E2: The Special Adjoint Functor Theorem

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22 November 2000

Section E2 of the course, on the Special Adjoint Functor Theorem, is examinable. Don't be put off by the length; there's lots of explanatory waffle in here. Like the General Adjoint Functor Theorem, this theorem provides sufficient conditions for a given functor to have a left adjoint.

A left adjoint to a functor $G : \mathcal{D} \longrightarrow \mathcal{C}$ consists of an initial object of $(A \Rightarrow G)$ for each $A \in \mathcal{C}$, as we saw in section B3. Expressed diagrammatically, this says that a left adjoint is an object $FA \in \mathcal{D}$ for each $A \in \mathcal{C}$, together with a map $\eta_A : A \longrightarrow GF(A)$, such that any map $f : A \longrightarrow G(B)$ factors uniquely as

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "f"'] & G(FA) \arrow[d, "G(?)"] \\ & G(B). \end{tikzcd}$$

The General Adjoint Functor Theorem says that under favourable conditions, it's not necessary to find a single object FA with this unique factorization property—it's enough just to find a set $\mathbb{V}$ of objects for each A, such that any f factors (not necessarily uniquely) as

$$\begin{tikzcd} A \arrow[r, "?"] \arrow[dr, "f"'] & G(V) \arrow[d, "G(?)"] \\ & G(B) \end{tikzcd}$$

for some $V \in \mathbb{V}$.

Consider, for instance, our construction of the free group functor (E1.7(a)). By using the GAFT, we didn't need to construct FA exactly; it was roughly speaking enough to realize that FA had to be a group of cardinality at most $\max\{\aleph_0, |A|\}$—and the collection of such groups (up to isomorphism) is small. More accurately, we showed that any function from a set A to a group had to factor through a group belonging to this small collection.

In the SAFT we use a similar strategy. We put extra hypotheses on a category $\mathcal{D}$ so that any object $B \in \mathcal{D}$ embeds as a 'subobject' of a certain larger object $K(B)$, and this K is defined in such a way that if $\mathcal{D} \underset{F}{\overset{G}{\rightleftarrows}} \mathcal{C}$ is an adjunction then $K(FA)$ can be expressed without reference to F. So given a limit-preserving functor $G : \mathcal{D} \longrightarrow \mathcal{C}$, we will show that it has a left adjoint by applying GAFT with $\mathbb{V} = \{\text{subobjects of } K(FA)\}$.

What is K? Perhaps the best example is when $\mathcal{D}$ is the category **CptHff** of compact Hausdorff spaces and continuous maps. (For those without the topological background, all you need to know for now is that the closed interval $[0,1] = \{t \in \mathbb{R} \mid 0 \leq t \leq 1\}$ is compact Hausdorff, and that compact Hausdorff spaces are closed under taking products.) Given a compact Hausdorff space B, we form the power $K(B) = [0,1]^{\mathcal{D}(B,[0,1])}$: that is, the product in $\mathcal{D}$ of rather a lot of copies of $[0,1]$, one copy for each map $B \longrightarrow [0,1]$. (Don't confuse this with the notation for exponentials in a cartesian closed category.) There's a canonical map

$$k : B \longrightarrow [0,1]^{\mathcal{D}(B,[0,1])},$$

whose component at $q \in \mathcal{D}(B,[0,1])$ is q itself. Moreover, it's a (non-trivial) fact that if $b_0 \neq b_1$ are points of B then there exists $q : B \longrightarrow [0,1]$ with $q(b_0) \neq q(b_1)$, and k is therefore injective. So we have constructed an embedding $k : B \longrightarrow K(B)$; of course, this is no great feat in itself (we could have taken $K(B) = B$), but the crucial point is that if $\mathcal{D} \underset{F}{\overset{G}{\rightleftarrows}} \mathcal{C}$ is an adjunction and $A \in \mathcal{C}$ then

$$K(FA) \cong [0,1]^{\mathcal{C}(A,G[0,1])}$$

—which formula, as asserted above, makes no reference to F. So if we're given just $G : \mathcal{D} \longrightarrow \mathcal{C}$, our strategy will be to realize FA as a subspace of $[0,1]^{\mathcal{C}(A,G[0,1])}$.

A similar example is $\mathcal{D} = \mathbf{Set}$, where we can consider the embedding

$$\begin{array}{ccc} B & \longrightarrow & \mathcal{P}(B) \\ b & \longmapsto & \{b\} \end{array}$$

and note that $\mathcal{P}(FA) \cong \mathbf{Set}(FA, 2) \cong \mathcal{C}(A, G(2))$; hence $FA \subseteq \mathcal{C}(A, G(2))$. However, this example is slightly misleading, as the K and k used in the proof of SAFT are actually

$$\begin{array}{ccc} B & \longrightarrow & \mathcal{P}\mathcal{P}(B) \\ b & \longmapsto & \{M \in \mathcal{P}(B) \mid b \in M\} \end{array}$$

in this case.

