The Reflexive Completion

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Plan

1. Isbell conjugacy and the reflexive completion
2. Characterizations of the reflexive completion
3. Limits in reflexive completions

1. Isbell conjugacy
and
the reflexive completion

Conjugacy: the definition

Let $\mathbf{A}$ be a small category.

Every $\mathbf{Set}$-valued functor on $\mathbf{A}$ has a conjugate:

$$
\begin{aligned}
X\colon \mathbf{A} \longrightarrow \mathbf{Set} \quad &\rightsquigarrow \quad \widehat{X}\colon \mathbf{A}^{\mathrm{op}} \longrightarrow \mathbf{Set}, \\
& \qquad \widehat{X}(a) = [\mathbf{A}, \mathbf{Set}]\big(X, \mathbf{A}(a, -)\big), \\
Y\colon \mathbf{A}^{\mathrm{op}} \longrightarrow \mathbf{Set} \quad &\rightsquigarrow \quad \check{Y}\colon \mathbf{A} \longrightarrow \mathbf{Set}, \\
& \qquad \widehat{Y}(a) = [\mathbf{A}^{\mathrm{op}}, \mathbf{Set}]\big(Y, \mathbf{A}(-, a)\big).
\end{aligned}
$$

These processes are adjoint:

$$
[\mathbf{A}, \mathbf{Set}](X, \check{Y}) \cong [\mathbf{A}^{\mathrm{op}}, \mathbf{Set}](Y, \widehat{X})
$$

$$
[\mathbf{A}, \mathbf{Set}]^{\mathrm{op}} \;\underset{\vee}{\overset{\wedge}{\rightleftarrows}}\; [\mathbf{A}^{\mathrm{op}}, \mathbf{Set}].
$$

E.g. In the conjugacy adjunction,

$$
\mathbf{A}(a, -) \;\substack{\longmapsto \\ \longleftarrow\!\shortmid}\; \mathbf{A}(-, a)
$$

for all $a \in \mathbf{A}$.

Conjugacy: examples

- Let G be a group, seen as a one-object category.
  If X is a left G-set then $\widehat{X}$ is the right G-set $\mathrm{Hom}(X, G_L)$,
  where G_L is the left regular representation of G.

Conjugacy is also defined for enriched categories.

- Let k be a field, seen as a one-object **Ab**-category.
  Then $[k, \mathbf{Ab}] = \mathbf{Vect}_k$, and if $V \in \mathbf{Vect}_k$ then $\widehat{V} = \check{V}$ is the dual of V.
- Let P be a poset, seen as a $\mathbf{2}$-category.
  Then $[P, \mathbf{2}]$ is the poset of upwards-closed subsets U of P,
  and $\widehat{U}$ is the downwards-closed set of lower bounds of U.

Interlude: how we teach the Yoneda lemma

Given $X\colon \mathbf{A} \longrightarrow \mathbf{Set}$, get $X'\colon \mathbf{A} \longrightarrow \mathbf{Set}$ defined by

$$X'(a) = [\mathbf{A}, \mathbf{Set}](\mathbf{A}(a, -), X).$$

Then $X' \cong X$ (Yoneda!).

Given $X\colon \mathbf{A} \longrightarrow \mathbf{Set}$, also get $\widehat{X}\colon \mathbf{A}^{\mathrm{op}} \longrightarrow \mathbf{Set}$, hence $\check{\widehat{X}}\colon \mathbf{A} \longrightarrow \mathbf{Set}$.

But in general, $\check{\widehat{X}} \not\cong X$.

We have the unit map $X \longrightarrow \check{\widehat{X}}$ of the adjunction, but nothing in the opposite direction.

The reflexive completion: definition

$X\colon \mathbf{A} \longrightarrow \mathbf{Set}$ is reflexive if the unit map $X \longrightarrow \check{\hat{X}}$ is iso.

$Y\colon \mathbf{A}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is reflexive if the unit map $Y \longrightarrow \hat{\check{Y}}$ is iso.

E.g. Representables are reflexive.

By adjointness, the full subcategory

$$(\text{reflexive functors } \mathbf{A} \longrightarrow \mathbf{Set}) \subseteq [\mathbf{A}, \mathbf{Set}]^{\mathrm{op}}$$

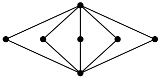
is equivalent to the full subcategory

$$(\text{reflexive functors } \mathbf{A}^{\mathrm{op}} \longrightarrow \mathbf{Set}) \subseteq [\mathbf{A}^{\mathrm{op}}, \mathbf{Set}].$$

The reflexive completion $\mathscr{R}(\mathbf{A})$ of $\mathbf{A}$ is either of these equivalent categories.

Remark The concept is self-dual: $\mathscr{R}(\mathbf{A}^{\mathrm{op}}) \simeq \mathscr{R}(\mathbf{A})^{\mathrm{op}}$.

