F4: The Monadicity Theorem

Tom Leinster

Department of Pure Mathematics, University of Cambridge
Email: leinster@dpmms.cam.ac.uk
Web: http://www.dpmms.cam.ac.uk/~leinster

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This section forms part of the course and is examinable—indeed, is full of likely exam questions. Conceptually it's not hard at all, but it is quite technical and demands a certain amount of effort in order to understand the terminology.

The Monadicity Theorem states (necessary and) sufficient conditions for an adjunction $\mathcal{D} \underset{F}{\overset{U}{\rightleftarrows}} \mathcal{C}$ (with $\top$) to be monadic. But why should we want to know that a given adjunction is monadic? Here are two possible reasons:

- Because we already have a lot of intuition about monadic adjunctions, from having done algebra for years—typical monadic adjunctions being $\mathbf{Gp} \overset{\top}{\rightleftarrows} \mathbf{Set}$ and $\mathbf{Vect} \overset{\top}{\rightleftarrows} \mathbf{Set}$. (In fact we have yet to show that all such 'algebraic' categories are monadic over **Set**, but this will be one of the consequences of the Monadicity Theorem.)
- Because various properties of monadic adjunctions are known, e.g. that U creates limits. So for instance, any category monadic over a complete category is complete. In particular, a great deal is known about categories monadic over **Set**, e.g. that such categories are also *co*complete. (This tells us, then, that it's possible to form the coproduct of groups, which is perhaps not instantly obvious.)

Having seen the trouble it takes to prove the Monadicity Theorem, and even to understand the terminology in the statement of it, one might be tempted to ignore it and prove monadicity of a given adjunction by showing directly that the comparison functor K is an equivalence. However, this is in general hard: often we don't have an explicit description of the left adjoint F in an adjunction $F \dashv U$, and only know that it exists by the use of an Adjoint Functor Theorem. This is why, for instance, we haven't actually yet been able to show that **Gp** is monadic over **Set**: the free group functor is a little tricky to describe explicitly. The magical thing about the Monadicity Theorem is that it gives conditions for an adjunction $F \dashv U$ to be monadic *without reference to F*.

Part One: Properties of U^T

We approach the Monadicity Theorem in two steps. In this first step, fix a monad (T, η, μ) on a category $\mathcal{C}$, and form the usual adjunction $\mathcal{C}^T \underset{F^T}{\overset{U^T}{\rightleftarrows}} \mathcal{C}$ (with $\top$). (For this step we're not concerned with the issue of comparison at all—in other words, you can forget $\mathcal{D}$, U, F and K for now.)

The functor U^T has various properties, including the following:

F4.1 Proposition

a. U^T *creates limits*

b. U^T *reflects isomorphisms*

c. U^T *is faithful.*

Proof See problem sheet 4. □

We don't actually need the results in the previous Proposition. But central to the Monadicity Theorem is the following property of U^T:

F4.2 Proposition U^T *creates coequalizers for* U^T*-absolute-coequalizer pairs.*

What can this possibly mean?

F4.3 Definitions

a. An *absolute coequalizer diagram* in a category $\mathcal{C}$ is a diagram of objects and arrows

$$A \underset{f_2}{\overset{f_1}{\rightrightarrows}} A' \xrightarrow{d} A'' \qquad (*)$$

in $\mathcal{C}$ such that for any functor $H : \mathcal{C} \longrightarrow \mathcal{E}$ (to any category $\mathcal{E}$), the image of $(*)$ under H is a coequalizer diagram in $\mathcal{E}$. (In particular we can take $H = 1_{\mathcal{C}}$, so $(*)$ itself is a coequalizer.)

b. An *absolute-coequalizer pair* in $\mathcal{C}$ is a parallel pair of arrows $A \underset{f_2}{\overset{f_1}{\rightrightarrows}} A'$ in $\mathcal{C}$ such that for some A'' and d,

$$A \underset{f_2}{\overset{f_1}{\rightrightarrows}} A' \xrightarrow{d} A''$$

is an absolute coequalizer diagram.

c. If $U : \mathcal{D} \longrightarrow \mathcal{C}$ is a functor, a *U-absolute-coequalizer pair* is a parallel pair $B \underset{g_2}{\overset{g_1}{\rightrightarrows}} B'$ in $\mathcal{D}$ such that $UB \underset{Ug_2}{\overset{Ug_1}{\rightrightarrows}} UB'$ is an absolute-coequalizer pair in $\mathcal{C}$.

