CATEGORY THEORY 4

M2000 | The Adjoint Functor Theorems & monads | Tom Leinster

The questions are divided into

- Exercises, to familiarize you with the basics
- Standard questions, which are very roughly of the kind that will make up the exam
- Further questions, which are 'optional' and of varying difficulty and importance.

Some questions require knowledge of other areas of mathematics; skip over any that you haven't the background for. Please report mistakes and obscurities to `leinster@dpmms.cam.ac.uk` .

Exercises

1. *The essence of the Adjoint Functor Theorems is that a limit-preserving functor on a category with limits has a left adjoint. In the general case this has to be qualified by size considerations, but in the case of small categories things are cleaner.*

   Let $g : Y \longrightarrow X$ be a map between (small) posets, and suppose that Y has all infima (meets, greatest lower bounds) and that g preserves them. Show that g has a left adjoint f, given by

$$f(x) = \bigwedge \{y \in Y \mid x \leq g(y)\}.$$

2. (i) What is a *monad* on a category? Explain how an adjunction gives rise to a monad, and show that every monad arises in this way.

   (ii) What does it mean for a functor to be *monadic*? Show that a monadic functor is faithful and reflects isomorphisms.

3. (i) What is an *absolute-coequalizer pair* in a category $\mathcal{C}$? What does it mean for a functor $U : \mathcal{D} \longrightarrow \mathcal{C}$ to preserve coequalizers for U-absolute-coequalizer pairs, or to reflect them, or to create them?

   (ii) Show that the following conditions on a functor $U : \mathcal{D} \longrightarrow \mathcal{C}$ are equivalent:

   a. U creates coequalizers for U-absolute-coequalizer pairs

   b. $\mathcal{D}$ has coequalizers of all U-absolute-coequalizer pairs, and U preserves and reflects such coequalizers

   c. $\mathcal{D}$ has coequalizers of all U-absolute-coequalizer pairs, U preserves such coequalizers, and U reflects isomorphisms.

4. A *reflexive pair* is a parallel pair $X \overset{p_1}{\underset{p_2}{\rightrightarrows}} Y$ of arrows for which there exists an arrow $X \xleftarrow{t} Y$ with $p_1 t = 1_Y = p_2 t$. Show that if (T, η, μ) is a monad and $(TA \xrightarrow{\theta} A)$ a T-algebra, then $T^2 A \overset{\mu_A}{\underset{T\theta}{\rightrightarrows}} TA$ is a reflexive split-coequalizer pair.

Standard Questions

5. Consider the following three conditions on a functor U from a locally small category $\mathcal{C}$ to **Set**:

   A. U has a left adjoint

   R. U is representable

   L. U preserves limits.

   (i) Show that $\mathbf{A} \Rightarrow \mathbf{R} \Rightarrow \mathbf{L}$.

   (ii) Show that if $\mathcal{C}$ has small coproducts then $\mathbf{R} \Rightarrow \mathbf{A}$.

   (iii) Show that if $\mathcal{C}$ is complete, well-powered and has a coseparating set then the three conditions are equivalent.

6. (i) State the General Adjoint Functor Theorem.

   (ii) Show that if a complete, locally small category has a weakly initial set of objects then it has an initial object.

   (iii) Explain in outline how the result in (ii) leads to a proof of the General Adjoint Functor Theorem.

7. What does it mean for a category to be *well-powered*? Show that if $\mathcal{C}$ is a well-powered category with all pullbacks then there is a functor $\mathrm{Sub} : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ assigning to each object its set of subobjects.

   A representation of this functor is called a subobject classifier*, and in fact it is possible to rephrase the definition of subobject classifier so that it makes sense in categories which aren't necessarily well-powered. A* topos *is a cartesian closed category with finite limits and a subobject classifier; typical examples are* **Set**, $[\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$, *and the category of sheaves of sets on a fixed space.*

8. State and prove the Special Adjoint Functor Theorem. (You may assume the General Adjoint Functor Theorem.)

9. Recall from sheet 3 the theories of widgets and chads.

   (i) Use the General Adjoint Functor Theorem to show that $U : \mathbf{Widget} \longrightarrow \mathbf{Set}$ has a left adjoint.

   (ii) Do the same for the forgetful functor $\mathbf{Widget} \longrightarrow \mathbf{Chad}$.

   (iii) Use the Monadicity Theorem to show that **Widget** is monadic over **Set**.

10. Let (T, η, μ) be a monad on a category $\mathcal{C}$, and $U^T : \mathcal{C}^T \longrightarrow \mathcal{C}$ the forgetful functor on the category of T-algebras.

    (i) Show that U^T creates limits.

    (ii) Show that U^T creates coequalizers for U^T-absolute-coequalizer pairs.

    (iii) Show that every T-algebra is a coequalizer of free T-algebras.

11. A monad (T, η, μ) is called *idempotent* if the multiplication $\mu : T^2 \longrightarrow T$ is an isomorphism. Show that every algebra for an idempotent monad is free.