The reflexive completion: examples

- $\mathscr{R}(\varnothing) = \mathbf{1} = \mathscr{R}(\mathbf{1})$.
- For a discrete category $\mathbf{A}$ with $\geqslant 2$ objects, $\mathscr{R}(\mathbf{A})$ is $\mathbf{A}$ with initial and terminal objects adjoined:  .
- For a group G with $\geqslant 3$ elements, $\mathscr{R}(G)$ is G with initial and terminal objects adjoined.
  But $\mathscr{R}(C_2)$ is the full subcategory of $[C_2, \mathbf{Set}]$ consisting of $\varnothing$, 1, C_2, *and* the free C_2-set on 2 generators.
- For a field k as an $\mathbf{Ab}$-category, $\mathscr{R}(k) =$ (fin-dim k-vector spaces).
- For a poset P, $\mathscr{R}(P)$ is the Dedekind–MacNeille completion of P.
  It is complete. E.g. $\mathscr{R}(\mathbb{Q}) = \mathbb{R} \cup \{\pm\infty\}$.
- The reflexive completion of a metric space is closely related to its tight span/injective envelope (Willerton).

An obstacle

The word 'completion' suggests that $\mathscr{R} \circ \mathscr{R} \simeq \mathscr{R}$. Isbell proved this.

But we defined $\mathscr{R}(\mathbf{A})$ only for *small* $\mathbf{A}$, and $\mathscr{R}(\mathbf{A})$ is not obviously small.

So how is $\mathscr{R}(\mathscr{R}(\mathbf{A}))$ even defined?

Some set-theoretic care is required...

Three open questions: Over **Set**,

- $\mathbf{A}$ small $\Rightarrow \mathscr{R}(\mathbf{A})$ small?
- $\mathbf{A}$ finite $\Rightarrow \mathscr{R}(\mathbf{A})$ finite?
- Is there an explicit construction of $\mathscr{R}(\mathbf{A})$?

The basic definitions, without smallness

Let $\mathscr{A}$ be a *locally* small category.

A **Set**-valued functor on $\mathscr{A}$ is small if it is a small colimit of representables.

When $X\colon \mathscr{A} \longrightarrow \mathbf{Set}$ is small, $\widehat{X}\colon \mathscr{A}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ can be defined as before (but need not be small). Similarly for $Y\colon \mathscr{A}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ and $\check{Y}\colon \mathscr{A} \longrightarrow \mathbf{Set}$.

X is reflexive if X is small, $\widehat{X}$ is small, and $X \xrightarrow{\text{unit}} \check{\widehat{X}}$ is iso. Similarly for Y.

The reflexive completion $\mathscr{R}(\mathscr{A})$ is the full subcategory

$$(\text{reflexive functors } \mathscr{A} \longrightarrow \mathbf{Set}) \subseteq [\mathscr{A}, \mathbf{Set}]^{\mathrm{op}}$$

or equivalently the full subcategory

$$(\text{reflexive functors } \mathscr{A}^{\mathrm{op}} \longrightarrow \mathbf{Set}) \subseteq [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}].$$

2. Characterizations of the reflexive completion

Density

Recall A functor $F\colon \mathscr{A} \longrightarrow \mathscr{B}$ is dense if the 'nerve' functor

$$\begin{array}{ccc} \mathscr{B} & \longrightarrow & [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}] \\ b & \mapsto & \mathscr{B}(F-, b) \end{array}$$

is full and faithful.

F is small-dense if also $\mathscr{B}(F-, b)$ is small for each $b \in \mathscr{B}$.

Codensity and small-codensity are defined dually.

A full, faithful, small-dense and small-codense functor will be called a snug embedding.

Nice property If $\mathscr{A} \xrightarrow{F} \mathscr{B} \xrightarrow{G} \mathscr{C}$ are full and faithful then

$$G \circ F \text{ is snug} \iff F \text{ and } G \text{ are snug.}$$

First characterization of the reflexive completion

Let $\mathscr{A}$ be a locally small category.

Since representables are reflexive, we have $\mathscr{A} \hookrightarrow \mathscr{R}(\mathscr{A})$ (Yoneda).

Fact This embedding is snug.

Theorem (essentially Isbell)
$\mathscr{R}(\mathscr{A})$ is the largest category into which $\mathscr{A}$ embeds snugly.

I.e.: if $\mathscr{A} \xrightarrow{G} \mathscr{B}$ is a snug embedding, there is a unique snug $\overline{G}$ such that

$$\begin{tikzcd} \mathscr{B} \arrow[rr, dotted, "\overline{G}"] & & \mathscr{R}(\mathscr{A}) \\ & \mathscr{A} \arrow[ul, "G"] \arrow[ur, "\text{Yoneda}"'] & \end{tikzcd}$$

commutes.

