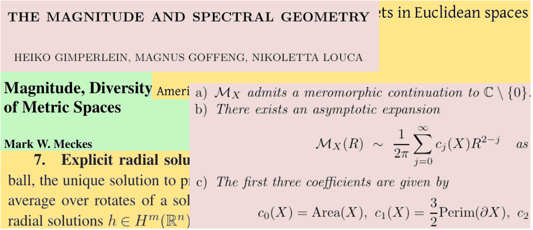

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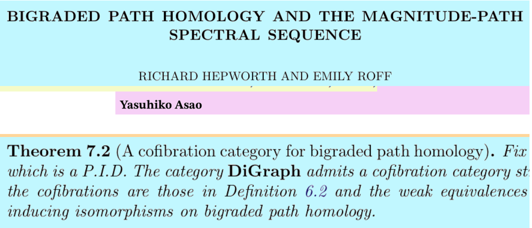

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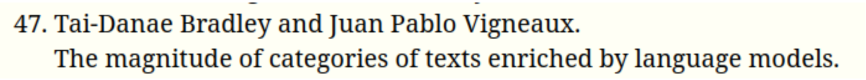

47. Tai-Danae Bradley and Juan Pablo Vigneaux.
The magnitude of categories of texts enriched by language models.

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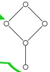

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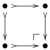

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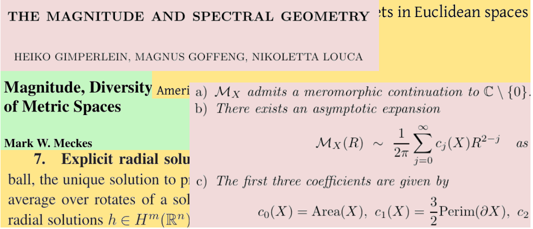

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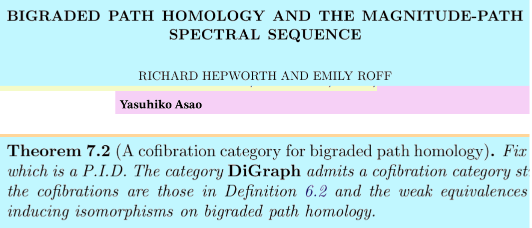

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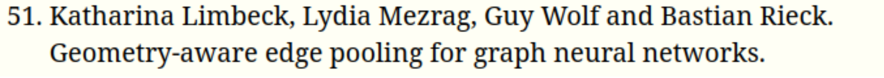
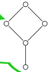


51. Katharina Limbeck, Lydia Mezrag, Guy Wolf and Bastian Rieck. Geometry-aware edge pooling for graph neural networks.

Some history

Magnitude arose from ideas such as:



Möbius inversion for posets

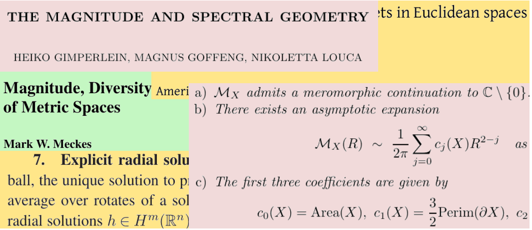
$$\begin{tikzcd}
\bullet \arrow[r, tail] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r, tail] & \bullet \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Formulas for cardinalities of colimits

Euler char is analogue of cardinality

It has developed thanks to some serious expertise in. . .

Analysis



THE MAGNITUDE AND SPECTRAL GEOMETRY

ts in Euclidean spaces

HEIKO GIMPERLEIN, MAGNUS GOFFENG, NIKOLETTA LOUCA

Magnitude, Diversity of Metric Spaces

Ameri

Mark W. Meckes

7. Explicit radial solu
ball, the unique solution to p
average over rotates of a sol
radial solutions $h \in H^m(\mathbb{R}^n)$

a) $\mathcal{M}_X$ *admits a meromorphic continuation to* $\mathbb{C} \setminus \{0\}$.

