

Lecture 2

Natural transformations and equivalence

Example 2.1 π_1 and H_1 both give ways of turning a space into a group. Precisely, there is a functor

$$\pi_1 : \mathbf{Top}_* \longrightarrow \mathbf{Gp}$$

where $\mathbf{Top}_*$ is the category of spaces with distinguished basepoint and continuous basepoint-preserving maps, and there is a functor

$$\begin{array}{ccc} H'_1 : \mathbf{Top}_* & \longrightarrow & \mathbf{Gp}, \\ (X, x) & \longmapsto & H_1(X). \end{array}$$

For each $(X, x) \in \mathbf{Top}_*$ there is a canonical map $\alpha_{(X, x)} : \pi_1(X, x) \longrightarrow H_1(X) = H'_1(X, x)$ (the Hurewicz map), and α defines a natural transformation $\pi_1 \longrightarrow H'_1$.

Definition 2.2 Let $\mathcal{A}$ and $\mathcal{B}$ be categories and $F, G : \mathcal{A} \longrightarrow \mathcal{B}$ functors. A **natural transformation** $\alpha :$

$F \longrightarrow G$ (sometimes drawn $\mathcal{A} \begin{array}{c} \xrightarrow{F} \\ \Downarrow \alpha \\ \xrightarrow{G} \end{array} \mathcal{B}$) is a family

$\left(FA \xrightarrow{\alpha_A} GA \right)_{A \in \mathcal{A}}$ of maps in $\mathcal{B}$ such that for every map $A \xrightarrow{f} A'$ in $\mathcal{A}$, the square

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FA' \\ \alpha_A \downarrow & & \downarrow \alpha_{A'} \\ GA & \xrightarrow{Gf} & GA' \end{array} \quad (2)$$

commutes.

Loosely, a natural transformation $\alpha : F \longrightarrow G$ is something that associates to each arrow $A \xrightarrow{f} A'$ in $\mathcal{A}$ precisely one arrow $FA \longrightarrow GA'$ in $\mathcal{B}$. When $f = 1_A$, this latter arrow is α_A . For general f , it is the diagonal of the naturality square (2), and ‘precisely one’ implies that the square commutes.

Example 2.3 For any vector space V , there is an evaluation map

$$\begin{array}{ccc} \varepsilon_V : V & \longrightarrow & V^{**}, \\ v & \longmapsto & (\text{evaluate at } v). \end{array}$$

(When V is finite-dimensional, ε_V is an isomorphism.) Since $(\)^*$ is contravariant, $(\)^{**}$ is covariant, and for any linear map $f : V \longrightarrow W$, the square

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \varepsilon_V \downarrow & & \downarrow \varepsilon_W \\ V^{**} & \xrightarrow{f^{**}} & W^{**} \end{array}$$

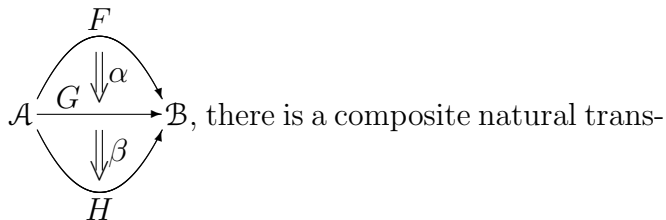
commutes. So we have a natural transformation

$$\begin{array}{ccc} & \text{id} & \\ \text{Vect} & \begin{array}{c} \curvearrowright \\ \Downarrow \varepsilon \\ \curvearrowleft \end{array} & \text{Vect} \\ & (\)^{**} & \end{array}$$

(There is no question of there being a natural transformation $\text{id} \longrightarrow (\)^*$, as $(\)^*$ is not even a functor $\mathbf{Vect} \longrightarrow \mathbf{Vect}$: it's contravariant.)

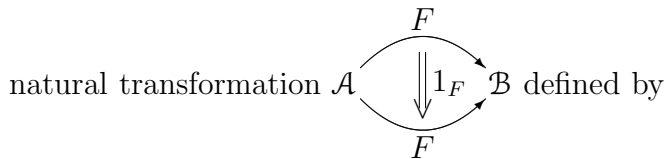
Natural transformations are a kind of map, so you would expect to be able to compose them. Fix categories $\mathcal{A}$ and $\mathcal{B}$.

- Given functors and natural transformations



formation $\mathcal{A} \xrightarrow{H} \mathcal{B}$ defined by $(\beta \circ \alpha)_A = \beta_A \circ \alpha_A$ ($A \in \mathcal{A}$).

- Given a functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$, there is an identity



$(1_F)_A = 1_{FA}$ ($A \in \mathcal{A}$).

