Terminal coalgebras via modules

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Plan

We'll give a new proof of an old theorem (Bird, Makkai–Paré, Barr, Adámek):

e.g. on **Set**: free group/ring/...

every finitary endofunctor
of a locally finitely presentable category
has a terminal coalgebra

e.g. **Set**, **Group**, **Ring**, **Vect**, ..., **Cat**, $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$

—and we'll *construct* the terminal coalgebra.

Plan

We'll give a new proof of an old theorem (Bird, Makkai–Paré, Barr, Adámek):

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—and we'll *construct* the terminal coalgebra.

Apostolos will show how the methods extend to a wider context.

For example, they cover the characterizations of the real numbers by Freyd and Pavlović–Pratt.

Coalgebras

Let $\Phi : \mathscr{C} \longrightarrow \mathscr{C}$ be an endofunctor of a category $\mathscr{C}$.

A Φ-coalgebra is a pair (X, ξ) with

$$X \in \mathscr{C}, \qquad \xi : X \longrightarrow \Phi(X).$$

There is an obvious notion of map of Φ-coalgebras.

Example

$\mathscr{C} = \mathbf{Set}$, $\Phi = 1 + - = \{*\} \amalg -$: then a Φ-coalgebra is a set X together with a partial map $\xi : X \longrightarrow X$.

The category of Φ-coalgebras may have a terminal object— a terminal coalgebra for Φ.

Example

In the previous example, the terminal coalgebra (T, τ) has $T = \mathbb{N} \cup \{\infty\}$.

Modules

Let $\mathscr{A}$ and $\mathscr{B}$ be categories.

A module (or bimodule, or profunctor, or distributor) $M : \mathscr{B} \nrightarrow \mathscr{A}$ is a functor $M : \mathscr{B}^{\mathrm{op}} \times \mathscr{A} \longrightarrow \mathbf{Set}$.

We write

$$m \in M(b, a) \qquad \text{as} \qquad b \overset{m}{\nrightarrow} a.$$

Example

A functor $\mathscr{A} \longrightarrow \mathbf{Set}$ is the same as a module $\mathbf{1} \nrightarrow \mathscr{A}$.

Tensor product of modules

There is a tensor product

$$\mathscr{C} \overset{N}{\relbar\mkern-5mu\mapstochar\mkern-2mu\rightarrow} \mathscr{B} \overset{M}{\relbar\mkern-5mu\mapstochar\mkern-2mu\rightarrow} \mathscr{A} \qquad \mapsto \qquad \mathscr{C} \overset{M\otimes N}{\relbar\mkern-5mu\mapstochar\mkern-2mu\rightarrow} \mathscr{A}$$

of modules, defined by

$$(M \otimes N)(c, a) = \int^{b} M(c, b) \times N(b, a) = \left(\coprod_{b} M(c, b) \times N(b, a) \right) \Big/ \sim .$$

Fix a module $\mathscr{A} \overset{M}{\relbar\mkern-5mu\mapstochar\mkern-2mu\rightarrow} \mathscr{A}$. Then $\in [\mathscr{A}, \mathbf{Set}]$

$$\mathbf{1} \overset{X}{\relbar\mkern-5mu\mapstochar\mkern-2mu\rightarrow} \mathscr{A} \overset{M}{\relbar\mkern-5mu\mapstochar\mkern-2mu\rightarrow} \mathscr{A} \qquad \mapsto \qquad \mathbf{1} \overset{M\otimes X}{\relbar\mkern-5mu\mapstochar\mkern-2mu\rightarrow} \mathscr{A},$$

so we have an endofunctor $M \otimes -$ of $[\mathscr{A}, \mathbf{Set}]$. Explicitly,

$$(M \otimes X)(a) = \left(\coprod_{b \in \mathscr{A}} M(b, a) \times X(b) \right) \Big/ \sim .$$

The equivalence class of $(b \overset{m}{\relbar\mkern-5mu\mapstochar\mkern-2mu\rightarrow} a,\ x \in X(b))$ is written $m \otimes x$.

