Categorical and metric density

Tom Leinster (Edinburgh)

On work by and with

Adrián Doña Mateo (Edinburgh)

Motivating questions

Metric spaces are enriched categories.

For general enriched categories, there is a notion of 'dense subcategory'.

But when specialized to metric spaces, this does *not* give the standard metric notion of density.

Two questions:

- Does metric density generalize usefully to enriched categories?
- How does categorical density behave for metric spaces?

Lawvere addressed both questions in his metric spaces paper—but only very briefly.

Plan

1. Metric density for categories
2. Categorical density for metric spaces

1. Metric density for categories

Almost all results here are from:

Adrián Doña Mateo, Cauchy density. ArXiv:2507.07869, 2025.

Cauchy density

Throughout, V is a complete, cocomplete symmetric monoidal closed category.
'Category' means V-category, etc.

Every functor $F\colon A \to B$ induces an adjunction $F_* \dashv F^*$ in V-**Prof**:

$$F_*\colon \begin{array}{ccc} B^{\mathrm{op}} \otimes A & \to & V \\ (b, a) & \mapsto & B(b, Fa) \end{array} \qquad F^*\colon \begin{array}{ccc} A^{\mathrm{op}} \otimes B & \to & V \\ (a, b) & \mapsto & B(Fa, b). \end{array}$$

Fact F is full and faithful $\iff$ the unit of this adjunction is an isomorphism.

Definition F is Cauchy dense $\iff$ the counit of this adjunction is an isomorphism.

So, F is Cauchy dense iff for all $b, b' \in B$, the canonical map

$$\int^{a \in A} B(b, Fa) \otimes B(Fa, b') \to B(b, b')$$

is an isomorphism.

Cauchy density for metric spaces

Lemma (basically Lawvere) *A map $F \colon A \to B$ of metric spaces is Cauchy dense if and only if its image FA is topologically dense in B.*

Sketch proof Cauchy density says: for all $b, b' \in B$,

$$\begin{aligned} \int^{a} B(b, Fa) \otimes B(Fa, b') &\xrightarrow{\sim} B(b, b') \\ \iff \inf_{a} \big(d_B(b, Fa) + d_B(Fa, b') \big) &= d_B(b, b'). \end{aligned}$$

Topological density (with respect to the *symmetric* topology induced by the metric) says: for all $b \in B$,

$$\inf_{a} \big(d_B(b, Fa) + d_B(Fa, b) \big) = 0.$$

So one implication is trivial. The other uses the triangle inequality.

So Cauchy density is a generalization of topological density to arbitrary enriched categories.

Cauchy density for monoids

Theorem (Doña Mateo) *A homomorphism of monoids is Cauchy dense (as a functor between one-object categories) if and only if it is epic in* **Mon**.

This is difficult! (And not in the paper.)

Note that there are non-surjective epics in **Mon**, e.g. $\mathbb{N} \hookrightarrow \mathbb{Z}$.

Cauchy density over an arbitrary V

Archetypal example of a full and faithful Cauchy dense functor: the inclusion $A \hookrightarrow \overline{A}$ of a category into its Cauchy completion $\overline{A}$.

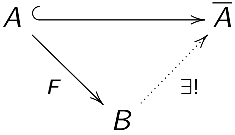
Lemma (Doña Mateo) *Let $F\colon A \to B$ be a full and faithful Cauchy dense functor. Then for all $b \in B$, the presheaf $B(F-, b)$ belongs to $\overline{A}$.*

Sketch proof Show that $B(b, F-)$ is right adjoint to $B(F-, b)$ in V-**Prof**.

Classification of full and faithful Cauchy dense functors

In fact, *$\overline{A}$ is the largest category containing A as a Cauchy dense full subcategory:*

Theorem (Doña Mateo) *Let $F\colon A \to B$ be a full and faithful Cauchy dense functor. Then*



Example The completion of a metric space A is the largest space containing A as a dense subspace.

