

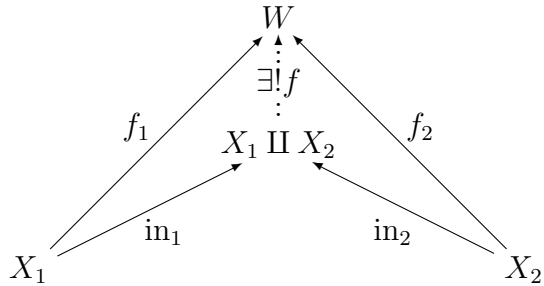
Solutions to exercises from Lecture 0

0.12 Here are three possibilities:

- a. Take sets X_1 and X_2 , their disjoint union $X_1 \amalg X_2$, and the inclusions

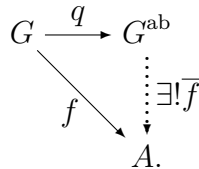
$$X_1 \xrightarrow{\text{in}_1} X_1 \amalg X_2 \xleftarrow{\text{in}_2} X_2.$$

Then the following universal property holds:



(Compare Example 0.5.)

- b. Take a group G , its abelianization $G^{\text{ab}} = G/[G, G]$, and the natural map $q : G \longrightarrow G^{\text{ab}}$. Then $G \xrightarrow{q} G^{\text{ab}}$ is the universal homomorphism from G to an abelian group. That is, if $G \xrightarrow{f} A$ is a homomorphism from G to an abelian group A , there is a unique homomorphism $\bar{f} : G^{\text{ab}} \longrightarrow A$ such that $\bar{f} \circ q = f$:

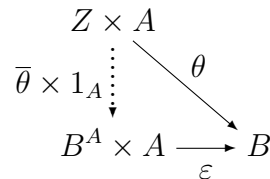


(Compare Example 0.1.)

- c. Take sets A and B , the set B^A of functions from A to B , and the evaluation map

$$\begin{aligned} \varepsilon : B^A \times A &\longrightarrow B \\ (f, a) &\longmapsto f(a). \end{aligned}$$

For any set Z and function $\theta : Z \times A \longrightarrow B$, there is a unique function $\bar{\theta} : Z \longrightarrow B^A$ such that



commutes. (This $\bar{\theta}$ is given by $(\bar{\theta}(z))(a) = \theta(z, a)$.)

0.13 I'll do (a). Fix sets X_1 and X_2 . Suppose that

$$X_1 \xrightarrow{i_1} Y \xleftarrow{i_2} X_2$$

has the universal property

$$\begin{array}{ccccc} & & W & & \\ & \nearrow f_1 & \vdots \exists! f & \nwarrow f_2 & \\ & \nearrow i_1 & Y & \nwarrow i_2 & \\ X_1 & & & & X_2 \end{array} \quad (1)$$

and that

$$X_1 \xrightarrow{i'_1} Y' \xleftarrow{i'_2} X_2$$

does too. Take $(W, f_1, f_2) = (Y', i'_1, i'_2)$ in (1): then we obtain $f : Y \longrightarrow Y'$ such that $f i_1 = i'_1$ and $f i_2 = i'_2$. Similarly, we obtain $f' : Y' \longrightarrow Y$ such that $f' i'_1 = i_1$ and $f' i'_2 = i_2$. But then $(f' f) i_1 = 1_Y i_1$ and $(f' f) i_2 = 1_Y i_2$; in other words, the following diagram commutes when the dotted arrow is either $f' f$ or 1_Y :

$$\begin{array}{ccccc} & & Y & & \\ & \nearrow i_1 & \vdots & \nwarrow i_2 & \\ & \nearrow i_1 & Y & \nwarrow i_2 & \\ X_1 & & & & X_2. \end{array}$$

So by the uniqueness part of the universal property, $f' f = 1_Y$. Similarly, $f f' = 1_{Y'}$. So $Y \cong Y'$.