So there is a category $[\mathcal{A}, \mathcal{B}]$ whose objects are functors

$\mathcal{A} \longrightarrow \mathcal{B}$ and whose maps are natural transformations between them. It is called the **functor category** from $\mathcal{A}$ to $\mathcal{B}$.

Example 2.4 If G is a group and k is a ring then $[G, k\text{-}\mathbf{Mod}]$ is the category of (left) k -linear representations of G .

Example 2.5 If X is a topological space then $[\mathbf{Open}(X)^{\text{op}}, \mathbf{Set}]$ is the category of presheaves (or more precisely, presheaves of sets) on X .

Everyday phrases such as ‘*the* cyclic group of order 6’ and ‘*the* product of two spaces’ reflect the fact that given two isomorphic objects of a category, we usually neither know nor care whether they are actually equal. In particular, given two functors $F, G : \mathcal{A} \longrightarrow \mathcal{B}$, we usually do not care whether they are equal (which would imply that the objects FA and GA of $\mathcal{B}$ were equal for all A). What really matters is whether they are naturally isomorphic.

Definition 2.6 A **natural isomorphism** of functors from $\mathcal{A}$ to $\mathcal{B}$ is an isomorphism in $[\mathcal{A}, \mathcal{B}]$.

There is an equivalent, more explicit, definition:

Lemma 2.7 *Let $\mathcal{A} \begin{array}{c} \xrightarrow{F} \\ \Downarrow \alpha \\ \xrightarrow{G} \end{array} \mathcal{B}$ be a natural transformation. Then α is a natural transformation if and only if $\alpha_A : FA \longrightarrow GA$ is an isomorphism for all $A \in \mathcal{A}$.*

Proof Elementary. □

If there is a natural isomorphism $\mathcal{A} \begin{array}{c} \xrightarrow{F} \\ \Downarrow \\ \xrightarrow{G} \end{array} \mathcal{B}$ then we write $F \cong G$. Another way of saying ‘ $F \cong G$ ’ is ‘ $FA \cong GA$ **naturally in A** ’.

Example 2.8 For finite-dimensional vector spaces V , we have $V \cong V^{**}$ naturally in V . That is, the two functors $\mathbf{FDVect} \xrightarrow[\text{()^{**}}]{\text{id}} \mathbf{FDVect}$ are naturally isomorphic. (Proof: 2.3 + 2.7.)

Example 2.9 Let G be a group and k a ring: then a functor $G \longrightarrow k\text{-}\mathbf{Mod}$ is a representation of G , and two such are naturally isomorphic if and only if there is an isomorphism between the two vector spaces compatible with the actions by G .

Example 2.10 There exist functors $\mathcal{A} \begin{smallmatrix} \xrightarrow{F} \\ \xleftarrow{G} \end{smallmatrix} \mathcal{B}$ such that $FA \cong GA$ for all A , but not *naturally* in A . Exercise 2.19 is an example from elementary combinatorics.

Two elements of a set are either equal or not. Two objects of a category might be equal, or isomorphic, or not. But for categories themselves, even isomorphism is an unreasonably strict relation: for if $\mathcal{A} \cong \mathcal{B}$ then there are functors

$$\mathcal{A} \begin{smallmatrix} \xrightarrow{F} \\ \xleftarrow{G} \end{smallmatrix} \mathcal{B}$$

such that $G(F(A))$ is *equal* to A for all $A \in \mathcal{A}$. The most useful notion of sameness of categories is ‘equivalence’.

Definition 2.11 An **equivalence** between categories $\mathcal{A}$ and $\mathcal{B}$ is a pair $\mathcal{A} \begin{smallmatrix} \xrightarrow{F} \\ \xleftarrow{G} \end{smallmatrix} \mathcal{B}$ of functors together

with natural isomorphisms $\eta : 1_{\mathcal{A}} \xrightarrow{\sim} G \circ F$ and $\varepsilon : F \circ G \xrightarrow{\sim} 1_{\mathcal{B}}$. We then say that $\mathcal{A}$ and $\mathcal{B}$ are **equivalent** and write $\mathcal{A} \simeq \mathcal{B}$. We also say that the functor F (or G) is an **equivalence**.

There is a more practical criterion for a functor to be an equivalence.

Definition 2.12 A functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ is

- **full** if for each $A, A' \in \mathcal{A}$, the function

$$\begin{array}{ccc} \mathcal{A}(A, A') & \longrightarrow & \mathcal{B}(FA, FA') \\ f & \longmapsto & Ff \end{array}$$

is surjective

- **faithful** if for each $A, A' \in \mathcal{A}$, this function is injective
- **essentially surjective on objects** if for all $B \in \mathcal{B}$, there exists $A \in \mathcal{A}$ such that $FA \cong B$.