For arbitrary V, this gives a complete description of the full and faithful Cauchy dense functors.

They are exactly the embeddings $A \hookrightarrow B$ where $A \subseteq B \subseteq_{\text{full}} \overline{A}$.

They can also be described as the functors $F\colon A \to B$ such that $- \circ F\colon [B, V] \to [A, V]$ is an equivalence.

Cauchy density vs. categorical density

The class of Cauchy dense functors has some properties that dense functors lack:

- it is closed under composition;
- it is self-dual: $A \xrightarrow{F} B$ is Cauchy dense iff $A^{\mathrm{op}} \xrightarrow{F^{\mathrm{op}}} B^{\mathrm{op}}$ is.

In fact,

$$\text{Cauchy dense} \Rightarrow \text{dense and codense}$$

(but not $\Leftarrow$).

Theorem (Lucatelli Nunes and Sousa; Doña Mateo) *The following are equivalent for $F\colon A \to B$:*

1. *F is Cauchy dense*
2. *F is absolutely dense: $\mathrm{Lan}_F\, F \cong 1$ (density) and this Kan extension is preserved by all functors*
3. *F is lax epi: the ordinary functor $- \circ F\colon [B, C]_0 \to [A, C]_0$ is full and faithful for all C.*

2. Categorical density for metric spaces

Unpublished work with Adrián Doña Mateo

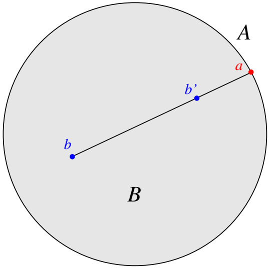
Categorical density in metric terms

Take a metric subspace A of a metric space B (that is, a V-full-and-faithful V-functor $A \to B$, where $V = \mathbb{R}^+$).

Unwinding the definition of dense functor, A is categorically dense in B iff:

for all $b, b' \in B$, for all $\varepsilon > 0$, there exists $a \in A$ such that

$$d(b, b') + d(b', a) \leq d(b, a) + \varepsilon.$$



When A is compact, this reduces to: for all $b, b' \in B$, there exists $a \in A$ such that

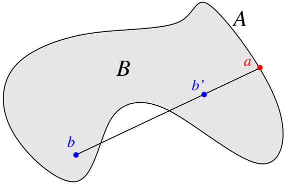
$$d(b, b') + d(b', a) = d(b, a).$$

Question Is this a significant condition in geometry?

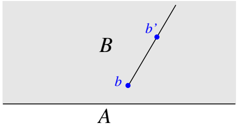
First examples

(Density of $A \subseteq B$ says: $\forall b, b' \in B$, $\exists a \in A$: $d(b,b') + d(b',a) = d(b,a)$, at least up to ε.)

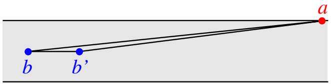
- Let B be a compact subset of $\mathbb{R}^n$. Then $A = \partial B$ is categorically dense in B.



- Let B be a closed half plane in $\mathbb{R}^2$. Then $A = \partial B$ is *not* categorically dense in B.



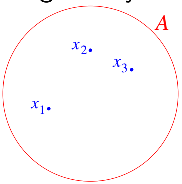
- Let $B = \mathbb{R} \times [0,1]$. Then $A = \partial B = \mathbb{R} \times \{0,1\}$ is categorically dense in B.



Surrounding sets

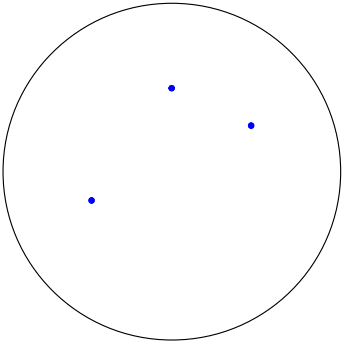
Given $A, X \subseteq \mathbb{R}^n$, let's say that A surrounds X if A is categorically dense in $A \cup X$.

