An Informal Introduction to Category Theory

Tom Leinster

Department of Pure Mathematics, University of Cambridge
Email: leinster@dpmms.cam.ac.uk
Web: http://www.dpmms.cam.ac.uk/∼leinster

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This is meant to give you a taste of some of the ideas and concerns of category theory. None of what follows is directly examinable, although most will eventually be made precise in lectures.

We begin (Section 1) by looking at various examples of 'universal properties', which are a major theme of category theory. One of the main services of category theory to mathematics at large is to provide a language in which many things can be expressed more easily, and in Section 2 we go through the basic categorical vocabulary. (This actually gives you a fairly complete outline of the course as a whole.) Finally, for casual browsing, Section 3 presents some of the further directions in which category theory can be taken, beyond this course. The choice of topics here is thoroughly subjective and somewhat random; I have omitted many research areas and applications.

1 Universal Properties

Example 1 Take a pair U, V of finite-dimensional vector spaces with chosen bases $(u_i)_{i\in I}$ and $(v_j)_{j\in J}$. There are 'natural' bijections

$$\begin{array}{rcl} \text{linear maps } U \longrightarrow V & \leftrightsquigarrow & \text{matrices of the appropriate size} \\ & \leftrightsquigarrow & \text{functions } I \longrightarrow V. \end{array}$$

(The ith column of the matrix tells you the value of the linear map at u_i, and of course the matrix determines the whole linear map.) So if we define a function

$$b : I \longrightarrow U$$

by

$$i \longmapsto u_i$$

then we have a 'unique factorization property':

$$\begin{tikzcd} I \arrow[r, hook, "b"] \arrow[dr, "f"'] & U \arrow[d, dotted, "\exists!\text{ linear }\alpha"] \\ & V. \end{tikzcd}$$

This means: for all vector spaces V and all functions $f : I \longrightarrow V$, there exists a unique linear map $\alpha : U \longrightarrow V$ such that $\alpha(u_i) = f(i)$ for all $i \in I$.

Example 2 Fix an integral domain R. Then its field of fractions $F(R)$ is the 'universal field containing R as a subring', in the following sense: given a field k and an injective homomorphism $f : R \longrightarrow k$, there's a unique embedding of fields $\overline{f} : F(R) \longrightarrow k$ such that $\overline{f}(r/1) = f(r)$ for all $r \in R$. This is depicted as

$$\begin{tikzcd} R \arrow[r, hook] \arrow[dr, hook, "\text{injective homomorphism } f"'] & F(R) \arrow[d, dotted, "\exists!\text{ embedding }\overline{f}\text{ of fields}"] \\ & k. \end{tikzcd}$$

Example 3 Given vector spaces U and V, there is a 'universal bilinear map out of $U \times V$'. In other words, there is a certain vector space T, and a certain bilinear map $h : U \times V \longrightarrow T$, with the following unique factorization property:

$$\begin{tikzcd} U \times V \arrow[r, "\text{bilinear } h"] \arrow[dr, "\text{bilinear } f"'] & T \arrow[d, dotted, "\exists!\text{ linear }\overline{f}"] \\ & W. \end{tikzcd} \qquad (*)$$

(In fact, T is the tensor product $U \otimes V$, for those who know what that means.)

A universal property always determines the object in question up to isomorphism (and who could want more?). For instance, in Example 3 we have:

Proposition *Let U and V be vector spaces. Suppose that $h : U \times V \longrightarrow T$ and $h' : U \times V \longrightarrow T'$ are both universal bilinear maps out of $U \times V$. Then $T \cong T'$.*

Proof Take $(U \times V \xrightarrow{f} W) = (U \times V \xrightarrow{h} T)$ in $(*)$ to obtain a linear map $i : T \longrightarrow T'$ satisfying $i \circ h = h'$. Similarly, use the universality of h' to obtain a linear map $i' : T' \longrightarrow T$ satisfying $i' \circ h' = h$.

We now have a linear map $i' \circ i : T \longrightarrow T$ with $(i' \circ i) \circ h = id_T \circ h$, as illustrated:

$$\begin{tikzcd}
 & T \arrow[d, "i"] \\
U \times V \arrow[ur, "h"] \arrow[r, "h'"] \arrow[dr, "h"'] & T' \arrow[d, "i'"] \\
 & T.
\end{tikzcd}$$

By the uniqueness part of $(*)$, this implies that $i' \circ i = id_T$. Similarly, $i \circ i' = id_{T'}$. Hence $T \cong T'$. $\square$

The lesson is that once we've found a universal property of an object, we can ignore its explicit construction; the universal property tells us everything we need to know. For instance, there are many possible ways of constructing the tensor product of vector spaces, but once you've proved $(*)$ above you can forget the construction entirely without losing any information. The same principle applies to the product of sets. Rather few mathematicians could write down the standard set-theoretic definition of the cartesian product of sets (that is, of ordered pair), but every mathematician knows that in practice it's not *necessary* to know this definition. All you actually need to know about the product is its universal property:

Example 4 Let X and Y be sets. Then there are projections

$$\begin{array}{ccccc}
X & \xleftarrow{p_1} & X \times Y & \xrightarrow{p_2} & Y \\
x & \longleftarrow\!\!\shortmid & (x, y) & \longmapsto & y
\end{array}$$

with the following universal property:

$$\begin{tikzcd}
 & W \arrow[d, dotted, "\exists! f"'] \arrow[ddl, "f_1"'] \arrow[ddr, "f_2"] & \\
 & X \times Y \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\
X & & Y.
\end{tikzcd}$$

That is: for any set W and pair of functions $f_1 : W \longrightarrow X$, $f_2 : W \longrightarrow Y$, there is a unique function $f : W \longrightarrow X \times Y$ such that $p_1 \circ f = f_1$ and $p_2 \circ f = f_2$. (Namely, $f(w) = (f_1(w), f_2(w))$.)

Example 5 The same universal property holds for the product of topological spaces X and Y, where this time p_i, f_i and f are *continuous* maps. The universal property holds because, by definition, the 'product topology' on $X \times Y$ is the *smallest* topology for which p_1 and p_2 are continuous.

Example 6 Let $f : A \longrightarrow B$ be a homomorphism of abelian groups; then we have a diagram

$$\ker(f) \overset{i}{\hookrightarrow} A \underset{0}{\overset{f}{\rightrightarrows}} B.$$

$i : \ker(f) \hookrightarrow A$ satisfies $f \circ i = 0 \circ i$ and 'is universal with this property', in a sense to be supplied by the diligent reader.

Universal properties arise in various different manners, and accordingly category theory provides various different pieces of language with which to describe them. Thus Examples 1 and 2 are most readily described in terms of 'adjoint functors', and Example 3 as a 'representable functor'. Examples 4–6 are all 'limits'. Here are some examples of 'colimits'—a concept which is dual to that of limit, in the sense that all the arrows get reversed.

Example 7 Let $f : A \longrightarrow B$ be a homomorphism of abelian groups. Then the quotient map

$$q : B \longrightarrow \operatorname{coker}(f) = B/\operatorname{im}(f)$$

has the universal property illustrated:

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & B \arrow[r, "q"] \arrow[dr, "g"'] & \operatorname{coker}(f) \arrow[d, dashed, "\exists!\overline{g}"] \\ & & C. \end{tikzcd}$$

In other words, given an abelian group C and a homomorphism $g : B \longrightarrow C$ satisfying $g \circ f = 0$, there is a unique homomorphism $\overline{g} : \operatorname{coker}(f) \longrightarrow C$ such that $g = \overline{g} \circ q$.

Example 8 Let X be a topological space and let $E \subseteq X \times X$ be an equivalence relation on X. Then the set X/E of equivalence classes can be given the quotient topology. We then have a diagram

$$E \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} X \xrightarrow{\;q\;} X/E$$

(where pr_i are the projections), and the quotient map $q : X \longrightarrow X/E$ has a universal property exactly analogous to that of the q in the previous example. This property holds because the quotient topology is *largest* such that q is continuous (cf. Example 5).

Example 9 The circle S^1 also has a universal property of this kind, by regarding it as a real interval with the endpoints identified:

$$\begin{tikzcd} \{\mathrm{pt}\} \arrow[r, shift left, "0"] \arrow[r, shift right, "1"'] & {[0,1]} \arrow[r] & S^1. \end{tikzcd}$$

Here $\{\mathrm{pt}\}$ is a ('the'?) one-point space.

Example 10 For a final example, let X be a topological space covered by two open subsets: $X = U \cup V$. Then the diagram of inclusion maps

$$\begin{tikzcd} U \cap V \arrow[r, "i"] \arrow[d, "j"'] & U \arrow[d, "j'"] \\ V \arrow[r, "i'"'] & X \end{tikzcd}$$

has a universal property in the world of topological spaces and continuous maps:

$$\begin{tikzcd} U \cap V \arrow[r, "i"] \arrow[d, "j"'] & U \arrow[d, "j'"] \arrow[ddr, "f"] & \\ V \arrow[r, "i'"'] \arrow[drr, "g"'] & X \arrow[dr, dotted, "\exists! h"] & \\ & & Y. \end{tikzcd}$$

That is, given Y, f, g such that $g \circ j = f \circ i$, there exists unique h such that $h \circ i' = g$ and $h \circ j' = f$.