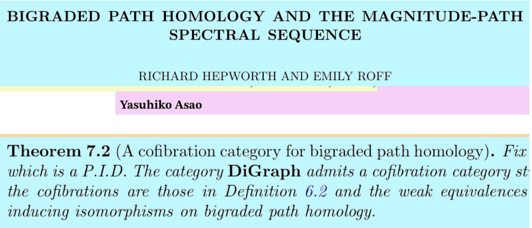
b) *There exists an asymptotic expansion*

$$\mathcal{M}_X(R) \sim \frac{1}{2\pi}\sum_{j=0}^{\infty} c_j(X)R^{2-j} \quad as$$

c) *The first three coefficients are given by*

$$c_0(X) = \mathrm{Area}(X), \;\; c_1(X) = \frac{3}{2}\mathrm{Perim}(\partial X), \;\; c_2$$

Algebra



BIGRADED PATH HOMOLOGY AND THE MAGNITUDE-PATH SPECTRAL SEQUENCE

RICHARD HEPWORTH AND EMILY ROFF

Yasuhiko Asao

Theorem 7.2 (A cofibration category for bigraded path homology). *Fix*
which is a P.I.D. The category **DiGraph** *admits a cofibration category st*
the cofibrations are those in Definition 6.2 and the weak equivalences
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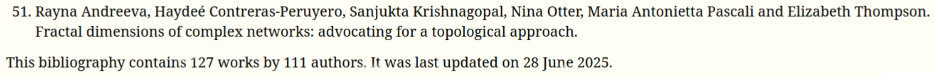


51. Rayna Andreeva, Haydeé Contreras-Peruyero, Sanjukta Krishnagopal, Nina Otter, Maria Antonietta Pascali and Elizabeth Thompson. Fractal dimensions of complex networks: advocating for a topological approach.

This bibliography contains 127 works by 111 authors. It was last updated on 28 June 2025.

From categories to presheaves

Very nearly all existing work is on magnitude of enriched *categories*.

This talk is about something new: magnitude of *presheaves* (functors from a **V**-category to **V**).

Analogy:

magnitude of categories	magnitude of presheaves
measure	integration

Plan

1. Definitions
2. Examples
3. Comagnitude

1. Definitions

The magnitude of a category

Let **A** be a finite category.

A weighting on **A** is a function $w_{\mathbf{A}} : \mathrm{ob}\, \mathbf{A} \to \mathbb{Q}$ such that

$$\text{for all } a \in \mathbf{A}, \quad \sum_{b \in \mathbf{A}} |\mathbf{A}(a,b)|\, w_{\mathbf{A}}(b) = 1.$$

A coweighting $w^{\mathbf{A}}$ on **A** is a weighting on $\mathbf{A}^{\mathrm{op}}$.

Tiny Lemma $\sum_{a \in \mathbf{A}} w_{\mathbf{A}}(a) = \sum_{a \in \mathbf{A}} w^{\mathbf{A}}(a)$.

The magnitude of **A** is

$$|\mathbf{A}| = \sum_{a} w_{\mathbf{A}}(a) = \sum_{a} w^{\mathbf{A}}(a) \ \in \mathbb{Q}.$$

Under mild hypotheses, $|A|$ equals $\chi(B\mathbf{A})$, the Euler characteristic of the classifying space.

The cardinality of a colimit

Let $\mathbf{A}$ be a finite category and $X\colon \mathbf{A} \to \mathbf{FinSet}$.

Can you compute $|\mathrm{colim}\, X|$ from $(|Xa|)_{a \in \mathbf{A}}$ alone?

In general, no. But for some X, yes:

Proposition *If X is a coproduct of representables then*

$$|\mathrm{colim}\, X| = \sum_a w_{\mathbf{A}}(a)\, |Xa|\,,$$

where $w_{\mathbf{A}}$ is any weighting on $\mathbf{A}$.

Examples

- Pushouts along injections: inclusion-exclusion formula for cardinality of a union.
- Free action of monoid M on set X: number of orbits is $|X|\,/\mathrm{order}(M)$.

The magnitude of a presheaf

Let $\mathbf{A}$ be a finite category and $X \colon \mathbf{A} \to \mathbf{FinSet}$.

We've seen that if X is a coproduct of representables then

$$|\operatorname{colim} X| = \sum_a w_{\mathbf{A}}(a)\, |Xa|\,.$$

Whether or not X has this property, define its magnitude to be

$$|X| = \sum_a w_{\mathbf{A}}(a)\, |Xa| \ \in \mathbb{Q}.$$

Magnitude of categories and magnitude of presheaves are special cases of each other:

- $|\mathbf{A}| = \left| \mathbf{A} \overset{\Delta 1}{\to} \mathbf{Set} \right|$.
- $|X| = |\mathbf{E}(X)|$, where $\mathbf{E}(X)$ is category of elements.