More generally, if 'purple' is some property, then a thing is said to be 'U-purple' if its image under U is purple. 'Pair' is an abbreviation for 'parallel pair of arrows'.

What it means for a functor U to *create* coequalizers for U-absolute-coequalizer pairs can be read off from the general definition of creation (D3.3). Spelling it out, it means that for any U-absolute-coequalizer pair $B \overset{g_1}{\underset{g_2}{\rightrightarrows}} B'$, there is a map $B' \xrightarrow{e} B''$ out of B' such that $UB \overset{Ug_1}{\underset{Ug_2}{\rightrightarrows}} UB' \xrightarrow{Ue} UB''$ is a coequalizer diagram; and moreover that any e with this property is a coequalizer of g_1 and g_2.

We actually show that U^T creates coequalizers for U^T-absolute-coequalizer pairs in the stronger (and more easily understandable?) sense of the word 'creates', as given in Example D3.2(c). That is, given a parallel pair

$$\begin{pmatrix} TA \\ \downarrow \theta \\ A \end{pmatrix} \overset{g_1}{\underset{g_2}{\rightrightarrows}} \begin{pmatrix} TA' \\ \downarrow \theta' \\ A' \end{pmatrix}$$

in $\mathcal{C}^T$ and an absolute coequalizer diagram

$$A \overset{g_1}{\underset{g_2}{\rightrightarrows}} A' \xrightarrow{e} A''$$

in $\mathcal{C}$, we show that

(i) there is a unique algebra structure $\theta'' : TA'' \longrightarrow A''$ on A'' such that e is an algebra map, and then

(ii) $\begin{pmatrix} TA \\ \downarrow \theta \\ A \end{pmatrix} \overset{g_1}{\underset{g_2}{\rightrightarrows}} \begin{pmatrix} TA' \\ \downarrow \theta' \\ A' \end{pmatrix} \xrightarrow{e} \begin{pmatrix} TA'' \\ \downarrow \theta'' \\ A'' \end{pmatrix}$ is a coequalizer diagram in $\mathcal{C}^T$.

Proof of F4.2 For (i), we have a diagram

$$\begin{tikzcd} TA \arrow[r, shift left, "Tg_1"] \arrow[r, shift right, "Tg_2"'] \arrow[d, "\theta"'] & TA' \arrow[r, "Te"] \arrow[d, "\theta'"] & TA'' \\ A \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & A' \arrow[r, "e"'] & A'' \end{tikzcd} \qquad (\dagger)$$

in $\mathcal{C}$, the left-hand half of which is *serially commutative*, i.e. the top square commutes ($\theta' \circ Tg_1 = g_1 \circ \theta$) and the bottom square commutes ($\theta' \circ Tg_2 = g_2 \circ \theta$). The bottom row is an *absolute* coequalizer, so both rows are coequalizers. Hence there's a unique map $\begin{pmatrix} TA'' \\ \downarrow \theta'' \\ A'' \end{pmatrix}$ making the right-hand half of ($\dagger$) commute.

To finish (i) we have to verify that θ'' satisfies the algebra axioms. Note that we have serially commutative diagrams

$$\begin{tikzcd}
T^2A \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "T\theta"'] & T^2A' \arrow[r] \arrow[d, "T\theta'"] & T^2A'' \arrow[d, "T\theta''"] \\
TA \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\theta"'] & TA' \arrow[r] \arrow[d, "\theta'"] & TA'' \arrow[d, "\theta''"] \\
A \arrow[r, shift left] \arrow[r, shift right] & A' \arrow[r] & A''
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2A \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\mu_A"'] & T^2A' \arrow[r] \arrow[d, "\mu_{A'}"] & T^2A'' \arrow[d, "\mu_{A''}"] \\
TA \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\theta"'] & TA' \arrow[r] \arrow[d, "\theta'"] & TA'' \arrow[d, "\theta''"] \\
A \arrow[r, shift left] \arrow[r, shift right] & A' \arrow[r] & A'',
\end{tikzcd}$$

and so in the diagram

$$\begin{tikzcd}
T^2A \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\theta\circ T\theta = \theta\circ\mu_A"'] & T^2A' \arrow[r] \arrow[d, "\theta'\circ T\theta' = \theta'\circ\mu_{A'}"] & T^2A'' \arrow[d, dotted] \\
A \arrow[r, shift left] \arrow[r, shift right] & A' \arrow[r] & A'',
\end{tikzcd}$$

taking the dotted arrow to be *either* $\theta''\circ T\theta''$ *or* $\theta''\circ\mu_{A''}$ makes the right-hand square commute. But the top row of this diagram is a coequalizer, which implies that there is only one dotted arrow making the right-hand square commute: hence $\theta''\circ T\theta'' = \theta''\circ\mu_{A''}$. The other algebra axiom ($\theta\circ\eta_{A''} = 1_{A''}$) holds by a similar argument.