Under favourable conditions, the induced diagram

$$\begin{tikzcd} \pi_1(U \cap V) \arrow[r, "i_*"] \arrow[d, "j_*"'] & \pi_1(U) \arrow[d, "j'_*"] \\ \pi_1(V) \arrow[r, "i'_*"'] & \pi_1(X) \end{tikzcd}$$

of fundamental groups has the same property in the world of groups and group homomorphisms. This is *van Kampen's Theorem.* In fact, van Kampen's original formulation (1935, before the birth of categories) was rather complicated, in terms of generators and relations; but using the language of colimits one can state the result transparently.

2 Language

One of the most immediate uses of category theory in mathematics is to help you express yourself more easily. Among the basic pieces of vocabulary are:

category —a category consists of some objects, some maps between the objects, and a way of composing strings of maps

functor —for instance, the Eilenberg-Steenrod axioms for homology start 'homology is a functor'; this was one of the first uses of categorical language

natural transformation —see 'naturality' below

adjoint functors —probably the most useful piece of categorical language for a mathematician to know (and still not known widely enough)

representability —see below

(co)limit —see below

monad —roughly, a monad is an algebraic theory; see below.

Adjoints, representables and (co)limits each provide a way of getting at the notion of 'universal property', as remarked before Example 7. Theoretically, anything which can be said in terms of one can be said in terms of either of the other two, but in practice it's useful to have all three modes of expression at our disposal.

Naturality This is a recurrent theme in category theory (and the rest of mathematics). In fact, it was an attempt to formalize the idea that led to the birth of category theory. Some examples:

a. A finite-dimensional vector space is 'naturally' isomorphic to its double dual: $V \cong V^{**}$. That is, we don't have to make any arbitrary choices (such as choosing a basis) in order to specify the isomorphism; it's somehow god-given.

b. For finite sets X, there are $|X|!$ permutations of X and $|X|!$ linear orderings of X, but there's *no* 'natural' correspondence between permutations and orderings. (A *linear* or *total ordering* of an n-element set X can be defined as a bijection between X and $\{1, \ldots, n\}$.)

c. For vector spaces U, V and W, there's a 'natural' correspondence

$$\{\text{bilinear maps } U \times V \longrightarrow W\} \cong \{\text{linear maps } U \otimes V \longrightarrow W\}.$$

d. For sets W, X and Y, there's again a 'natural' correspondence

$$\{\text{functions } W \longrightarrow X \times Y\} \cong \{\text{functions } W \longrightarrow X\} \times \{\text{functions } W \longrightarrow Y\}.$$

e. An analogous 'natural' correspondence can be written down for each of the other eight examples of universal properties listed above.

Algebraic Theories Theories of algebraic structures—e.g. the theory of groups, the theory of vector spaces, the theory of Lie algebras—can be expressed in the language of *monads*. In the monadic approach we describe a real vector space as consisting of a set V together with a suitable function

$$\{\text{formal } \mathbb{R}\text{-linear combinations } \sum \lambda_i v_i\} \longrightarrow V,$$

and similarly we describe a group as consisting of a set G together with a suitable function

$$\{\text{words in } G, \text{ such as } x^{-2}y^3xz^{-1}y^4 \ (x,y,z \in G)\} \longrightarrow G.$$

Contrast this with the traditional (or universal algebra) approach, where one uses a somewhat arbitrary selection of primitive operations (e.g. binary $+$ and scalar $\cdot$, for vector spaces) and a somewhat arbitrary selection of axioms (e.g. associativity, distributivity). By 'arbitrary' I mean that one could equally well use a different selection of primitive operations (e.g. ternary addition, $(u,v,w) \longmapsto u+v+w$) and by imposing appropriate axioms, arrive at the same end result (in our example, the same notion of vector space). From the point of view of monads, the operation $(u,v) \longmapsto 2u-v$ has exactly the same status in the theory of vector spaces as the operation $(u,v) \longmapsto u+v$. As usual, category theory makes no arbitrary choices.

3 Further Directions

Here are some further aspects and applications of category theory, all well off the course but perhaps interesting. Don't worry if none of it makes sense.

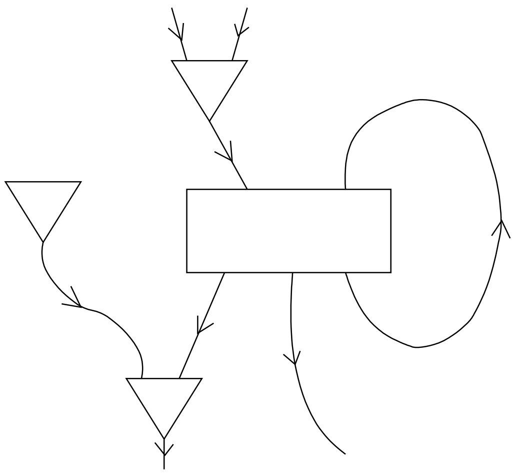
Homological Algebra This is where category theory started. Homological algebra is done in the language of 'abelian categories' (a typical one of which is the category of modules over a fixed commutative ring). In particular, this language makes possible a general definition of many kinds of homology simultaneously: group homology and cohomology, sheaf cohomology, Ext and Tor, ...