E.g. For any subset X of a disk, the bounding circle $A = S^1$ surrounds X.



So does any topologically dense subset of S^1.

Question Take non-collinear points x_1, x_2, x_3 in the open disk. If $A \subseteq S^1$ surrounds $\{x_1, x_2, x_3\}$, must A be topologically dense in S^1?



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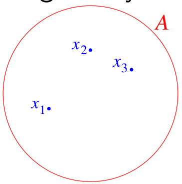

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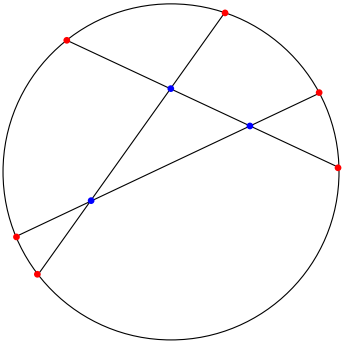

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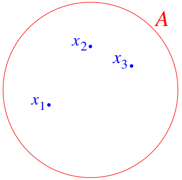

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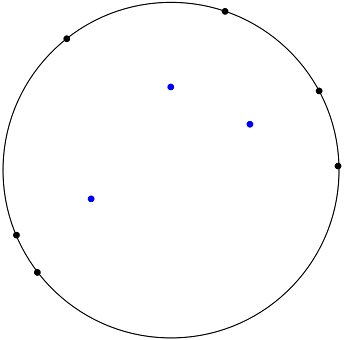

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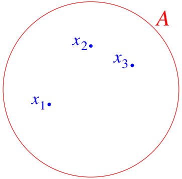

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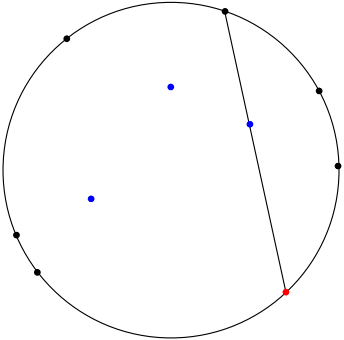

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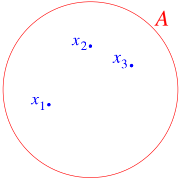

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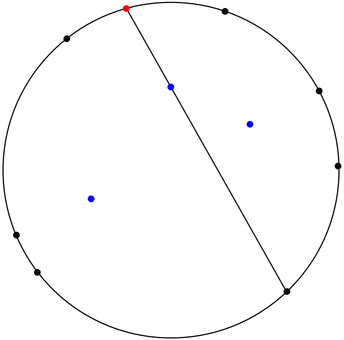

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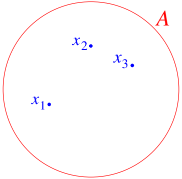

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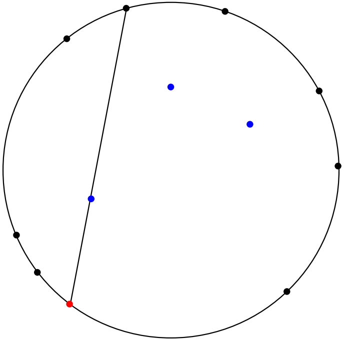

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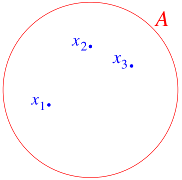

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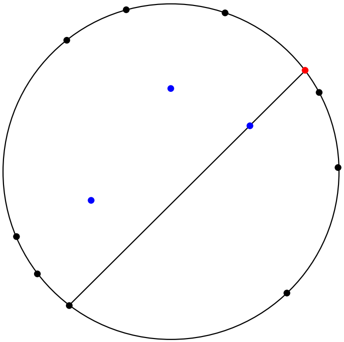

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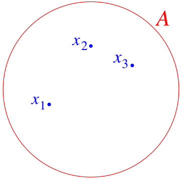