We now have to show (ii), that

$$\begin{pmatrix} TA \\ \downarrow\theta \\ A \end{pmatrix} \underset{g_2}{\overset{g_1}{\rightrightarrows}} \begin{pmatrix} TA' \\ \downarrow\theta' \\ A' \end{pmatrix} \xrightarrow{e} \begin{pmatrix} TA'' \\ \downarrow\theta'' \\ A'' \end{pmatrix}$$

is a coequalizer in $\mathcal{C}^T$. So, take

$$\begin{pmatrix} TA \\ \downarrow\theta \\ A \end{pmatrix} \underset{g_2}{\overset{g_1}{\rightrightarrows}} \begin{pmatrix} TA' \\ \downarrow\theta' \\ A' \end{pmatrix} \xrightarrow{h} \begin{pmatrix} TC \\ \downarrow\phi \\ C \end{pmatrix}$$

with $h\circ g_1 = h\circ g_2$. Since $A \underset{g_2}{\overset{g_1}{\rightrightarrows}} A' \xrightarrow{e} A''$ is a coequalizer, there's a unique map $\overline{h} : A'' \longrightarrow C$ in $\mathcal{C}$ such that

$$\begin{tikzcd}
A' \arrow[r, "e"] \arrow[dr, "h"'] & A'' \arrow[d, "\overline{h}"] \\
& C
\end{tikzcd}$$

commutes. We now only need to show that $\overline{h}$ is a map of algebras, and this is true because

$$\phi\circ T\overline{h}\circ Te = \phi\circ Th = h\circ\theta' = \overline{h}\circ e\circ\theta' = \overline{h}\circ\theta''\circ Te$$

and Te is (regular) epic. $\square$

So far we haven't seen any examples of absolute coequalizers, and indeed you might feel it's rather unlikely that any coequalizer is preserved by all the functors in the world. However:

F4.4 Crucial Fact *For any T-algebra* $\begin{pmatrix} TA \\ \downarrow \theta \\ A \end{pmatrix}$,

$$T^2A \overset{\mu_A}{\underset{T\theta}{\rightrightarrows}} TA \xrightarrow{\theta} A$$

is an absolute coequalizer diagram.

Proof We actually show rather more: that this is a 'split coequalizer diagram'.

a. A *split coequalizer diagram* is a diagram

$$\begin{tikzcd} X \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & Y \arrow[l, bend left, "v"] \arrow[r, "r"] & Z \arrow[l, bend left, "u"] \end{tikzcd} \qquad (\ddagger)$$

such that

$$rp_1 = rp_2, \qquad ru = 1_Z, \qquad p_1v = 1_Y, \qquad p_2v = ur.$$

b. It's easy to check that

$$\begin{tikzcd} T^2A \arrow[r, shift left, "\mu_A"] \arrow[r, shift right, "T\theta"'] & TA \arrow[l, bend left, "\eta_{TA}"] \arrow[r, "\theta"] & A \arrow[l, bend left, "\eta_A"] \end{tikzcd}$$

is a split coequalizer diagram in $\mathcal{C}$; in fact this says *exactly* that $(TA \xrightarrow{\theta} A)$ is an algebra.

c. Any split coequalizer diagram ($\ddagger$) is an absolute coequalizer diagram. Since equations are preserved by functors, 'absolute' is clear just as long as we can prove that

$$X \overset{p_1}{\underset{p_2}{\rightrightarrows}} Y \xrightarrow{r} Z$$

is a coequalizer. We're given that $rp_1 = rp_2$. Take

$$X \overset{p_1}{\underset{p_2}{\rightrightarrows}} Y \xrightarrow{h} W$$

with $hp_1 = hp_2$: then for maps $\overline{h} : Z \longrightarrow W$,

$$\overline{h}r = h \Rightarrow \overline{h}ru = hu \Rightarrow \overline{h} = hu$$

and conversely

$$(hu)r = hp_2v = hp_1v = h.$$

So there is a unique $\overline{h}$ satisfying $\overline{h}r = h$. $\square$

You might compare the terminology 'split monic'. A split monic is a map which is monic by virtue of a certain equation holding, and so a split monic is an absolute monic—that is, remains monic on application of any functor.

