Solutions to exercises from Lecture 4

4.13 Take an isomorphism $\alpha : H_A \xrightarrow{\sim} H_B$. Put

$$\begin{aligned} f &= \alpha_A(1_A) \in H_B(A) = \mathcal{A}(A, B), \\ g &= \alpha_B^{-1}(1_B) \in H_A(B) = \mathcal{A}(B, A). \end{aligned}$$

By naturality of α, the square

$$\begin{tikzcd}
H_A(A) \arrow[r, "\alpha_A"] \arrow[d, "H_A(g)"'] & H_B(A) \arrow[d, "H_B(g)"] \\
H_A(B) \arrow[r, "\alpha_B"'] & H_B(B)
\end{tikzcd}$$

commutes. This square is

$$\begin{tikzcd}
\mathcal{A}(A, A) \arrow[r, "\alpha_A"] \arrow[d, "g^*"'] & \mathcal{A}(A, B) \arrow[d, "g^*"] \\
\mathcal{A}(B, A) \arrow[r, "\alpha_B"'] & \mathcal{A}(B, B),
\end{tikzcd}$$

and we have

$$\begin{tikzcd}
1_A \arrow[r, mapsto] \arrow[dd, mapsto] & f \arrow[d, mapsto] \\
 & f \circ g \\
g \arrow[r, mapsto] & 1_B,
\end{tikzcd}$$

so $f \circ g = 1_B$. Dually, $g \circ f = 1_A$. Hence $A \cong B$.

4.14 Remain calm.