Corollary (Isbell) $\mathscr{R}^2(\mathscr{A}) \simeq \mathscr{R}(\mathscr{A})$.

Definition $\mathscr{A}$ is reflexively complete if the only reflexive presheaves on $\mathscr{A}$ are the representables.

Then $\mathscr{A}$ is reflexively complete $\iff$ $\mathscr{A} \simeq \mathscr{R}(\mathscr{B})$ for some $\mathscr{B}$.

Second characterization of the reflexive completion

We just showed: $\mathscr{R}(\mathscr{A})$ is the largest category into which $\mathscr{A}$ embeds snugly.

Similarly: the completion $\widetilde{A}$ of a metric space A is the largest metric space into which A embeds densely.

But alternatively: $\widetilde{A}$ is the *unique complete* metric space into which A embeds densely.

Similarly:

Theorem $\mathscr{R}(\mathscr{A})$ is the unique reflexively complete category into which $\mathscr{A}$ embeds snugly.

(More precisely: $\mathscr{A} \hookrightarrow \mathscr{R}(\mathscr{A})$ is the unique-up-to-equivalence snug embedding of $\mathscr{A}$ into a reflexively complete category.)

When do two categories have equivalent reflexive completions?

By the second characterization theorem, any snug embedding $\mathscr{A} \longrightarrow \mathscr{B}$ induces an equivalence $\mathscr{R}(\mathscr{A}) \simeq \mathscr{R}(\mathscr{B})$:

$$\begin{tikzcd}
\mathscr{R}(\mathscr{A}) \arrow[r, dotted, "\simeq"] & \mathscr{R}(\mathscr{B}) \\
\mathscr{A} \arrow[r] \arrow[u] & \mathscr{B}. \arrow[u]
\end{tikzcd}$$

So:

Theorem The following are equivalent for categories $\mathscr{A}$ and $\mathscr{B}$:

- $\mathscr{R}(\mathscr{A}) \simeq \mathscr{R}(\mathscr{B})$
- there is a cospan $\mathscr{A} \longrightarrow \cdot \longleftarrow \mathscr{B}$ of snug embeddings
- there is a zigzag $\mathscr{A} \longrightarrow \cdot \longleftarrow \ \cdots \ \longrightarrow \cdot \longleftarrow \mathscr{B}$ of snug embeddings.

3. Limits in reflexive completions

A trap

The reflexive (Dedekind–MacNeille) completion of a poset is complete.

So, we might guess that the reflexive completion of any category is complete.

False!

E.g. Take any non-posetal finite $\mathbf{A}$ such that $\mathscr{R}(\mathbf{A})$ is finite. Then $\mathscr{R}(\mathbf{A})$ is a non-posetal finite category, so does not have finite products (Freyd).

For posets, complete $\iff$ reflexively complete.

For general categories, only $\Rightarrow$ holds.

Which limits exist in reflexive completions?

For a small category $\mathbf{A}$:

- $\mathscr{R}(\mathbf{A})$ is Cauchy-complete, that is, has absolute (co)limits. (It also contains the Cauchy-completion of $\mathbf{A}$.)
- $\mathscr{R}(\mathbf{A})$ has terminal and initial objects. (Viewing $\mathscr{R}(\mathbf{A})$ as a subcategory of $[\mathbf{A}^{\mathrm{op}}, \mathbf{Set}]$, the terminal object is the terminal presheaf, but the initial object is $\mathrm{Cone}(-, \mathrm{id}_{\mathbf{A}})$.)

Theorem The following are equivalent for a small category $\mathbf{J}$:

- $\mathbf{J}$-limits exist in $\mathscr{R}(\mathbf{A})$ for every small category $\mathbf{A}$
- $\mathbf{J}$ is empty or $\mathbf{J}$-limits are absolute.

Summary

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- Isbell conjugacy is a canonical adjunction $[\mathbf{A}, \mathbf{Set}]^{\mathrm{op}} \rightleftarrows [\mathbf{A}^{\mathrm{op}}, \mathbf{Set}]$.
- The reflexive completion $\mathscr{R}(\mathbf{A})$ is the invariant part of this adjunction.
- $\mathscr{R}(\mathbf{A})$ is the largest category
  containing $\mathbf{A}$ as a full, small-dense and small-codense subcategory.
- $\mathscr{R}(\mathbf{A})$ is the unique reflexively complete category
  containing $\mathbf{A}$ as a full, small-dense and small-codense subcategory.
- $\mathscr{R}(\mathbf{A}) \simeq \mathscr{R}(\mathbf{B})$ iff $\mathbf{A}$ and $\mathbf{B}$ can be joined by a zigzag of full, faithful, small-dense and small-codense functors.
- $\mathscr{R}(\mathbf{A})$ has initial and terminal objects and absolute (co)limits,
  and in general, no other (co)limits.

But we lack both an *explicit construction* and a *universal characterization* of the reflexive completion.

Someone should find them.