F4.5 Corollary *Every algebra is a coequalizer of free algebras. Precisely, if* $\begin{pmatrix} TA \\ \downarrow \theta \\ A \end{pmatrix} \in \mathcal{C}^T$ *then*

$$\begin{pmatrix} T^3A \\ \downarrow \mu_{TA} \\ T^2A \end{pmatrix} \underset{T\theta}{\overset{\mu_A}{\rightrightarrows}} \begin{pmatrix} T^2A \\ \downarrow \mu_A \\ TA \end{pmatrix} \xrightarrow{\theta} \begin{pmatrix} TA \\ \downarrow \theta \\ A \end{pmatrix} \qquad (\S)$$

is a coequalizer diagram in $\mathcal{C}^T$*.*

Remark In 'everyday algebra', every algebra is a quotient of a free algebra. Coequalizers are the categorical version of quotients. (In the context of monads, an algebra is called *free* if it is isomorphic to one of the form $F^T(A)$.)

Proof It's easy to check that the horizontal arrows in (§) are maps of algebras. F4.4 says that U^T of (§) is an absolute coequalizer diagram. So F4.2 tells us that (§) itself is a coequalizer. $\square$

Part Two: Analysis of K

We've just shown that given a monad T, the forgetful functor U^T creates coequalizers for U^T-absolute-coequalizer pairs. We now show that this exactly characterizes monadic adjunctions.

Fix an adjunction $\mathcal{D} \underset{F}{\overset{U}{\rightleftarrows}} \mathcal{C}$ ($\top$) with unit η and counit ε, let (T, η, μ) be the induced monad on $\mathcal{C}$, and let $K : \mathcal{D} \longrightarrow \mathcal{C}^T$ be the comparison functor.

F4.6 Lemma *Let* $\begin{pmatrix} TA \\ \downarrow \theta \\ A \end{pmatrix} \in \mathcal{C}^T$*, let* $B \in \mathcal{D}$*, and let* $f : A \longrightarrow UB$ *be a map in* $\mathcal{C}$*. Then the following conditions on* f *are equivalent:*

- f *is an algebra map* $\begin{pmatrix} TA \\ \downarrow \theta \\ A \end{pmatrix} \longrightarrow K(B)$
- $FUFA \underset{F\theta}{\overset{\varepsilon_{FA}}{\rightrightarrows}} FA \xrightarrow{\overline{f}} B$ *is a* fork *(i.e.* $\overline{f} \circ \varepsilon_{FA} = \overline{f} \circ F\theta$*).*

Proof f is an algebra map if and only if the square

$$\begin{array}{ccc} UFA & \xrightarrow{UFf} & UFUB \\ \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle U\varepsilon_B} \\ A & \xrightarrow[f]{} & UB \end{array}$$

commutes. The transposes of the composites $\downarrow_{\rightarrow}$ and ${}^{\rightarrow}\!\downarrow$ around this square are given by

$$\frac{UFA \xrightarrow{\theta} A \xrightarrow{f} UB}{FUFA \xrightarrow{F\theta} FA \xrightarrow{\overline{f}} B}$$

and

$$\frac{UFA \xrightarrow{1} UFA \xrightarrow{UFf} UFUB \xrightarrow{U\varepsilon_B} UB}{FUFA \xrightarrow{\varepsilon_{FA}} FA \xrightarrow{Ff} FUB \xrightarrow{\varepsilon_B} B,}$$

and in the bottom line, $\varepsilon_B \circ Ff = \overline{f}$. $\square$

We now prove a pair of propositions which together give sufficient conditions for K to be an equivalence. Recall that a functor is an equivalence just when it is full, faithful and essentially surjective on objects.

F4.7 Proposition *Suppose that for all $B \in \mathcal{D}$,*

$$FUFUB \underset{\varepsilon_{FUB}}{\overset{FU\varepsilon_B}{\rightrightarrows}} FUB \xrightarrow{\varepsilon_B} B$$

is a coequalizer (in $\mathcal{D}$). Then K is full and faithful.

Remarks The condition says 'every $\mathcal{D}$-object is a coequalizer of free $\mathcal{D}$-objects'. In fact the converse of both this Proposition and the next are true, as may be seen; but we do not need to know this for the present purposes.

Proof K being full and faithful says:

> for all $B, B' \in \mathcal{D}$ and $f : KB \longrightarrow KB'$ in $\mathcal{C}^T$, there is a unique $g : B \longrightarrow B'$ in $\mathcal{D}$ such that $Kg = f$.