Before we embark on the statement and proof of SAFT, we need some terminology.

Let B be an object of a category $\mathcal{D}$. Given monics $M \overset{m}{\rightarrowtail} B$ and $M' \overset{m'}{\rightarrowtail} B$, we write $(M, m) \leq (M', m')$ if there is an arrow $M \xrightarrow{n} M'$ such that

$$\begin{tikzcd} M \arrow[rr, "n"] \arrow[dr, tail, "m"'] & & M' \arrow[dl, tail, "m'"] \\ & B & \end{tikzcd}$$

commutes. (In fact, n is necessarily monic, and unique such that this diagram commutes. We use $\rightarrowtail$ to denote a monic.) Two monics (M, m) and (M', m') into B are *equivalent* if $(M, m) \leq (M', m') \leq (M, m)$.

E2.1 Definition A *subobject* of an object B is an equivalence class of monics into B.

(The passage from {monics into B} to {subobjects of B} is an example of the passage from a preordered set to the ordered set of equivalence classes.)

E2.2 Examples (a) For injections $M \overset{m}{\rightarrowtail} B$ and $M' \overset{m'}{\rightarrowtail} B$ in **Set**, $(M, m) \leq (M', m')$ if and only if

$$(\text{image of } m) \subseteq (\text{image of } m'),$$

so the two injections are equivalent if and only if their images are equal. Moreover, every subset of B arises as the image of some monic. Hence a subobject of B is just a subset of B.

(b) Similarly, subobjects in **Gp** are subgroups, and so on for all the usual categories of algebras.

In **Set**, a map $g : B' \longrightarrow B$ induces various maps between $\mathcal{P}(B)$ and $\mathcal{P}(B')$ (see sheet 1, q. 12). We don't need to conduct a full investigation of the extent to which this behaviour is reproduced in other categories, but we do need to look at the general analogue of the inverse-image function $g^* : \mathcal{P}(B) \longrightarrow \mathcal{P}(B')$.

E2.3 Lemma The pullback of a monic is monic. That is, if

$$\begin{tikzcd} M' \arrow[r] \arrow[d, "m'"'] & M \arrow[d, "m"] \\ B' \arrow[r, "g"'] & B \end{tikzcd}$$

is a pullback square in a category, then m monic $\Rightarrow$ m' monic.

Proof Exercise. □

E2.4 Example In **Set**, if $M \overset{m}{\rightarrowtail} B$ is the inclusion of a subset M of B, then $M' \overset{m'}{\rightarrowtail} B$ is the inclusion of $g^{-1}M$ (the usual inverse image) into B'.

E2.5 Definition A category is *well-powered* if for every object B, the collection of subobjects of B is small (in other words, there is some *set* of monics into B which contains an element of each equivalence class).

So in any well-powered category $\mathcal{D}$ with pullbacks, a map $g : B' \longrightarrow B$ induces a map of sets

$$g^* : \{\text{subobjects of } B\} \longrightarrow \{\text{subobjects of } B'\},$$

which in the case $\mathcal{D} = \mathbf{Set}$ is the inverse-image function.

E2.6 Definition A *coseparating set* in a category $\mathcal{D}$ is a set $\mathbb{S} \subseteq \mathrm{ob}(\mathcal{D})$ such that for any maps $B' \underset{g_1}{\overset{g_0}{\rightrightarrows}} B$,

$$\begin{array}{rl} & q \circ g_0 = q \circ g_1 \text{ for all } S \in \mathbb{S} \text{ and } q : B \longrightarrow S \\ \Rightarrow & g_0 = g_1. \end{array}$$

Put differently, if $g_0 \neq g_1$ then there's a map q from B to some element of $\mathbb{S}$ such that $q \circ g_0 \neq q \circ g_1$. The dual notion, *separating set*, concerns maps *out* of elements of $\mathbb{S}$. The name *(co)generating set* is more often used, but this terminology is very misleading.

E2.7 Examples (a) In **Set**, $\{2\}$ is a coseparating set, where 2 is a two-element set.

