THE THOMPSON GROUPS

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Thm The automorphism group of the $\underbrace{\text{generic idempotent object}}_{\S 2}$ $\underbrace{\text{is}}_{\S 3}$ $\underbrace{\text{Thompson's group } F}_{\S 1}$.

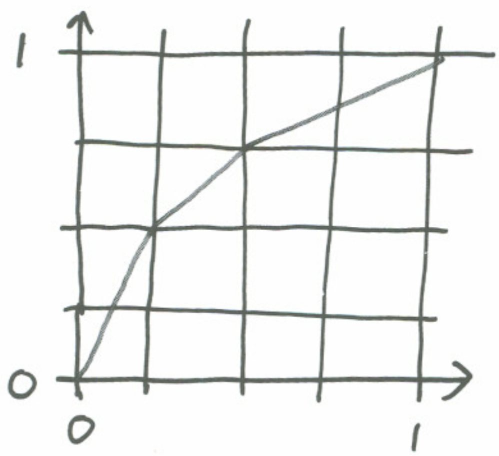
1. THOMPSON'S GROUP F

Defn: F is the group of bijections $[0,1] \xrightarrow{f} [0,1]$ satisfying

(i) f is piecewise linear

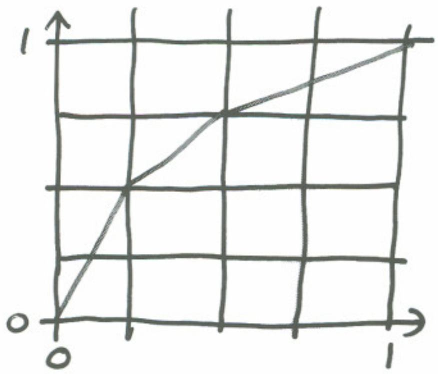
(ii) gradient of each piece is 2^n (some $n \in \mathbb{Z}$)

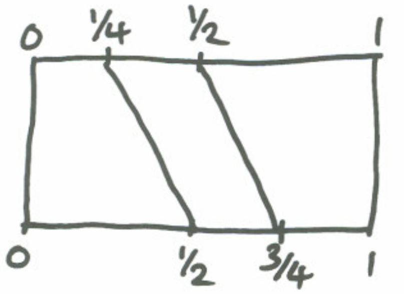
(iii) coords of endpoints of each piece are dyadic rationals.

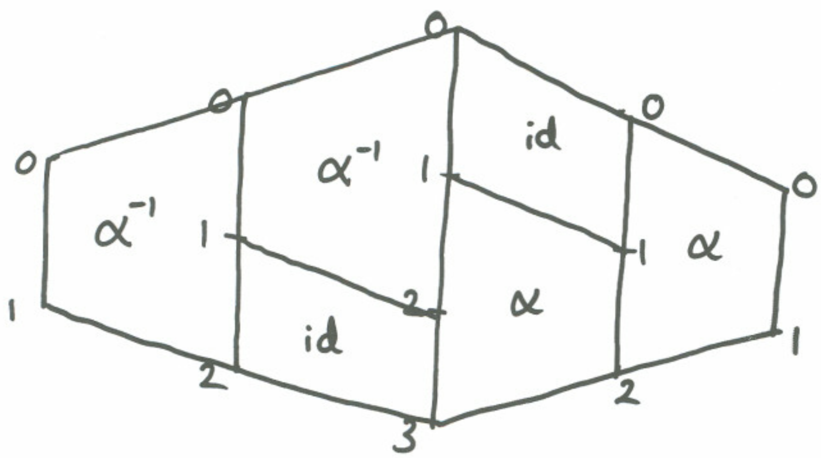
E.g.:  $\in F$

Thompson's group F (continued)

Ways of representing elements of F:

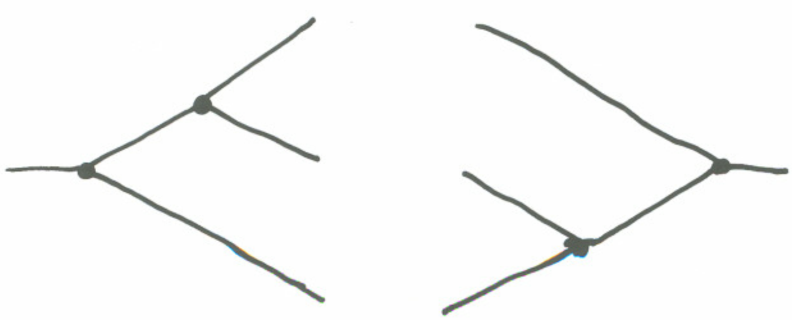






where

$\alpha = \left([0,2] \xrightarrow{\div 2} [0,1]\right)$



2. THE GENERIC IDEMPOTENT OBJECT

Some free cats with structure:

- The free monoidal cat on a monoid is (finite ordinals)

(In this talk, mon cat = strict mon cat.)