By the Lemma, this says:

> for all $B, B' \in \mathcal{D}$ and $f : UB \longrightarrow UB'$ in $\mathcal{C}$ such that
>
> $$FUFUB \underset{\varepsilon_{FUB}}{\overset{FU\varepsilon_B}{\rightrightarrows}} FUB \xrightarrow{\overline{f}} B'$$
>
> is a fork, there is a unique $g : B \longrightarrow B'$ in $\mathcal{D}$ with $Ug = f$.

Then just note that for maps $g : B \longrightarrow B'$,

$$\begin{tikzcd} FUB \arrow[r, "\varepsilon_B"] \arrow[dr, "\overline{f}"'] & B \arrow[d, "g"] \\ & B' \end{tikzcd}$$

commutes if and only if $\overline{Ug} = \overline{f}$, if and only if $Ug = f$. □

F4.8 Proposition *Suppose that for all* $\begin{pmatrix} TA \\ \downarrow \theta \\ A \end{pmatrix} \in \mathcal{C}^T$*, there exists a map* $g : FA \longrightarrow B$ *in* $\mathcal{D}$ *such that*

$$T^2A \underset{T\theta}{\overset{\mu_A}{\rightrightarrows}} TA = UFA \xrightarrow{Ug} UB$$

is a coequalizer (in $\mathcal{C}$*). Then* K *is essentially surjective on objects.*

Proof Let $\begin{pmatrix} TA \\ \downarrow \theta \\ A \end{pmatrix} \in \mathcal{C}^T$, and take g as above. Then we have a fork

$$\begin{tikzcd} T^3A \arrow[r, shift left, "T\mu_A"] \arrow[r, shift right, "T^2\theta"'] \arrow[d, "\mu_{TA}"'] & T^2A = UFUFA \arrow[r, "UFUg"] \arrow[d, "\mu_A = U\varepsilon_{FA}"] & UFUB \arrow[d, "U\varepsilon_B"] \\ T^2A \arrow[r, shift left, "\mu_A"] \arrow[r, shift right, "T\theta"'] & TA = UFA \arrow[r, "Ug"'] & UB \end{tikzcd} \qquad (¶)$$

in $\mathcal{C}^T$. If we can show that it is a coequalizer then, by comparison with the standard coequalizer diagram of F4.5, we can deduce that $K(B) \cong \begin{pmatrix} TA \\ \downarrow \theta \\ A \end{pmatrix}$ and be done. In order to do this we could just calculate with our bare hands, but don't need to: for the left-hand half of (¶) is a U^T-absolute-coequalizer pair (F4.4), and U^T reflects coequalizers for such pairs (F4.2). □

Part Three: Conclusion

Finally, the payoff.

F4.9 Theorem (The Monadicity Theorem) *Let* $U : \mathcal{D} \longrightarrow \mathcal{C}$ *be a functor with a left adjoint. Then the following are equivalent:*

a. U *is monadic*

b. U *creates coequalizers for* U*-absolute-coequalizer pairs*

c. $\mathcal{D}$ *has coequalizers of* U*-split-coequalizer pairs, and* U *preserves and reflects coequalizers of such pairs.*

Proof (a $\Rightarrow$ b): F4.2, plus the fact that an equivalence creates all (co)limits.

(b $\Rightarrow$ c): this is trivial once you've worked out what the terminology means; compare Exercise D3.4. (For that matter, (c $\Rightarrow$ b) is equally trivial.)

(c $\Rightarrow$ a): F4.7 together with 'reflects' and F4.4 (the Crucial Fact) implies that K is full and faithful. F4.8 together with 'has', 'preserves' and F4.4 (again) implies that K is essentially surjective on objects. $\square$

There are many versions of the Monadicity Theorem (see problem sheet 4 or any book on category theory), according to various different ways of dressing up the hypotheses of Propositions F4.7 and F4.8. However, you can regard the basic statement as (a $\Leftrightarrow$ b); and in fact, part (b) is often the most sensible condition to use when trying to prove that a given adjunction is monadic.

F4.10 Examples

a. $U : \mathbf{Gp} \longrightarrow \mathbf{Set}$ (or any other similar forgetful functor on a category of algebras) is monadic. We've already seen that by the General Adjoint Functor Theorem, U has a left adjoint, so it suffices to check that U creates coequalizers of U-absolute-coequalizer pairs.