Aside $\mathbf{E}(X)$ is the *colax* colimit of X (and as Thomason showed, homotopy equivalent to hocolim X). So $|X|$ measures the size of the colax/homotopy (not strict) colimit.

Simplifying assumptions

To avoid technicalities, I'll make two simplifying assumptions in this talk.

Every finite category has a unique weighting and coweighting

Everything is finite if necessary

Magnitude of presheaves is a size-like invariant

Magnitude of presheaves has decent properties, e.g.:

- $\left|\mathbf{B} \overset{G}{\to} \mathbf{A} \overset{X}{\to} \mathbf{Set}\right| = |X|$ if G is an equivalence (or more generally, a right adjoint).
- For a pushout square

$$\begin{tikzcd}
X_1 \arrow[r, tail] \arrow[d, tail] & X_2 \arrow[d, tail] \\
X_3 \arrow[r, tail] \arrow[ur, phantom, "\ulcorner", very near end] & X_4
\end{tikzcd}$$

  of presheaves,

$$|X_4| = |X_2| + |X_3| - |X_1| \, .$$

Enriching the definitions

To make the definitions **V**-enriched, replace the cardinality function

$$|\cdot| : \mathrm{ob}\,(\mathbf{FinSet}) \to \mathbb{Q}$$

by an isomorphism-invariant function

some field

$$|\cdot| : \mathrm{ob}\,(\mathbf{V}) \to k$$

satisfying $|X \otimes Y| = |X| \cdot |Y|$ and $|I| = 1$.

Most important case: metric spaces. We use

$$\begin{array}{rccc} |\cdot| : & [0, \infty] & \to & \mathbb{R} \\ & x & \mapsto & e^{-x}. \end{array}$$

The magnitude of a finite metric space can be understood as the 'effective number of points'.

The definition extends to many *compact* metric spaces.

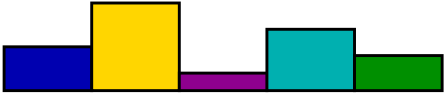
2. Examples

First examples

- When $X\colon \mathbf{A} \to \mathbf{Set}$ is a coproduct of representables,
$$|X| = |\operatorname{colim} X|,$$
by construction.
- For an action of a monoid M on a set X, the magnitude of the corresponding functor $M \to \mathbf{Set}$ is
$$\frac{\operatorname{card}(X)}{\operatorname{order}(M)} \notin \mathbb{Z}.$$

Entropy from magnitude

The Shannon entropy of a finite probability distribution $\mathbf{p} = (p_1, \ldots, p_n)$ is $H(\mathbf{p}) = -\sum p_i \log p_i$.



The definition is extended to arbitrary measure spaces by homogeneity:

$$H(c\mathbf{p}) = cH(\mathbf{p}) \qquad (c \in \mathbb{R}^+).$$

An integer-valued measure μ on a finite set I 'is' a map of sets $S \to I$.

It gives rise to an inclusion of monoids $\boxed{\mathrm{End}_I(S)} \hookrightarrow \mathrm{End}(S)$.

$\{$endomorphisms of S over $I\}$

Like any homomorphism of monoids, this induces an action of the domain on the codomain, hence a functor $\mathrm{End}(S) \to \mathbf{Set}$.

Theorem *The magnitude of this functor is* $e^{H(\mu)}$, *the exponential of the entropy of the measure.*

Relative and conditional entropy

A pair of integer-valued measures on the same set I, with the same total mass, can be seen as a diagram $S \rightrightarrows I$ in **Set**.

A similar derivation produces exp(−relative entropy) as the magnitude of a presheaf.

An integer-valued measure on a product $I \times J$ can be seen as a diagram $I \leftarrow S \rightarrow J$ in **Set**.

Another similar derivation produces exp(conditional entropy) as the magnitude of a presheaf.

Counting primes

Every functor $\mathbf{A} \xrightarrow{F} \mathbf{B}$ induces a new functor

$$\begin{array}{rcl} \mathbf{B} & \to & [\mathbf{A}^{\mathrm{op}}, \mathbf{Set}] \\ b & \mapsto & \mathbf{B}(F-, b), \end{array}$$

hence a function

$$\begin{array}{rcl} \mathrm{ob}\,\mathbf{B} & \to & \mathbb{Q} \\ b & \mapsto & \left|\mathbf{B}(F-, b)\right|. \end{array}$$

