Lecture 6

Adjoints, representables and limits

Definition 6.1 A functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ **preserves limits** if for any diagram $D : \mathbb{I} \longrightarrow \mathcal{A}$ and cone $(L \xrightarrow{p_I} DI)_{I \in \mathbb{I}}$,

$$\begin{aligned} &\left(L \xrightarrow{p_I} DI\right)_{I\in\mathbb{I}} \text{ is a limit cone on } D \text{ in } \mathcal{A} \\ \Rightarrow \quad &\left(FL \xrightarrow{Fp_I} FDI\right)_{I\in\mathbb{I}} \text{ is a limit cone on } F \circ D \text{ in } \mathcal{B}. \end{aligned}$$

If $F : \mathcal{A} \longrightarrow \mathcal{B}$ preserves limits and $D : \mathbb{I} \longrightarrow \mathcal{A}$ has a limit then so does $F \circ D$, and

$$\lim_{\leftarrow I} F(DI) \cong F(\lim_{\leftarrow I} DI).$$

In fact the whole limit cone is preserved (not just the limit object), but we won't worry about this level of detail here.

Remarks 6.2 A category $\mathbb{I}$ is **finite** if it has only finitely many objects and arrows. A **finite limit** is a limit of a diagram on a finite category $\mathbb{I}$. In homological algebra, an additive functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ between abelian categories is **left exact** if it preserves finite limits, and **right exact** if it preserves finite colimits.

For any diagram $D : \mathbb{I} \longrightarrow \mathcal{A}$, there is a functor

$$\begin{array}{rccc} \mathrm{Cones}(-, D) : & \mathcal{A}^{\mathrm{op}} & \longrightarrow & \mathbf{Set}, \\ & A & \longmapsto & \{\text{cones on } D \text{ with vertex } A\}. \end{array}$$

A limit of D amounts to an object $\lim_{\leftarrow I} DI$ of $\mathcal{A}$ and an isomorphism

$$\mathrm{Cones}(A, D) \cong \mathcal{A}(A, \lim_{\leftarrow I} DI) \qquad (6)$$

natural in $A \in \mathcal{A}$, in other words, a representation of $\mathrm{Cones}(-, D)$. (This follows from Corollary 4.8.)

Lemma 6.3 *For any diagram $D : \mathbb{I} \longrightarrow \mathcal{A}$ in a category $\mathcal{A}$,*

$$\mathrm{Cones}(A, D) \cong \lim_{\leftarrow I} \mathcal{A}(A, DI)$$

naturally in $A \in \mathcal{A}$.

Example: when $\mathbb{I}$ is the discrete category $\{1, 2\}$, this says that $\text{Cones}(A, D) \cong \mathcal{A}(A, D1) \times \mathcal{A}(A, D2)$.

Proof **Set** has all limits (5.9), so the right-hand side makes sense. Now by 5.9,

$$\begin{aligned}
\lim_{\leftarrow I} \mathcal{A}(A, DI) &\cong \{(q_I)_{I \in \mathbb{I}} \in \prod_{I \in \mathbb{I}} \mathcal{A}(A, DI) \mid (Du)_*(q_I) = q_J \text{ for} \\
&\qquad \text{each } I \xrightarrow{u} J \text{ in } \mathbb{I}\} \\
&\cong \{(A \xrightarrow{q_I} DI)_{I \in \mathbb{I}} \mid (Du) \circ q_I = q_J \text{ for each } I \xrightarrow{u} \\
&= \text{Cones}(A, D).
\end{aligned}$$

□

Theorem 6.4 *Representables preserve limits.*

Example: in the case of binary products, this says that $\mathcal{A}(A, B_1 \times B_2) \cong \mathcal{A}(A, B_1) \times \mathcal{A}(A, B_2)$. 'A map into $B_1 \times B_2$ is a map into B_1 together with a map into B_2.'

Sketch proof Let $\mathcal{A}$ be a locally small category, $A \in \mathcal{A}$, and $D : \mathbb{I} \longrightarrow \mathcal{A}$; suppose that D has a limit. Then by (6) and 6.3,

$$\mathcal{A}(A, \lim_{\leftarrow I} DI) \cong \lim_{\leftarrow I} \mathcal{A}(A, DI). \tag{7}$$

In other words, $H^A(\lim_{\leftarrow I} DI) \cong \lim_{\leftarrow I}(H^A(DI))$. $\Box$

The dual of (7) (changing $\mathcal{A}$ to $\mathcal{A}^{\mathrm{op}}$) is

$$\mathcal{A}(\lim_{\rightarrow I} DI, A) \cong \lim_{\leftarrow I} \mathcal{A}(DI, A). \qquad (8)$$

For instance,

$$\mathcal{A}(B_1 + B_2, A) \cong \mathcal{A}(B_1, A) \times \mathcal{A}(B_2, A).$$

Theorem 6.5 *Left adjoints preserve colimits; right adjoints preserve limits.*

This is an excellent test for existence of an adjoint: if you meet a functor that preserves limits/colimits, the chances are very high that it has a left/right adjoint. The Adjoint Functor Theorems (Lecture 3) make this precise.