Lemma 2.13 *A functor is an equivalence if and only if it is full, faithful, and essentially surjective on objects.*

Proof Omitted. ‘If’ uses a strong form of the axiom of choice. $\square$

Example 2.14 A subcategory $\mathcal{B}$ of a category $\mathcal{A}$ is **full** if the inclusion functor $\mathcal{B} \hookrightarrow \mathcal{A}$ is full, that is, $\mathcal{B}(B, B') = \mathcal{A}(B, B')$ for all $B, B' \in \mathcal{B}$. (So a full subcategory is determined by its objects.) Suppose that $\mathcal{B}$ is a full subcategory of $\mathcal{A}$ containing at least one object from every isomorphism class of objects of $\mathcal{A}$. Then the inclusion $\mathcal{B} \hookrightarrow \mathcal{A}$ is full, faithful (like any inclusion of subcategories), and essentially surjective on objects, so $\mathcal{B} \simeq \mathcal{A}$.

Example 2.15 A **total order** on a set A is an order $\leq$ such that for all $a, a' \in A$, either $a \leq a'$ or $a' \leq a$. A function $f : (A, \leq) \longrightarrow (B, \leq)$ is **order-preserving** if $a \leq a' \Rightarrow f(a) \leq f(a')$.

Let $\mathcal{D}$ be the category of nonempty finite totally ordered sets and order-preserving maps. Write $[n]$ for the set $\{0, \dots, n\}$ with its usual ordering, and let Δ be the full subcategory of $\mathcal{D}$ with objects $[0], [1], \dots$ (A functor $\Delta^{\text{op}} \longrightarrow \mathbf{Set}$ is called a **simplicial set**.) Then $\Delta \simeq \mathcal{D}$ by the previous example.

Example 2.16 More substantially, if k is an algebraically closed field then

$$\begin{aligned} & (\text{affine varieties over } k)^{\text{op}} \\ \simeq & (\text{finitely generated reduced } k\text{-algebras}). \end{aligned}$$

Example 2.17 Let $\mathcal{C}$ be the full subcategory of **Cat** consisting of the small categories with precisely one object. There is a functor $U : \mathcal{C} \longrightarrow \mathbf{Monoid}$ sending a one-object category to its unique hom-set, and U is an equivalence. This makes precise the statement (1.7) that one-object categories are the same as monoids.

Example 2.18 This example shows that there's no point having a category of sets-with-structure if the maps don't preserve the structure. Let $\mathcal{A}$ be the category whose objects are groups and whose maps are all functions between them. Every nonempty set admits at least one group structure, so the forgetful functor from $\mathcal{A}$ to the category of nonempty sets is an equivalence.

Exercises

2.19 Let $\mathcal{B}$ be the category of finite sets and bijections. For $X \in \mathcal{B}$, let $\mathbf{Sym}(X)$ be the set of bijections $X \longrightarrow X$ and let $\mathbf{Ord}(X)$ be the set of total orders on X .

- a. Give a definition of $\mathbf{Sym}$ on morphisms of $\mathcal{B}$ so that $\mathbf{Sym}$ becomes a functor $\mathcal{B} \longrightarrow \mathbf{Set}$. Do the same for $\mathbf{Ord}$. (Both your definitions should be canonical. This means, roughly, that you should not use the word ‘choose’.)
- b. Show that there is no natural transformation $\mathbf{Sym} \longrightarrow \mathbf{Ord}$. (Hint: consider the identity permutation.)

Conclude that $\mathbf{Sym}(X) \cong \mathbf{Ord}(X)$ for all $X \in \mathcal{B}$, but not naturally in $X \in \mathcal{B}$.

2.20 Let G be a group. Homomorphisms $\mathbb{Z} \longrightarrow G$ can be identified with elements of G (by considering where $1 \in \mathbb{Z}$ is sent). So regarding groups as one-object categories, an element of G is the same as a

functor $\mathbb{Z} \longrightarrow G$; hence natural isomorphism defines an equivalence relation on the elements of G . What is that equivalence relation? (One-word answer.)

2.21 (Linear algebra can be done equivalently with matrices or linear transformations...)

Let k be a field. Let **Mat** be the category whose objects are natural numbers and with

$$\mathbf{Mat}(m, n) = \{n \times m \text{ matrices over } k\}.$$

Prove that **Mat** is equivalent to **FDVect**, the category of finite-dimensional vector spaces over k . (Hint: use 2.13.) Does your equivalence involve a *canonical* functor **Mat** $\longrightarrow$ **FDVect**, or **FDVect** $\longrightarrow$ **Mat**?