CATEGORY THEORY 2

M2000 | Adjoints & representables | Tom Leinster

The questions are divided into

- Exercises, to familiarize you with the basics
- Standard questions, which are very roughly of the kind that will make up the exam
- Further questions, which are 'optional' and of varying difficulty and importance.

Some questions require knowledge of other areas of mathematics; skip over any that you haven't the background for. Please report mistakes and obscurities to `leinster@dpmms.cam.ac.uk` .

Exercises

1. Let $\mathcal{C}$ be a locally small category. What does it mean for a functor $X : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ to be *representable*? What is a *representation* of such a functor?

2. Write down five examples of representable functors.

3. Let $\mathcal{C}$ be a locally small category. Define the *Yoneda embedding* $H_\bullet : \mathcal{C} \longrightarrow [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$. Prove each of the following directly (without using the Yoneda Lemma):

   (i) $H_A \cong H_B \Rightarrow A \cong B$, for $A, B \in \mathcal{C}$

   (ii) $H_\bullet$ is full

   (iii) $H_\bullet$ is faithful

   (iv) if $X : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$, $A \in \mathcal{C}$, and $X(A)$ has an element u with the property $(*)$ below, then $X \cong H_A$.

   for any $B \in \mathcal{C}$ and $x \in X(B)$, there is a unique map $f : B \longrightarrow A$ such that $x = (Xf)(u)$ $\qquad (*)$

4. What does it mean for a functor to *reflect isomorphisms*? Show that a full and faithful functor reflects isomorphisms.

Standard Questions

5. (i) Fix a topological space S, and let $\mathcal{O}(S)$ be the poset of open subsets of S, ordered by inclusion. Let

$$\Delta : \mathbf{Set} \longrightarrow [\mathcal{O}(S)^{\mathrm{op}}, \mathbf{Set}]$$

be the functor assigning to a set A the presheaf ΔA with constant value A. Exhibit a chain of adjoints

$$\Lambda \dashv \Pi \dashv \Delta \dashv \Gamma \dashv \nabla.$$

(It is enough simply to describe the behaviour of these functors on objects.)

(ii) Let $O : \mathbf{Cat} \longrightarrow \mathbf{Set}$ be the functor taking a small category to its set of objects. Exhibit a chain of adjoints

$$C \dashv D \dashv O \dashv I.$$

(iii) Do either of these chains of adjoints extend further in either direction? (Hint: recall question 11 on sheet 1.)

6. (i) Given an adjunction $\mathcal{C} \underset{G}{\overset{F}{\underset{\perp}{\rightleftarrows}}} \mathcal{D}$ and a category $\mathcal{S}$, show how there arises an adjunction $[\mathcal{C}, \mathcal{S}] \underset{F^*}{\overset{G^*}{\underset{\perp}{\rightleftarrows}}} [\mathcal{D}, \mathcal{S}]$. (Hint: use the description of an adjunction by its unit and counit.)

(ii) Let $\mathbf{n}$ denote an n-element totally ordered set, so that a functor from $\mathbf{n}$ to a category $\mathcal{S}$ is a composable string of $n-1$ morphisms in $\mathcal{S}$. Show that there are functors

$$F_1, \ldots, F_n : [\mathbf{n}, \mathcal{S}] \longrightarrow [\mathbf{n+1}, \mathcal{S}]$$

and

$$G_0, \ldots, G_n : [\mathbf{n+1}, \mathcal{S}] \longrightarrow [\mathbf{n}, \mathcal{S}]$$

which form a chain of adjunctions

$$G_0 \dashv F_1 \dashv \cdots \dashv F_n \dashv G_n.$$

(iii) Suppose now that $\mathcal{S}$ has both an initial object 0 and a terminal object 1. Prove that G_0 has a left adjoint F_0, and that G_n has a right adjoint F_{n+1}.

(iv) Finally, supposing that $0 \not\cong 1$, prove that the chain of adjunctions extends no further.

7. *This question is about the Cayley representation theorem, which originally concerned groups but generalizes to small categories.*

Let $\mathbb{C}$ be a small category and $\mathbb{C}_0$ its set of objects.

(i) Describe a left adjoint F to the forgetful functor $[\mathbb{C}, \mathbf{Set}] \longrightarrow \mathbf{Set}^{\mathbb{C}_0}$. (Here $\mathbf{Set}^{\mathbb{C}_0}$ is the product of $\mathbb{C}_0$ copies of $\mathbf{Set}$.)

(ii) Let 1 be the terminal object of $\mathbf{Set}^{\mathbb{C}_0}$. Describe the *Cayley embedding* $\mathbf{Cay} = F(1) : \mathbb{C} \longrightarrow \mathbf{Set}$, and show that it is faithful.

(iii) Use (ii) to show that any small category is isomorphic to a subcategory of $\mathbf{Set}$.

(iv) Also use (ii) to show that if G is a finite group with n elements then G is isomorphic to a subgroup of the symmetric group S_n.