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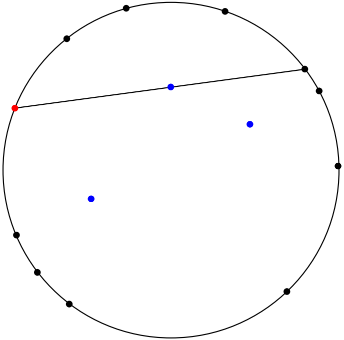

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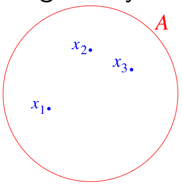

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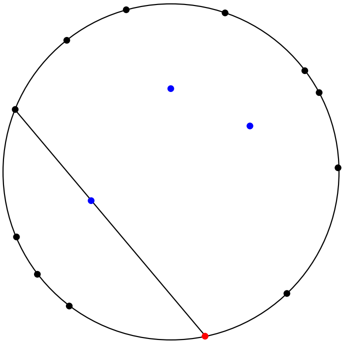

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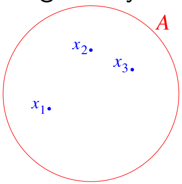

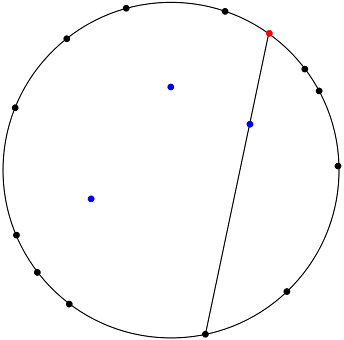

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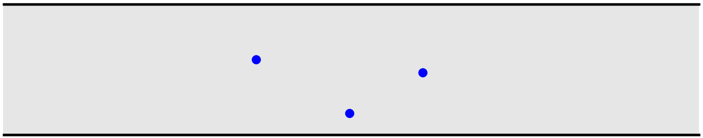
Surrounding sets and slabs

The question: if $A \subseteq S^1$ surrounds three non-collinear points, must A be dense in S^1?

We couldn't answer this. But something similar is true for 'slabs' instead of disks:

Theorem (I think) *Let x_1, x_2, x_3 be non-collinear points in $\mathbb{R} \times (0,1)$. If $A \subseteq \mathbb{R} \times \{0,1\}$ surrounds $\{x_1, x_2, x_3\}$ then A is topologically dense in $\mathbb{R} \times \{0,1\}$.*



A similar statement holds for $\mathbb{R}^n \times \{0,1\}$ for arbitrary n, with points $x_1, \ldots, x_{n+2}$.

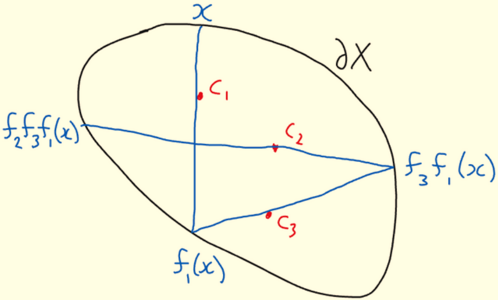
Proof Careful analysis of affine transformations, lots of εs and δs, plus Dirichlet's approximation theorem: for any $x \in \mathbb{R}$ and $\varepsilon > 0$, some integer multiple of x has fractional part $< \varepsilon$.

Surrounding sets and disks

As we couldn't answer our disk question, I asked on MathOverflow.

Let X be a compact convex subset of the plane. Let c_1, c_2 and c_3 be non-collinear points in the interior of X.

For every point x on the boundary ∂X, you can draw the line from x through c_1 and continue it until it meets ∂X in another point, which I'll call $f_1(x)$. This gives a homeomorphism $f_1 : \partial X \to \partial X$. Define $f_2, f_3 : \partial X \to \partial X$ similarly. For example:



Let G be the subgroup of the group of self-homeomorphisms of ∂X generated by f_1, f_2 and f_3. Then G acts in a natural way on ∂X.

