CATEGORY THEORY 1

M2000 Categories, functors, natural transformations & adjoints Tom Leinster

There will be four problem sheets. On each, the questions are divided into

- Exercises, to familiarize you with the basics
- Standard questions, which are very roughly of the kind that will make up the exam
- Further questions, which are 'optional' and of varying difficulty and importance.

Some questions require knowledge of other areas of mathematics; skip over any that you haven't the background for. Please report mistakes and obscurities to `leinster@dpmms.cam.ac.uk` .

Exercises

1. Write down three examples each of (a) categories, (b) functors, (c) natural transformations, and (d) adjunctions, that weren't given in lectures.

2. (i) Show that in any category, a map with both a left inverse and a right inverse is an isomorphism, and that the two inverses are equal.

   (ii) Show that functors preserve isomorphisms.

3. (i) What are the subcategories of a poset?

   (ii) What are the subcategories of a group?

4. (i) What is the opposite of a group G? Establish the existence of an isomorphism $G \cong G^{\mathrm{op}}$.

   (ii) Give an example of (a) a poset and (b) a monoid not isomorphic to its dual.

5. Consider the homotopy category **Hty**: its objects are topological spaces and its maps are homotopy classes of maps between spaces. (It is good to contemplate this example for long enough to convince yourself that categorical maps need not be remotely like mathematical functions.) Characterize the objects isomorphic in **Hty** to the one-element space.

6. Is there a functor $Z : \mathbf{Gp} \longrightarrow \mathbf{Gp}$ taking a group to its centre?

7. Show that a natural transformation is a natural isomorphism if and only if each of its components is an isomorphism.

8. (i) A *pointed set* is a set X equipped with an element ('basepoint') $x \in X$. Let $\mathbf{Set}_*$ be the category of pointed sets and basepoint-preserving functions. Let 1 be a one-element set. Show that $\mathbf{Set}_*$ is isomorphic to $1/\mathbf{Set}$.

   (ii) What does it mean for two categories to be *equivalent*? Show that $\mathbf{Set}_*$ is equivalent to **Par**, the category of sets and partial functions.

9. *Linear algebra can be expressed in terms either of matrices or of linear transformations...*

Fix a field k. Let $\mathbf{Mat}_k$ be the category whose objects are the natural numbers, and with

$$\mathbf{Mat}_k(m, n) = \{n \times m \text{ matrices over } k\}.$$

Prove that $\mathbf{Mat}_k$ is equivalent to $\mathbf{FDVect}_k$, the category of finite-dimensional vector spaces over k. Does your equivalence involve a *canonical* functor from $\mathbf{Mat}_k$ to $\mathbf{FDVect}_k$, or from $\mathbf{FDVect}_k$ to $\mathbf{Mat}_k$?

10. Let $F : \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{C}$ be a functor. For $A \in \mathcal{A}$, there is an induced functor $F^A : \mathcal{B} \longrightarrow \mathcal{C}$ with $F^A(B) = F(A, B)$ and $F^A(g) = F(1_A, g)$. Similarly, for $B \in \mathcal{B}$ there is an induced functor $F_B : \mathcal{A} \longrightarrow \mathcal{C}$.

The functors F^A and F_B satisfy the following:

- $F^A(B) = F_B(A)$
- if $f : A \longrightarrow A'$ and $g : B \longrightarrow B'$ then $F^{A'}(g) \circ F_B(f) = F_{B'}(f) \circ F^A(g)$

Show that given families of functors $F^A : \mathcal{B} \longrightarrow \mathcal{C}$ and $F_B : \mathcal{A} \longrightarrow \mathcal{C}$ satisfying the above conditions, there is a unique functor $F : \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{C}$ giving rise to them.

11. What is an *adjunction*? Show that left adjoints preserve initial objects: that is, if $\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left, "G"] \end{tikzcd}$ and $\mathcal{C}$ has an initial object 0, then $F(0)$ is an initial object of $\mathcal{D}$. Dually, show that right adjoints preserve terminal objects. *(Later we'll see this as part of a larger picture: right adjoints preserve all limits, and left adjoints all colimits.)*

Standard Questions

12. (i) Let $\begin{tikzcd} X \arrow[r, shift left, "f"] & Y \arrow[l, shift left, "g"] \end{tikzcd}$ be a pair of order-preserving maps between posets. Describe in explicit terms what an adjunction $f \dashv g$ is.

(ii) Let $p : A \longrightarrow B$ be a map of sets, and let $p^* : \mathcal{P}(B) \longrightarrow \mathcal{P}(A)$ be the induced map of power-sets. Exhibit left and right adjoints to p^*.

13. Let $\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left, "G"] \end{tikzcd}$ be an adjunction.

(i) Define the *unit* η and the *counit* ε of the adjunction.

(ii) Prove the triangle identities, $(\varepsilon F) \circ (F\eta) = 1_F$ and $(G\varepsilon) \circ (\eta G) = 1_G$.

(iii) Prove that given functors $\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] & \mathcal{D} \arrow[l, shift left, "G"] \end{tikzcd}$ and natural transformations $1 \xrightarrow{\eta} GF$, $FG \xrightarrow{\varepsilon} 1$ satisfying the triangle identities, there is a unique adjunction between F and G with η as its unit and ε as its counit.

Further Question

14. *Given an n-element set X, there are $n!$ permutations of X and $n!$ total orderings of X, but there is no 'natural' way of matching up permutations with orderings. The following exercise makes this statement precise using the notion of natural isomorphism.*

    Let $\mathcal{P}$ be the category of finite sets and bijections. For $X \in \mathcal{P}$, let $\mathbf{Sym}(X)$ be the set of permutations of X (i.e. bijections $X \longrightarrow X$) and let $\mathbf{Ord}(X)$ be the set of total orderings of X (i.e. reflexive transitive relations $\leq$ on X, such that if $x \neq y$ then precisely one of $x \leq y$ and $y \leq x$ is true).

    (i) Give a definition of $\mathbf{Sym}$ on morphisms of $\mathcal{P}$, so that $\mathbf{Sym}$ becomes a functor $\mathcal{P} \longrightarrow \mathbf{Set}$. Do the same for $\mathbf{Ord}$. (In each case there is only one sensible such definition.)

    (ii) Show that there is no natural transformation $\mathbf{Sym} \longrightarrow \mathbf{Ord}$. (Hint: consider what would happen to the identity permutation.)

    Conclude that the functors $\mathbf{Sym}, \mathbf{Ord} : \mathcal{P} \longrightarrow \mathbf{Set}$ are pointwise isomorphic, but not naturally isomorphic.