Proof Take $\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$ with $F \dashv G$. Suppose that $D : \mathbb{I} \longrightarrow \mathcal{A}$ has a limit. Then

$$\begin{aligned} \mathcal{A}(A, G(\lim_{\leftarrow I} DI)) &\cong \mathcal{B}(FA, \lim_{\leftarrow I} DI) \\ &\cong \lim_{\leftarrow I} \mathcal{B}(FA, DI) \\ &\cong \lim_{\leftarrow I} \mathcal{A}(A, GDI) \\ &\cong \mathrm{Cones}(A, G \circ D) \end{aligned}$$

naturally in A (the last step by Lemma 6.3), so by the characterization (6) of limits, $G(\lim_{\leftarrow I} DI)$ is a limit of $G \circ D$. So G preserves limits; dually, F preserves colimits. □

Example 6.6 All the usual forgetful functors (**Gp** $\longrightarrow$ **Set**, **AssAlg** $\longrightarrow$ **LieAlg**, **Top** $\longrightarrow$ **Set**, etc) preserve limits. In particular, monics in categories of algebras are injective (see 5.7).

Example 6.7 Free functors preserve colimits.

Example 6.8 In homological algebra, left adjoints are right exact, and dually. For instance, we have $- \otimes B \dashv \mathbf{Hom}_k(B, -)$ (see 3.4), so $- \otimes B$ is right exact and $\mathbf{Hom}_k(B, -)$ left exact.

Digression 6.9 Under suitable conditions, a right exact functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ between abelian categories gives rise to a sequence of **left derived functors** $L_0F \cong F$, L_1F, L_2F, ... from $\mathcal{A}$ to $\mathcal{B}$. Dually, a left exact functor has **right derived functors**. For instance, if $B \in k$-**Mod** then $L^n(- \otimes B) = \mathrm{Tor}_n(B, -)$ and $R^n(\mathbf{Hom}_k(B, -)) = \mathrm{Ext}^n(B, -)$. This also gives easy definitions of group homology and cohomology and of sheaf cohomology.

A category is **cartesian closed** if it has finite products and for each $B \in \mathcal{A}$, the functor

$$- \times B : \mathcal{A} \longrightarrow \mathcal{A}$$

has a right adjoint $(\ )^B$. There is then an isomorphism

$$\mathcal{A}(A \times B, C) \cong \mathcal{A}(A, C^B) \qquad (9)$$

natural in $A, B, C \in \mathcal{A}$. The object C^B is called an **exponential**.

Example 6.10 **Set** is cartesian closed, with $C^B = \mathbf{Set}(B, C)$. The isomorphism (9) can be written $C^{A \times B} \cong (C^B)^A$.

Example 6.11 **Ab** is *not* cartesian closed. If it were, $B \times -$ would preserve colimits, and in particular initial objects, but $\times = \oplus$ and $B \oplus 0 \cong B \not\cong 0$ in general.

Digression 6.12 A **topos** is a cartesian closed category with finite limits and one further property. A topos can be regarded as a generalized topological space: for any space X, the category of sheaves of sets on X is a topos and contains very nearly all the information about X. A topos can also be regarded as a generalized universe of sets, for logic: **Set** is a topos, and there are other toposes that demonstrate the independence of the Axiom of Choice and the Continuum Hypothesis from the ZF axioms of set theory.

We now consider the category $[\mathbb{A}^{\mathrm{op}}, \mathbf{Set}]$ of presheaves on a small category $\mathbb{A}$.

Proposition 6.13 *Let $\mathbb{A}$ be a small category and $\mathcal{S}$ a category with all limits. Then $[\mathbb{A}, \mathcal{S}]$ has all limits, and they are* **computed pointwise***: given $D : \mathbb{I} \longrightarrow [\mathbb{A}, \mathcal{S}]$ and $A \in \mathbb{A}$,*

$$(\lim_{\leftarrow I} DI)A \cong \lim_{\leftarrow I}((DI)A).$$

(And dually for colimits.)