   So, let $G \underset{f_2}{\overset{f_1}{\rightrightarrows}} G'$ be a pair of homomorphisms between groups, and suppose that $G \underset{f_2}{\overset{f_1}{\rightrightarrows}} G' \xrightarrow{d} S$ is an absolute coequalizer in $\mathbf{Set}$. (Strictly speaking I should have written $U(G)$ rather than G in the second half of that sentence, but I'm adopting the usual mathematical custom of confusing a group with its underlying set.) We have to show that S carries a unique group structure such that d is a homomorphism, and that the resulting fork in $\mathbf{Gp}$ is a coequalizer.

   Consider the diagram

$$\begin{tikzcd}
G \times G \arrow[r, shift left, "f_1 \times f_1"] \arrow[r, shift right, "f_2 \times f_2"'] \arrow[d, "\cdot"'] & G' \times G' \arrow[r, "d \times d"] \arrow[d, "\cdot"] & S \times S \arrow[d, dotted] \\
G \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & G' \arrow[r, "d"] & S
\end{tikzcd}$$

   in $\mathbf{Set}$. Since the bottom row is an *absolute* coequalizer, the top row is a coequalizer, and so there is a unique dotted arrow $\cdot : S \times S \longrightarrow S$ making the diagram commute. We obtain $(\ )^{-1} : S \longrightarrow S$ and $u : 1 \longrightarrow S$ similarly. By an argument like the proof of F4.2, the operations $\cdot$, $(\ )^{-1}$ and u obey the group axioms, and d is the coequalizer of f_1 and f_2 in $\mathbf{Gp}$.

b. *Not* $U : \mathbf{Top} \longrightarrow \mathbf{Set}$: for as asserted in F4.1, any monadic functor reflects isomorphisms, and a continuous bijection needn't be a homeomorphism.

c. The forgetful functor $U : \mathbf{CptHff} \longrightarrow \mathbf{Set}$ is monadic: see Mac Lane, *Categories for the Working Mathematician*, VI.9. (Notice in particular that a continuous bijection between compact Hausdorff spaces *is* a homeomorphism.)

One might wonder what the induced monad on the category of sets is, and indeed 'why' compact Hausdorff spaces are at all like an algebraic theory. The answer I have for you will only make precise sense if you know a fair amount of general topology, but I hope that in any case the basic idea will come across. *Ultrafilters* on a topological space play very roughly the same role as sequences in a metric space: e.g. you can talk about an ultrafilter converging to a point, and knowing which ultrafilters converge to which points tells you a great deal about the structure of the space. A space is compact Hausdorff if and only if every ultrafilter on it converges to a unique point. So it makes some sense that the monad T on **Set** induced by U and its left adjoint is

$$TA = \{\text{ultrafilters on } A\};$$

thus a T-algebra is a set equipped with a function saying 'which point each ultrafilter converges to'.

d. Another non-example: the forgetful functor from the category of posets to the category of sets does not reflect isomorphisms, so is not monadic.

e. The forgetful functor $U : \bigvee\text{-}\mathbf{Latt} \longrightarrow \mathbf{Set}$ is monadic. Here $\bigvee$-**Latt** is the category of *sup-lattices* (or *join-lattices*, or *complete semi-lattices*), whose objects are posets in which every subset S has a sup or least upper bound $\bigvee S$ (or equivalently, all colimits exist), and whose maps are functions preserving $\bigvee$ (equivalently, colimits). The left adjoint to U sends a set S to its power-set $\mathcal{P}(S)$ with the usual ordering. Monadicity can either be proved with one's bare hands (as in Borceux, *Handbook of Categorical Algebra*, Vol. 2) or by a rather delicate application of the Monadicity Theorem.

Again, one might wonder 'why' the theory of $\bigvee$-lattices is in any sense algebraic, especially when the theory of plain posets is not. An answer lies in the fact that a $\bigvee$-lattice structure on a set A can be described as consisting of a function

$$\bigvee\nolimits_I : A^I \longrightarrow A$$

for each set I (or each cardinal, if you prefer), obeying certain equations: for instance, one saying that $\bigvee_1(a) = a$, and another saying that $\bigvee_I(a_i)_{i\in I} = \bigvee_I(b_i)_{i\in I}$ if $(b_i)_{i\in I}$ is got from $(a_i)_{i\in I}$ by a permutation of I. A map of $\bigvee$-lattices is a function commuting with each $\bigvee_I$. This is, then, an 'infinitary algebraic theory': it looks like an ordinary (finitary) algebraic theory, presented by operations and equations, except for the unusual feature that the sets I can be infinite.