12. Let $\mathbb{C}$ be a small category. The forgetful functor $U : [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}] \longrightarrow \mathbf{Set}^{\mathrm{ob}\,\mathbb{C}}$ has a left adjoint F, which is given on $X \in \mathbf{Set}^{\mathrm{ob}\,\mathbb{C}}$ and $C \in \mathbb{C}$ by

$$(FX)(C) = \coprod_{D \in \mathbb{C}} XD \times \mathbb{C}(C, D)$$

(as you may have discovered on sheet 2, q. 7).

(i) Show that the adjunction is monadic.

(ii) Deduce that U creates limits and reflects isomorphisms. The latter statement is actually a rather familiar fact from early in the course: what?

(iii) Show that any presheaf of the form FX is a coproduct of representables, and deduce from a standard result on coequalizers of free algebras that every presheaf is a colimit of representables.

13. *There are many versions of the Monadicity Theorem. The following Huge Monadicity Theorem encompasses many of them, and should give you a feel for where the real substance of the Theorem lies. Part* **5** *would not ordinarily be included in a Monadicity Theorem, because it refers to F; it is only included here as an intermediate step.*

Let $U : \mathcal{D} \longrightarrow \mathcal{C}$ be a functor with a left adjoint F. Prove the equivalence of the following conditions on U:

1. U is monadic

2a. U creates coequalizers for U-absolute-coequalizer pairs

2b. $\mathcal{D}$ has coequalizers of all U-absolute-coequalizer pairs, U preserves such coequalizers, and U reflects isomorphisms

2c. $\mathcal{D}$ has coequalizers of all U-absolute-coequalizer pairs, and U preserves and reflects such coequalizers

3a,b,c. As for **2**, but with 'split' in place of 'absolute'

4a,b,c. As for **3**, but with 'reflexive U-split-coequalizer' in place of 'U-split-coequalizer'

5. Both of the following hold:

(i) for each $B \in \mathcal{D}$, $FUFUB \overset{FU\varepsilon_B}{\underset{\varepsilon_{FUB}}{\rightrightarrows}} FUB \xrightarrow{\varepsilon_B} B$ is a coequalizer

(ii) for each T-algebra $(TA \xrightarrow{\theta} A)$ (where T is the induced monad), there exists a map $g : FA \longrightarrow B$ in $\mathcal{D}$ such that $UFUFA \overset{U\varepsilon_{FA}}{\underset{UF\theta}{\rightrightarrows}} UFA \xrightarrow{Ug} UB$ is a coequalizer.

Hints: Use q. 3(ii) to show that **2a,b,c** are equivalent, and similarly **3** and **4**. Then do **1** $\Rightarrow$ **2** $\Rightarrow$ **3** $\Rightarrow$ **4** $\Rightarrow$ **5** $\Rightarrow$ **1**.

14. *This is an alternative way of presenting the Monadicity Theorem, although the methods used to prove it are just the ones we're used to.*

    Let $U : \mathcal{D} \longrightarrow \mathcal{C}$ be a functor with a left adjoint, T the induced monad on $\mathcal{C}$, and $K : \mathcal{D} \longrightarrow \mathcal{C}^T$ the comparison functor. Show that:

    (i) if $\mathcal{D}$ has coequalizers of U-absolute-coequalizer pairs then K has a left adjoint J

    (ii) if also U preserves such coequalizers then the unit of the adjunction $J \dashv K$ is an isomorphism

    (iii) if, in addition to this, U reflects isomorphisms, then the counit is also an isomorphism, and so U is monadic.

Further Questions

15. The objects of the category $\mathbf{\Delta}$ are the natural numbers $0, 1, 2, \ldots$, and a map $m \longrightarrow n$ is an order-preserving map from $\{0, \ldots, m-1\}$ to $\{0, \ldots, n-1\}$. There is a functor $\mathbf{\Delta} \longrightarrow \mathbf{Top}$ which sends n to the standard n-dimensional simplex Δ^n, and is defined on maps via the standard face and degeneracy maps (although you don't need to know about them to do this question.) This induces a functor $S : \mathbf{Top} \longrightarrow [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$. Use the Special Adjoint Functor Theorem to show that S has a left adjoint.

    An object of $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$ is called an augmented simplicial set, and $S(X)$ is the standard 'singular complex' of a space X (from which the homotopy, homology and cohomology of X can be derived). The left adjoint to S is called geometric realization.

16. Let (T, η, μ) be a monad on a category $\mathcal{C}$, and suppose that T has a right adjoint G. Show that G has the structure of a comonad in a canonical way, and that the category of G-coalgebras is isomorphic to the category of T-algebras. (Hint: this is most efficiently done using the theory of mates—see sheet 2, q. 14, especially part (iv).)

17. True or false?

    (i) the category of fields is monadic over $\mathbf{Set}$

    (ii) the category of posets is monadic over $\mathbf{Set}$

    (iii) the category of totally ordered sets is monadic over $\mathbf{Set}$

    (iv) the category of lattices is monadic over $\mathbf{Set}$ (a *lattice* being a poset with sups and infs of all finite subsets)

    (v) the category of topological groups is monadic over $\mathbf{Top}$.