Examples

- Inclusion of posets $\{\text{primes}\} \hookrightarrow (\mathbb{Z}^+, |)$: get

$$\begin{array}{rcl} \mathbb{Z}^+ & \to & \mathbb{Q} \\ n & \mapsto & \text{number of distinct prime factors of } n. \end{array}$$

- Inclusion $\mathbf{Field}^{\mathrm{op}} \hookrightarrow \mathbf{Ring}^{\mathrm{op}}$: get

$$\begin{array}{rcl} \mathrm{ob}\,(\mathbf{Ring}) & \to & \mathbb{Q} \\ R & \mapsto & \text{number of prime ideals of } R. \end{array}$$

The potential function

Generally, given a functor $\mathbf{A} \xrightarrow{F} \mathbf{B}$, it's interesting to consider the function

$$\begin{array}{ccc} \mathrm{ob}\,(\mathbf{B}) & \xrightarrow{h_F} & \mathbb{Q} \\ b & \mapsto & \left|\mathbf{B}(F-, b)\right|, \end{array}$$

the potential function of F.

More examples

- When F is an opfibration, $h_F(b) = \left|F^{-1}(b)\right|$ (magnitude of fibre).
- What about $\Delta \hookrightarrow \mathbf{Cat}$? Cut down to $\Delta_{\text{inj}} \hookrightarrow$ (categories + functors reflecting isos).
  The value of the potential function h on a category $\mathbf{C}$ is

$$\begin{aligned} h(\mathbf{C}) &= \sum_{n \in \mathbb{N}} (-1)^n \left| \{\text{nondegenerate paths } c_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} c_n \text{ in } \mathbf{C}\} \right| \\ &= \chi(N\mathbf{C}) \\ &= |\mathbf{C}|, \end{aligned}$$

  under finiteness hypotheses.

Potential functions on metric spaces

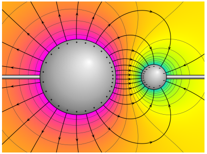
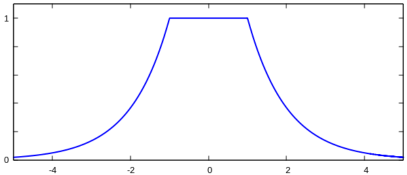


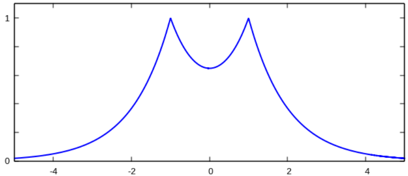
Image: Geek3

The name *potential function* comes from the metric case.

Potential function of $[-1,1] \hookrightarrow \mathbb{R}$



Potential function of $\{-1,1\} \hookrightarrow \mathbb{R}$



For a general $\mathbf{V}$-functor $\mathbf{A} \xrightarrow{F} \mathbf{B}$:

- Potential function determines magnitude: $|\mathbf{A}| = \sum_b h_F(b) w_{\mathbf{B}}(b)$ (or $|A| = \int_B h_F \, dw_{\mathbf{B}}$)
- If F is full and faithful then $h_F \equiv 1$ on $\mathrm{im}(F)$.

How potential functions came to the rescue

In the beginning, no one knew how to calculate the magnitude of more or less anything.

State of the art in 2013:

That is: apart from line segments, there is no compact convex set whose magnitude is known.

Then Mark Meckes introduced the method of potential functions, and suddenly people could prove results like this:

$$\underset{\text{5-dim Euclidean ball of radius } R}{\left|RB^5\right|} = \frac{R^6 + 18R^5 + 135R^4 + 525R^3 + 1080R^2 + 1080R + 360}{120(R+3)}$$

(Barceló and Carbery, 2018).

How?

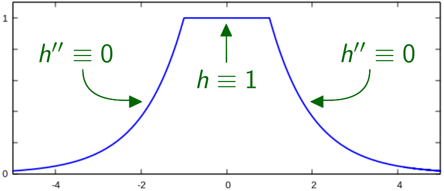
The method of potential functions

Goal Given a metric space $A \subseteq \mathbb{R}^n$, find the magnitude $|A|$.

Method

- The potential function $h\colon \mathbb{R}^n \to \mathbb{R}$ is unknown but satisfies a PDE:

$$\begin{cases} h \equiv 1 & \text{on } A \\ (I - \Delta)^{(n+1)/2} h \equiv 0 & \text{on } \mathbb{R}^n \setminus A. \end{cases}$$



- Solve the PDE to find h.
- Then voilà! $|A| = \int_{\mathbb{R}^n} h$.