- The free braided mon cat on an object is $\sum_{n \in \mathbb{N}} B_n$, the braid groups

- The free finite product cat containing a group is the Lawvere theory of groups (etc)

- The free symmetric mon cat containing a commutative Frobenius algebra is (1-manifolds + 2-cobordisms).

We'll add one to this list.

The generic idempotent object (continued)

An idempotent object in a mon cat $\mathcal{M}$ is a pair (M, μ) where $M \in \mathcal{M}$ and $\mu : M \otimes M \xrightarrow{\sim} M$.

Q. What is the free mon cat on an idempotent object?

Call it $(\mathcal{A}, A, \alpha)$, and (A, α) the generic idempotent object.

$$\mathcal{A} = \left(\begin{tikzcd} I & A \arrow[loop below, "\sim"] \arrow[r, bend right, "\sim"', "\alpha^{-1}"'] & A^{\otimes 2} \arrow[l, "\alpha"', "\sim"] \arrow[loop below, "\sim"] \arrow[r, shift right, "\sim"'] & A^{\otimes 3} \arrow[l, shift right, "\sim"'] \arrow[loop below, "\sim"] & \cdots \end{tikzcd} \right)$$

E.g.

$$(A \xrightarrow{\alpha^{-1}} A^{\otimes 2} \xrightarrow{\alpha^{-1} \otimes 1} A^{\otimes 3} \xrightarrow{1 \otimes \alpha} A^{\otimes 2} \xrightarrow{\alpha} A) \in \mathcal{A}(A, A)$$

A. $1 + F$ (where $+$ is coproduct in $\mathbf{Cat}$)

This answer is equivalent to:

Thm: $\mathrm{Aut}(A) = F$.

3. THE PROOF

Concrete proof: Define $\mathcal{A}$ by

$$\text{ob}\,\mathcal{A} = \mathbb{N},$$

$$\mathcal{A}(m,n) = \{\text{bijections } [0,m] \to [0,n] \text{ satisfying (i)–(iii)}\}$$

and verify universal property directly.

Abstract proof:

$$\begin{tikzcd}
 & & \mathbf{MonGpd} \arrow[d, shift left=0] & \\
 & & \mathbf{MonCat} \arrow[u, shift left=4] \arrow[u, shift right=4] \arrow[u, phantom, shift left=2, "\dashv"] \arrow[u, phantom, shift right=2, "\dashv"] \arrow[d, shift left=2] & \mathcal{A} \\
 & \mathbf{Operad} \arrow[r, hook] \arrow[d, shift left=2] & \mathbf{Multicat} \arrow[u, shift left=2] \arrow[u, phantom, "\dashv"] & \\
 \mathbb{B} \arrow[uuurrr, bend left=40, mapsto] & \mathbf{Set}^{\mathbb{N}} \arrow[u, shift left=2] \arrow[u, phantom, "\dashv"] & &
\end{tikzcd}$$

$$\mathbb{B}_n = \begin{cases} 1 & \text{if } n=2 \\ \emptyset & \text{otherwise} \end{cases}$$

- Universal property of $\mathcal{A}$ immediate from adjointness
- Explicit description of adjointness gives $\mathcal{A} \cong 1+F$.

4. FREYD-HELLER

A conjugacy-idempotent on a group G is a pair (γ, g) where $\gamma : G \longrightarrow G$ and $g : \gamma \circ \gamma \xrightarrow{\sim} \gamma$, i.e.

$$g \in G \quad \text{and} \quad \forall x \in G, \quad \gamma(\gamma(x)) = g \cdot \gamma(x) \cdot g^{-1}.$$

Thm: Let (G, γ, g) be the initial group with a conjugacy-idempotent. Then $G \cong F$.

Q. Relation between the two Thms?