Proof Omitted. □

In the case of binary products, this just says that given $\mathbb{A} \underset{X_2}{\overset{X_1}{\rightrightarrows}} \mathcal{S}$, the product $X_1 \times X_2$ in $[\mathbb{A}, \mathcal{S}]$ exists and is given by

$$(X_1 \times X_2)A = X_1 A \times X_2 A.$$

Corollary 6.14 *Let $\mathbb{A}$ be a small category. Then all limits and colimits exist in $[\mathbb{A}^{\mathrm{op}}, \mathbf{Set}]$ and are computed pointwise.* $\square$

Corollary 6.15 *Let $\mathbb{A}$ be a small category. Then the Yoneda embedding $H_\bullet : \mathbb{A} \longrightarrow [\mathbb{A}^{\mathrm{op}}, \mathbf{Set}]$ preserves limits.*

Proof Suppose that $D : \mathbb{I} \longrightarrow \mathbb{A}$ has a limit. Then

$$\begin{aligned} (H_\bullet(\lim_{\leftarrow I} DI))A &= \mathcal{A}(A, \lim_{\leftarrow I} DI) \\ &\cong \lim_{\leftarrow I} \mathcal{A}(A, DI) \\ &= \lim_{\leftarrow I}(H_{DI}(A)) \\ &\cong (\lim_{\leftarrow I} H_{DI})A \\ &= (\lim_{\leftarrow I} H_\bullet(DI))A \end{aligned}$$

naturally in A (the penultimate step by 6.14). $\square$

On the other hand, $H_\bullet$ does *not* usually preserve colimits. Suppose, for instance, that $\mathbb{A}$ has an initial object 0: then $H_0(0) = \{1_0\} \not\cong \emptyset$, but the initial object 0 of $[\mathbb{A}^{\mathrm{op}}, \mathbf{Set}]$ satisfies $0(A) = \emptyset$ for all A. In some sense, $H_\bullet$ fails as badly as possible to preserve colimits:

by taking colimits of representables, we can obtain *any* presheaf.

Theorem 6.16 *Every presheaf is a colimit of representables.*

Proof Omitted. □

The theorem says that for any small category $\mathbb{A}$ and presheaf $X : \mathbb{A}^{\mathrm{op}} \longrightarrow \mathbf{Set}$, there is some category $\mathbb{E}(X)$ and functor $P_X : \mathbb{E}(X) \longrightarrow \mathbb{A}$ such that

$$X \cong \varinjlim_{E} H_{P_X(E)}.$$

Example 6.17 Let G be a group, regarded as a one-object category. A presheaf on G is a set with a right G-action, and the unique representable presheaf on G is the set G acted on by right multiplication. Any set X acted on by G is the disjoint union of its orbits, and each orbit is of the form $[G : H]$ (the set of right cosets of some subgroup H of G, acted on by multiplication). Roughly speaking, this implies that X is a coproduct of quotients of representables, hence a colimit of representables.

Analogies to Theorem 6.16: every analytic function can be written as a power series, e.g.

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots.$$

Every positive integer is a product of primes.

Corollary 6.18 *Let* $\mathbb{A}$ *be a small category. Then* $[\mathbb{A}^{\mathrm{op}}, \mathbf{Set}]$ *is the* **free cocompletion** *of* $\mathbb{A}$*: that is,* $[\mathbb{A}^{\mathrm{op}}, \mathbf{Set}]$ *is* **cocomplete** *(has all colimits) and* $H_\bullet : \mathbb{A} \longrightarrow [\mathbb{A}^{\mathrm{op}}, \mathbf{Set}]$ *is universal among all functors from* $\mathbb{A}$ *to a cocomplete category.*

Proof Omitted. □

So the process $\mathbb{A} \longmapsto [\mathbb{A}^{\mathrm{op}}, \mathbf{Set}]$ takes $\mathbb{A}$ and adjoins formally all possible colimits; 'free' means that it ignores any colimits that $\mathbb{A}$ might already have.

You may meet the **ind-completion** of a category $\mathbb{A}$, which is $\mathbb{A}$ with just the *filtered* colimits freely adjoined; it is a subcategory of $[\mathbb{A}^{\mathrm{op}}, \mathbf{Set}]$. The dual process (using cofiltered limits) is called **pro-completion**.

Exercises

6.19 Show that if $\mathcal{A}$ is **Gp**, k-**Mod**, **LieAlg** or **Ring** then the forgetful functor $\mathcal{A} \longrightarrow$ **Set** does not have a right adjoint. (Hint: Theorem 6.5.)

6.20 Dualization: given that (7) holds for all $\mathcal{A}$, A, and D for which it makes sense, show that (8) holds for all $\mathcal{A}$, A, and D for which it makes sense.

6.21 Let $\mathbb{A}$ be a small category. What are the epics in $[\mathbb{A}^{\mathrm{op}}, \mathbf{Set}]$?