(b) In **CptHff**, $\{[0,1]\}$ is a coseparating set. You can take this on trust; or, if you have the requisite background, recall that

- Urysohn's Lemma says that for disjoint closed subsets M_0, M_1 of a normal space B, there is a continuous map $B \longrightarrow [0,1]$ which takes value 0 on all of M_0 and 1 on all of M_1
- a compact Hausdorff space is normal
- a one-point subset of a Hausdorff space is closed,

and so for any compact Hausdorff B and distinct $b_0, b_1 \in B$, there is a map $q : B \longrightarrow [0,1]$ with $q(b_0) = 0$ and $q(b_1) = 1$.

(c) Also by a non-trivial argument, $\{\mathbb{Q}/\mathbb{Z}\}$ is a coseparating set in **Ab**.

(d) If $\mathbb{D}$ is a small category then $\{H_D \mid D \in \mathbb{D}\}$ is a *separating* set in $[\mathbb{D}^{\mathrm{op}}, \mathbf{Set}]$, by a Yoneda argument (exercise).

(e) $[\mathbb{D}^{\mathrm{op}}, \mathbf{Set}]$ also has a *co*separating set, but this time the argument is rather beyond our reach.

(f) If $\mathcal{D}$ is a poset then $\emptyset$ is a coseparating set.

E2.8 Theorem (The Special Adjoint Functor Theorem) Let $G : \mathcal{D} \longrightarrow \mathcal{C}$ be a functor, and suppose that

- $\mathcal{D}$ and $\mathcal{C}$ are locally small
- $\mathcal{D}$ is small-complete
- $\mathcal{D}$ is well-powered
- $\mathcal{D}$ has a (small) coseparating set $\mathbb{S}$
- G preserves limits.

Then G has a left adjoint.

Proof Fix $A \in \mathcal{C}$, and let $\mathbb{V}$ be a set of monics into the object

$$\prod_{S \in \mathbb{S}} S^{\mathcal{C}(A,GS)} = \prod(S \mid S \in \mathbb{S}, p \in \mathcal{C}(A,GS))$$

of $\mathcal{D}$ which contains an element of each equivalence class. This product exists since $\mathbb{S}$ is small, $\mathcal{C}$ is locally small and $\mathcal{D}$ is small-complete; and such a set $\mathbb{V}$ exists since $\mathcal{D}$ is well-powered. (This product is notionally $K(FA)$, except we don't yet know that F exists.) So by the GAFT (and the note after it), it's enough to show that any map $A \xrightarrow{f} G(B)$ factorizes as

$$\begin{tikzcd} A \arrow[r, "d"] \arrow[dr, "f"'] & G(M) \arrow[d, "G(h)"] \\ & G(B) \end{tikzcd}$$

for some object $M \in \mathcal{D}$ for which there exists a monic $M \rightarrowtail \prod_{S \in \mathbb{S}} S^{\mathcal{C}(A,GS)}$, and some maps d and h.

So, take $A \xrightarrow{f} G(B)$. There is a canonical map

$$k : B \longrightarrow \prod_{S \in \mathbb{S}} S^{\mathcal{D}(B,S)}$$

defined by $\mathrm{pr}_q \circ k = q$ for $q \in \mathcal{D}(B,S)$ (where pr means projection). There is also a map

$$f_* : \prod_{S \in \mathbb{S}} S^{\mathcal{C}(A,GS)} \longrightarrow \prod_{S \in \mathbb{S}} S^{\mathcal{D}(B,S)}$$

defined by $\mathrm{pr}_q \circ f_* = \mathrm{pr}_{Gq \circ f}$ for $q \in \mathcal{D}(B,S)$. (Notionally, $\prod_{S \in \mathbb{S}} S^{\mathcal{D}(B,S)}$ is $K(B)$ and f_* is $K(\overline{f})$.) Form the pullback

$$\begin{tikzcd} M \arrow[r, "h"] \arrow[d, "m"'] & B \arrow[d, "k"] \\ \prod_{S \in \mathbb{S}} S^{\mathcal{C}(A,GS)} \arrow[r, "f_*"'] & \prod_{S \in \mathbb{S}} S^{\mathcal{D}(B,S)}. \end{tikzcd}$$

k is monic since $\mathbb{S}$ is a coseparating set, and so m is monic by Lemma E2.3. Moreover, G preserves limits (and therefore monics—D1.8), so we obtain a pullback square

$$\begin{tikzcd} G(M) \arrow[r, "G(h)"] \arrow[d, tail, "G(m)"'] & G(B) \arrow[d, tail, "G(k)"] \\ \prod_{S \in \mathbb{S}} (GS)^{\mathcal{C}(A,GS)} \arrow[r, "G(f_*)"'] & \prod_{S \in \mathbb{S}} (GS)^{\mathcal{D}(B,S)}. \end{tikzcd} \qquad (*)$$