8. Let $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ be an adjunction with unit η and counit ε. Let $\mathrm{Fix}(GF)$ be the full subcategory of $\mathcal{C}$ whose objects are those A for which η_A is an isomorphism, and dually $\mathrm{Fix}(FG) \subseteq \mathcal{D}$. Prove that the adjunction $(F, G, \eta, \varepsilon)$ restricts to an equivalence $(F', G', \eta', \varepsilon')$ between $\mathrm{Fix}(GF)$ and $\mathrm{Fix}(FG)$.

9. Let $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ be an adjunction with unit η and counit ε. Show that the following conditions are equivalent:

i $F\eta$ is a natural isomorphism

ii εF is a natural isomorphism

iii $G\varepsilon F$ is a natural isomorphism

iv $GF\eta = \eta GF$

v $GF\eta G = \eta GFG$

vi–x the duals of **i–v**, e.g. **ix** says that $FG\varepsilon = \varepsilon FG$.

(Hint: do the implications in cyclic order, **i** $\Rightarrow$ ... $\Rightarrow$ **x** $\Rightarrow$ **i**.) Such an adjunction is called *idempotent*.

Use question 8 to show that there is an equivalence of categories $\mathcal{I}(F) \simeq \mathcal{I}(G)$. Here $\mathcal{I}(F)$ is the 'essential full image of F': the full subcategory of $\mathcal{D}$ whose objects are those B such that $B \cong F(A)$ for some $A \in \mathcal{C}$.

Show that any adjunction between preorders is idempotent.

10. Let $G : \mathcal{D} \longrightarrow \mathcal{C}$ be a functor.

(i) For $A \in \mathcal{C}$, define the comma category $(A \Rightarrow G)$.

(ii) Suppose that G has a left adjoint F, and let η be the unit of the adjunction. Show that for each $A \in \mathcal{C}$, η_A is an initial object of $(A \Rightarrow G)$.

(iii) Conversely, suppose that for each $A \in \mathcal{C}$, the category $(A \Rightarrow G)$ has an initial object. Show that G has a left adjoint.

11. State and prove the Yoneda Lemma. Deduce:

(i) that the Yoneda embedding is full and faithful

(ii) that for objects A, B of a locally small category $\mathcal{C}$, $A \cong B$ if and only if $\mathcal{C}(C, A) \cong \mathcal{C}(C, B)$ naturally in $C \in \mathcal{C}$

(iii) that a functor $X : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is representable if and only if there exist an object $A \in \mathcal{C}$ and an element $u \in X(A)$ which is 'universal' in a sense you should make precise.

Further Questions

12. (i) Let G be a group and let G-**Set** be the category of left G-sets. What 'interesting' functors are there between **Set** and G-**Set** (in either direction)? What adjunctions are there between these functors?

    (Hints: I have six functors in mind. It might help to know that for any functor $T : \mathbb{D} \longrightarrow \mathbb{C}$ between small categories, the induced functor $T^* : [\mathbb{C}, \mathbf{Set}] \longrightarrow [\mathbb{D}, \mathbf{Set}]$ always has both a left and a right adjoint.)

    (ii) (For those into algebra.) What happens if we replace **Set** by **Vect**, so that we're looking for adjunctions between **Vect** and the category $[G, \mathbf{Vect}]$ of linear representations of G?

13. *Quantifiers as adjoints: for the logically inclined.*

    Let X and Y be sets and let $p : X \times Y \longrightarrow X$ be first projection. Regard a subset R of $X \times Y$ as a predicate $R(x, y)$ in two variables, and a subset S of X as a predicate $S(x)$. What, in terms of predicates, are the effect of p^* and its left and right adjoints, as constructed in sheet 1, question 12? For each of the two adjunctions, interpret the unit and counit as logical implications.

    The left adjoint to p^* is often written $\exists_p$, and the right adjoint $\forall_p$.

14. *The theory of mates, an Australian creation.*

    (i) Let $(F, G, \eta, \varepsilon)$ and $(F', G', \eta', \varepsilon')$ be adjunctions between the same categories, i.e. $\mathcal{C} \underset{G,G'}{\overset{F,F'}{\rightleftarrows}} \mathcal{D}$. Show that a natural transformation $\alpha : F \longrightarrow F'$ gives rise to a natural transformation $\overline{\alpha} : G' \longrightarrow G$ (the *mate* of α).

    (ii) Show that if $(F'', G'', \eta'', \varepsilon'')$ is another adjunction from $\mathcal{C}$ to $\mathcal{D}$, and if $\alpha' : F' \longrightarrow F''$, then $\overline{\alpha' \circ \alpha} = \overline{\alpha} \circ \overline{\alpha'}$. Show also that $\overline{1_F} = 1_G$.

    (iii) Deduce that any two right adjoints to a functor are naturally isomorphic.

    (iv) Part (ii) says that taking the mate preserves (or rather, reverses) vertical composition of natural transformations. Write down some other likely-looking equations in the theory of mates, and attempt to prove them.

15. Let $A : \mathbf{Cat} \longrightarrow \mathbf{Set}$ be the functor taking a small category to its set of arrows. Show that A has a left adjoint T, but that T does not have a left adjoint and A does not have a right adjoint. *(One of these non-existence proofs is hard with our current methods. When we've done limits it will be easier.)*