Question Is the G-orbit of every point of ∂X dense in ∂X?

Surrounding sets and disks

As we couldn't answer our disk question, I asked on MathOverflow.

And I got the answer no (thanks, Moishe Kohan!):

1 Answer   Sorted by: Highest score (default)

4

Consider the case when X is a closed round disk in the plane with the interior (open disk) H and boundary circle C. I will regard H as a Klein model on the hyperbolic plane, whose isometries are projective transformations preserving H. Then, given a point $c \in H$ and defining the corresponding involution $\theta_c : C \to C$ as in your question, one checks that θ_c is the restriction of a projective involution of X, a hyperbolic isometry of H. (I will leave it to you to verify this.) Let Γ be a subgroup of the isometry group of H generated by three involutions $\theta_{c_i}, i = 1, 2, 3$. For every point $x \in C$, the accumulation set of the orbit Γx is the *limit set* Λ of Γ. It turns out that Λ does not depend on x and if Γ is discrete, then Λ equals C if and only if H/Γ has finite area (Γ is a lattice). If Γ is discrete then Λ either equals C, or is homeomorphic to the Cantor set or consists of at most two points.

Let H_i be hyperbolic half-planes in H bounded by lines L_i in H through the points c_i. If one can find H_i's which are pairwise disjoint, then the complement F to their union is a fundamental domain of Γ acting on H (in particular, Γ is discrete). The hyperbolic area of F is the area of H/Γ. If L_i's are not just disjoint but (at least) two of them have disjoint closures in X, then F has infinite area. Thus, to find a counter-example to your conjecture, it suffices to take c_i's which are sufficiently close to three pairwise distinct points in C.

At the same time, to find examples when $\Lambda = C$, you can take points c_i such that for some choice of the lines L_i you get an triangle inscribed in C. (The group Γ will be still discrete in this case.)

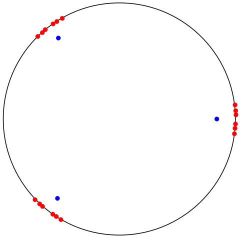
In my answer I used standard results on Fuchsian groups. You can find detailed proofs for instance in Beardon's book "Geometry of discrete groups." (Apart from the discussion of the Klein model.)

Share Cite Improve this answer Follow   edited Oct 29 at 14:59   answered Oct 29 at 1:49 Moishe Kohan

Someone then gave an elementary argument and counterexample (thanks, Christian Remling!).

Surrounding sets and disks

Here's the counterexample (roughly drawn):



The set A of red points on the circle (a Cantor set) surrounds the three blue points x_1, x_2, x_3 in the interior.

That is, A is categorically dense in $A \cup \{x_1, x_2, x_3\}$.

I then asked: is there *any* choice of x_1, x_2, x_3 such that every surrounding set $A \subseteq S^1$ is topologically dense in S^1?

Moishe Kohan gave me an answer of yes:

@MoisheKohan I've accepted your answer - thanks! While we're at it, do you know a straightforward description of the set of triples (c_1, c_2, c_3) in the open disk such that all G-orbits are dense?
– Tom Leinster Nov 4 at 22:13

@TomLeinster Yes, equip the unit disk H with hyperbolic metric of curvature -1 (the standard Hilbert metric on the open unit disk), then take points c_i in a hyperbolic disk in H of (hyperbolic) radius 0.13. But you have to learn a lot of hyperbolic geometry to understand why. – Moishe Kohan Nov 4 at 22:30

Is categorical density geometrically significant?

We asked:

> *Is categorical density, when specialized to metric spaces, geometrically significant?*

I still don't know!

But at least it leads to some nontrivial geometric/dynamical questions.

Conclusions

- When topological density is generalized from metric spaces to arbitrary enriched categories, it becomes *Cauchy density*, which is intimately related to Cauchy completion.
- When categorical density is specialized from enriched categories to metric spaces, it leads us into nontrivial aspects of geometric dynamics.

Thanks very much