Coalgebras and modules

Fix a module $\mathscr{A} \overset{M}{\nrightarrow} \mathscr{A}$. A coalgebra for $[\mathscr{A}, \mathbf{Set}]^{\circlearrowleft M\otimes -}$ consists of

a functor $\mathscr{A} \xrightarrow{X} \mathbf{Set}$ and a natural transformation $X \xrightarrow{\xi} M \otimes X$.

The components of ξ are maps

$$\xi_a : X(a) \longrightarrow (M \otimes X)(a) = (\coprod_b M(b,a) \times X(b))/\sim .$$

Take an element $x \in X(a)$. Then:

- $\xi_a(x) \quad = m_1 \otimes x_1$ for some $a_1 \overset{m_1}{\nrightarrow} a$ and $x_1 \in X(a_1)$
- $\xi_{a_1}(x_1) = m_2 \otimes x_2$ for some $a_2 \overset{m_2}{\nrightarrow} a_1$ and $x_2 \in X(a_2)$

$$\vdots$$

So x gives rise to a diagram

$$\cdots \overset{m_3}{\nrightarrow} a_2 \overset{m_2}{\nrightarrow} a_1 \overset{m_1}{\nrightarrow} a \qquad (*)$$

—a complex ending in a. But many choices were involved!

Essential uniqueness of the complex

Q. To what extent is the complex $(*)$ uniquely determined by the element x?

A. All the complexes arising from x in this way are in the same connected-component of the category **Complex**(a) of complexes ending at a

—*as long as* $X : \mathscr{A} \longrightarrow \mathbf{Set}$ is a sum of flat functors.
(If $\mathscr{A}$ has finite limits, this just means that X preserves pullbacks.)

So then:
Given a coalgebra (X, ξ) and an object $a \in \mathscr{A}$, every element $x \in X(a)$ gives rise *canonically* to a connected-component of **Complex**(a).

A terminal coalgebra theorem

Let $\mathscr{A} \overset{M}{\longrightarrow\!\!\!\!\!\!+\;\;} \mathscr{A}$ be a flat module.

Then $[\mathscr{A}, \mathbf{Set}]^{\circlearrowleft M\otimes -}$ restricts to an endofunctor $\mathbf{Flat}(\mathscr{A}, \mathbf{Set})^{\circlearrowleft M\otimes -}$.

There is a particular coalgebra (T, τ) for this restricted endofunctor, with

$$T(a) = \{\text{connected-components of the category } \mathbf{Complex}(a)\}.$$

We know: for every coalgebra (X, ξ), there is a canonical map $X \longrightarrow T$.

Theorem

Suppose that the category $\mathbf{Complex}(M)$ *of all complexes is cofiltered.*

Then (T, τ) *is the terminal coalgebra for* $\mathbf{Flat}(\mathscr{A}, \mathbf{Set})^{\circlearrowleft M\otimes -}$.

A terminal coalgebra theorem

Let $\mathscr{A}$ be a category with finite limits.

Let $\mathscr{A} \overset{M}{\longrightarrow\!\!\!\!\!\!\!+\;} \mathscr{A}$ be a module such that for each $a \in \mathscr{A}$, the functor $M(a,-) : \mathscr{A} \longrightarrow \mathbf{Set}$ preserves finite limits.

Then $[\mathscr{A}, \mathbf{Set}]^{\circlearrowleft M\otimes -}$ restricts to an endofunctor $\mathbf{FinLim}(\mathscr{A}, \mathbf{Set})^{\circlearrowleft M\otimes -}$.

There is a particular coalgebra (T, τ) for this restricted endofunctor, with

$$T(a) = \{\text{connected-components of the category } \mathbf{Complex}(a)\}.$$

We know: for every coalgebra (X, ξ), there is a canonical map $X \longrightarrow T$.

Theorem

every finitary endofunctor is of this form

(T, τ) is the terminal coalgebra for $\mathbf{FinLim}(\mathscr{A}, \mathbf{Set})^{\circlearrowleft M\otimes -}$

every LFP category is of this form