Here are a few more applications of the General Adjoint Functor Theorem, continuing from Example E1.7(a) (free groups). They all require a little specialist knowledge, so aren't particularly examinable.

Example E1.7(b) Any forgetful functor between categories of algebras has a left adjoint, arguing more or less as for free groups. For instance, the forgetful functor

$$U : (\text{associative algebras over } k) \longrightarrow (\text{Lie algebras over } k)$$

has a left adjoint—'universal enveloping algebra'—for any field k. This time we use

$$\Phi(R) = R \cup \{r_1 \cdot r_2 \mid r_1, r_2 \in R\} \cup \{1\} \cup \{cr \mid r \in R, c \in k\}$$

and so have

$$|\langle S \rangle| \leq \max\{\aleph_0, |k|, |S|\}.$$

(c) Similarly topological-algebraic examples: e.g.

$$U : (\text{topological groups \& continuous homomorphisms}) \longrightarrow \mathbf{Top}.$$

(d) A more simple size argument tells you that the inclusion

$$(\text{Hausdorff spaces \& continuous maps}) \hookrightarrow \mathbf{Top}$$

has a left adjoint ('Hausdorffification'). This uses the fact that any subspace of a Hausdorff space is Hausdorff, but note that this fails if we substitute the word 'compact' for 'Hausdorff'. So a similar argument cannot be applied for compact Hausdorff spaces; instead, we can treat this case with the *Special* Adjoint Functor Theorem.