Where does that PDE come from?

[Möbius] inversion can be carried out by the analog of the “difference operator” relative to a partial ordering

Gian-Carlo Rota, 1964

3. Comagnitude

The comagnitude of a presheaf

Let $\mathbf{A}$ be a finite category and $X \colon \mathbf{A} \to \mathbf{Set}$.

We've seen that when $X \cong \sum_a S_a \times \mathbf{A}(a, -)$ for some family of sets $(S_a)_{a \in \mathbf{A}}$,

$$\left|\operatorname{colim} X\right| = \sum_a w_{\mathbf{A}}(a) \left|Xa\right|,$$

and we defined the magnitude $|X|$ to be the right-hand side for *arbitrary* X.

There's a dual theorem: when $X \cong \prod_a S_a^{\mathbf{A}(-,a)}$ for some family of sets $(S_a)_{a \in \mathbf{A}}$,

$$\left|\lim X\right| = \prod_a \left|Xa\right|^{w^{\mathbf{A}}(a)}.$$

And we define the comagnitude of X to be the right-hand side for *arbitrary* X.

Comagnitude and random presheaves

Suppose I secretly choose a functor $X\colon \mathbf{A} \to \mathbf{Set}$, and I tell you $\mathbf{A}$ and $|Xa|$ for each $a \in \mathbf{A}$, but not the action of X on morphisms.

Can you calculate the cardinality of the limit of X?

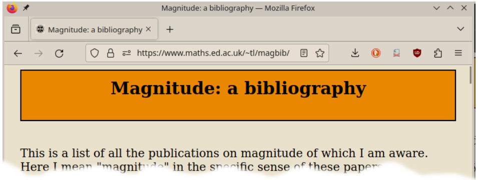
No.

Your best guess: Run through *all* functors $Y\colon \mathbf{A} \to \mathbf{Set}$ such that $Ya = Xa$ for all a, compute $|\lim Y|$ for each such Y, then take the average.

That is, take the expected cardinality of the limit of a random presheaf satisfying the constraints.

Theorem *The result is the comagnitude of X, assuming that $\mathbf{A}$ is the free category on a graph.*

References



Magnitude: a bibliography

This is a list of all the publications on magnitude of which I am aware.
Here I mean "magnitude" in the specific sense of these papers

www.maths.ed.ac.uk/∼tl/magbib

Blog posts on the *n*-Category Café (2025):

- Comagnitude 1, 2
- Potential functions and the magnitude of functors 1, 2

Thanks for listening

Conceivably asked questions

What about magnitude *homology* of presheaves?

There is probably a sensible notion of the magnitude homology of a functor $X\colon \mathbf{A} \to \mathbf{Set}$.

Start by considering the magnitude homology of the category of elements $\mathbf{E}(X)$.

The magnitude homology paper by Leinster and Shulman (arXiv version 2, Section 3) has something in this direction.

What about mutual information?

Define the discrepancy of $X\colon \mathbf{A} \to \mathbf{Set}$ as $|\operatorname{colim} X| - |X|$ (which is 0 if X is a coproduct of representables).

Similarly, the codiscrepancy is $|\lim X| / \operatorname{comag}(X)$.

For an integer-valued joint measure as a diagram of sets $I \leftarrow S \rightarrow J$, the codiscrepancy of the presheaf

$$\begin{tikzcd}
 & \mathrm{End}_I(A) \arrow[d, tail] \\
\mathrm{End}_J(A) \arrow[r, tail] & \mathrm{End}(A)
\end{tikzcd}$$

is the exponential of the mutual information.

What about the magnitude of a bimodule?

The magnitude of a bimodule $M \colon \mathbf{A}^{\mathrm{op}} \times \mathbf{B} \to \mathbf{Set}$ can reasonably be defined as

$$|M| = \sum_{a,b} w^{\mathbf{A}}(a)\,|M(a,b)|\,w_{\mathbf{B}}(b) \in \mathbb{Q}.$$

It is equal to the magnitude of the two-sided category of elements $\mathbf{E}(M)$:

$$\begin{tikzcd}
 & \mathbf{E}(M) \arrow[dl] \arrow[dr] & \\
\mathbf{A} & & \mathbf{B}.
\end{tikzcd}$$

The special case $\mathbf{A} = \mathbf{1}$ or $\mathbf{B} = \mathbf{1}$ reduces to the definition of the magnitude of a presheaf.