On the other hand, we have a square

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "l"'] & G(B) \arrow[d, tail, "G(k)"] \\
\prod_{S \in \mathbb{S}} (GS)^{\mathcal{C}(A,GS)} \arrow[r, "G(f_*)"'] & \prod_{S \in \mathbb{S}} (GS)^{\mathcal{D}(B,S)}
\end{tikzcd}
$$

in which l is defined by $\mathrm{pr}_p \circ l = p$ for $p \in \mathcal{C}(A, GS)$; and this square commutes since

$$\mathrm{pr}_q \circ G(f_*) \circ l = \mathrm{pr}_{G(q) \circ f} \circ l = G(q) \circ f = \mathrm{pr}_q \circ G(k) \circ f$$

for all $q \in \mathcal{D}(B, S)$. And since $(*)$ is a pullback square, there is a map $d : A \longrightarrow G(M)$ such that

$$
\begin{tikzcd}
A \arrow[dr, "d"'] \arrow[drr, "f"] & & \\
& G(M) \arrow[r, "G(h)"] & G(B)
\end{tikzcd}
$$

commutes. □

E2.9 Examples (a) The best example of SAFT is when G is the inclusion **CptHff** $\hookrightarrow$ **Top**. Let's verify the hypotheses.

- **CptHff** and **Top** are certainly locally small.
- **CptHff** has small limits and G preserves them. Firstly, the product of compact spaces is compact (Tychonoff's Theorem) and the product of Hausdorff spaces is Hausdorff (easily); so **CptHff** has small products and G preserves them. Secondly, if $g_0, g_1 : B \longrightarrow B'$ are maps in **CptHff** then the subset $\{b \mid g_0(b) = g_1(b)\}$ of B is closed and therefore compact, and trivially Hausdorff; so **CptHff** has equalizers and G preserves them.
- **CptHff** is well-powered. Monics in **CptHff** are just injections (since the one-point space is compact Hausdorff), so a subobject of a compact Hausdorff space B is just a subset equipped with a suitable topology; and there is only a small collection of such things for each fixed B.
- $\{[0, 1]\}$ is a coseparating set for **CptHff**, as we've seen.

The left adjoint F is called *Stone-Čech compactification.* Close examination of the proof reveals that the Stone-Čech compactification $F(A)$ of a space A is the closure of the image of the canonical map

$$A \longrightarrow [0, 1]^{\mathbf{Top}(A,[0,1])}.$$

Topologists often only talk about the Stone-Čech compactification in the case where A is completely regular (a certain separation condition). As it happens, A is completely regular if and only if the unit map $\eta_A : A \longrightarrow GF(A)$ is an embedding, so this amounts to the view that a compactification ought to contain the original space as a subspace.

(b) Any limit-preserving functor $G : \mathbf{Set} \longrightarrow \mathbf{Set}$ has a left adjoint F. By Proposition C1.2, this implies that G is representable—that is, $G \cong \mathbf{Set}(S, -)$ for some set S. Hence $F \cong S \times -$. So we've shown that the only adjunctions from **Set** to itself are those arising from its cartesian closed structure.

(c) Let $T : \mathbb{C} \longrightarrow \mathbb{D}$ be a functor between small categories. By the dual of SAFT, the functor

$$T^* : [\mathbb{D}^{\mathrm{op}}, \mathbf{Set}] \longrightarrow [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$$

(compose with T) has a right adjoint. I leave the verification of the hypotheses as a recommended exercise.

(d) By the unproved assertion that $[\mathbb{D}^{\mathrm{op}}, \mathbf{Set}]$ has a coseparating set (E2.7(e)), T^* also has a left adjoint. The left and right adjoints to T^* are called *left* and *right Kan extension along* T. The adjoint functor theorem approach doesn't give much insight into Kan extensions, but with the appropriate language ('ends') they can be described explicitly and better understood.

(e) A laughable example: the inclusion $\mathbf{Ab} \hookrightarrow \mathbf{Gp}$ satisfies the hypotheses, and so has a left adjoint. But of course the left adjoint, abelianization, is very simply described (B1.3(a)), whereas the proof that $\{\mathbb{Q}/\mathbb{Z}\}$ is coseparating is hard.

(f) Let $g : D \longrightarrow C$ be a map of (small) posets. Suppose that D is complete, i.e. every subset S of D has a meet (inf, greatest lower bound) $\bigwedge S$, and that g preserves $\bigwedge$. Then the hypotheses of SAFT are trivially verified, so g has a left adjoint.