Logic and Computer Science Along with the more obvious categories such as sets + functions, groups + homomorphisms, etc, there are examples such as (states of a machine) + computations, or (statements in a formal language) + deductions. A morphism (map) in a category can be seen, in this context, as a 'process'. One of the basic connections between category theory, logic and computer science is that cartesian closed categories are intimately related to the λ-calculus, which in turn is a close relative of functional programming languages. Many category theorists work in computer science departments, and might claim (especially if asked by their funding body) to be working towards understanding the structure of programming languages.

Higher categorical structures In ordinary category theory we have diagrams of objects and arrows such as

$$\bullet \longrightarrow \bullet \longrightarrow \bullet$$

(as we've seen already). We can also consider more complex category-like structures, in which there are diagrams such as



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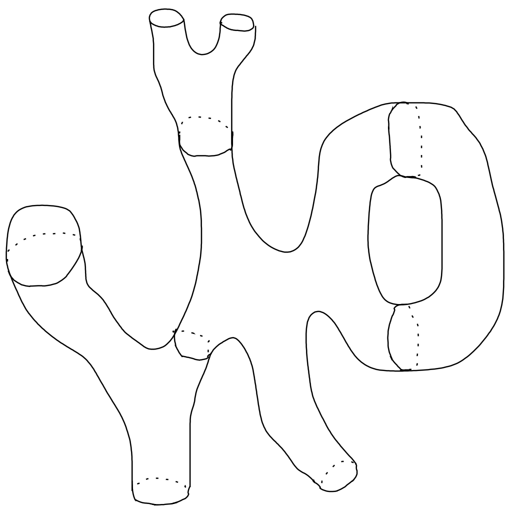
This looks like an electronic circuit diagram or a flow chart; the unifying idea is that of 'information flow'. It can also be redrawn as



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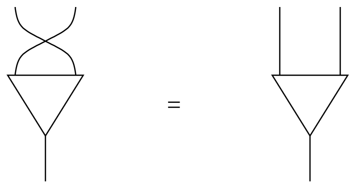
which looks like a topological surface or a diagram from quantum field theory.

You can also use diagrams like this to express algebraic laws, e.g. commutativity:



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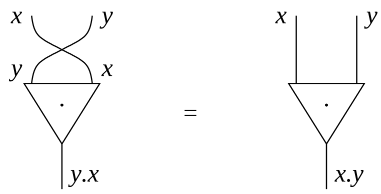
This is perhaps more clear when labels are added:



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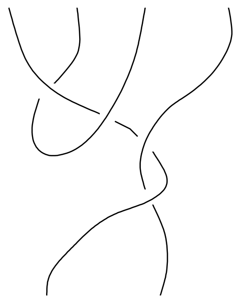
If you know about topological quantum field theory, and the fact that two-dimensional TQFTs are essentially Frobenius algebras, then you will be able to see an explicit link between the spatial and algebraic aspects of these diagrams.

Moreover, if you allow crossings, as in the commutativity diagrams or as in



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then you start getting pictures which look like knots; and there are indeed relations between knot theory and higher categorical structures.

Topos Theory A topos is a category with various special properties. It is both

A a generalized topological space ('a topos is what a topologist needs to study', SGA4, p. 301)

B a generalized universe of sets (for logicians—e.g. topos theory can be used to prove the independence of the axiom of choice and the continuum hypothesis from the ZF axioms of set theory).

The basic flavour of the topos-theoretic connection between **A** and **B** (topology and logic) is as follows. Fix a set X, and let $\mathcal{P}(X)$ be the set of all subsets of X. Then the complement operation

$$\begin{array}{rccc} \neg : & \mathcal{P}(X) & \longrightarrow & \mathcal{P}(X) \\ & U & \longmapsto & X \backslash U \end{array}$$

satisfies the law $\neg\neg U = U$, which can be read as 'not not $U = U$', the law of the excluded middle from classical logic. Now suppose X carries a topology, and let $\mathcal{O}(X)$ be the set of *open* subsets of X. Then the nearest we can get to a 'complement' operation on $\mathcal{O}(X)$ is

$$\begin{array}{rccc} \neg : & \mathcal{O}(X) & \longrightarrow & \mathcal{O}(X) \\ & U & \longmapsto & \text{interior}(X \backslash U). \end{array}$$

In this case $\neg\neg U \neq U$ in general (for instance, when $X = \mathbb{R}$ and $U = \mathbb{R}\backslash\{0\}$). This says that 'excluded middle fails', which is what happens in intuitionistic (constructive) logic: proof by contradiction is not permitted.