TOPOLOGICAL QUALITIES OF ENRICHED CATEGORY MAGNITUDE AND CLASSIFICATION OF MONOIDAL-TYPE CATEGORIES

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2015

DECLARATION:

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

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ABSTRACT:

This dissertation consists of two stand-alone parts. We begin with an exploration of the *magnitude* of an enriched category. Introduced by Leinster in [8], magnitude is a rig valued invariant of an enriched category sharing many properties with the Euler characteristic of topological spaces. When the enriching category is simply the category of sets, magnitude – like the Euler characteristic of topological spaces – obeys a sort of multiplicativity over fibrations, and also can be computed in many cases via an associated homology theory. First we show that multiplicativity over fibrations again holds when the enriching category is the 2-category of categories, Cat. Second, motivated in large part by [4], we give a survey of the problem of computing magnitude via a homology theory in a general enriched categorical setting. Though we present no new results in this section, we do outline some successes and many of the difficulties encountered in this area as well as provide a promising direction for future research.

The second part of the dissertation extends from the work of [2]. In that paper it is shown that strict monoidal, monoidal, and skew monoidal categories are in one to one correspondence with maps from the *Catalan simplicial set* – so named because its simplices can be counted by the Catalan numbers – into one of three carefully chosen nerves of Cat. We extend and simplify this result by showing that there is a single nerve, $N_\Delta(N_2\mathsf{Cat})$, such that the three monoidal-type categories already mentioned as well as lax monoidal categories and Σ-monoidal categories can each be understood as maps from the Catalan simplicial set to $N_\Delta(N_2\mathsf{Cat})$. This provides a general framework for defining and studying monoidal-type categories. A detailed examination of these maps actually provides for the definition of a new monoidal-type category which is a joint generalization of the five mentioned above. This part can presently be found on the arxiv at *arXiv:1507.05205*, and is also currently submitted for publication in a slightly modified form.

ORGANIZATION:

This dissertation consists of two stand-alone parts.

The first part – *Topological qualities of enriched category magnitude* – is concerned with the magnitude of enriched categories. The second part – *The classification of monoidal-type categories* – is concerned with the Catalan simplicial set and its classification properties with respect to monoidal-type categories.

This second part can presently be found on the arXiv at *arXiv:1507.05205*, and is also currently submitted for publication in a slightly modified form.

ACKNOWLEDGMENTS:

I would like to thank Tom Leinster for introducing me to magnitude and for the many idea-forming meetings we have shared throughout my exploration of this intriguing invariant and its various aspects. I would also like to thank Richard Hepworth, who helped me to understand many aspects of the homological side of magnitude. With respect to the second part of the paper, I would especially like to extend my gracious thanks to Richard Garner for originally posing the problem which drove this research, for patiently helping me to understand many of the simplicial constructions in this area, and for his immeasurable assistance in putting it all to paper. Finally, I would like to thank the Mathematics department of the University of Edinburgh.

Contents

1. Topological qualities of enriched category magnitude

The magnitude of an enriched category was first described in the context of *finite categories* – categories with a finite number of objects and finite hom-sets – where it bore the name 'Euler characteristic'. The Euler characteristic of a category – a rational number associated to that category – earns its namesake due to a number of shared properties with the familiar Euler characteristic of topological spaces. Properly understood, the Euler characteristic of a category is:

(i) a homotopy invariant;

(ii) multiplicative over cartesian product;

(iii) additive over disjoint union;

(iv) multiplicative over fibrations;

(v) and is computable via an associated homology theory.

More generally, for *any enriching monoidal category* $\mathcal{V}$, the magnitude of a $\mathcal{V}$-enriched category also (properly understood) shares properties (i)–(iii). It is presently unknown whether the final two properties, (iv) and (v), are also shared for any enriching $\mathcal{V}$.

In part 1 of this dissertation, we explore the properties (iv) and (v). In section 1.1, we will define the magnitude of an enriched category as well as its basic properties including (i)–(iii) above. In section 1.2 we explain the multiplicativity of the Euler characteristic on the total space of a fibration for finite categories, and show that the magnitude of Cat-enriched categories also shares this property. Finally, in section 1.3 we take up the problem of computing magnitude of an enriched category via a homology theory associated to that enriched category. I have been unable to produce a homology theory for enriched categories which can be used to compute magnitude in general. Rather, this section gives a survey of the problem, including: a description of the finite categorical case; a success story in the context of the magnitude of finite graphs due to [4]; an account of the difficulties in generalizing the approach used there; and lastly, a promising direction for future research.

1.1. **The magnitude of an enriched category.** Suppose $(\mathcal{V}, \otimes, I)$ is a monoidal category and write $\mathcal{V}$-Cat for the category of $\mathcal{V}$-enriched categories with enriched functors. In many cases, the objects of $\mathcal{V}$ come with a natural notion of 'size' which respects the monoidal product: if $\mathcal{V} = \mathsf{FinSet}$, the category of finite sets, then the size of an object $X \in \mathsf{FinSet}$ is simply its cardinality; if $\mathcal{V} = \mathsf{FinVect}$, the category of finite dimensional vector spaces, then the size is just dimension; if $\mathcal{V}$ is a suitable category of topological spaces, then size can be taken to be Euler characteristic [14]. The magnitude of a *finite $\mathcal{V}$-category* – that is, a category enriched in $\mathcal{V}$ with a finite number of objects – can then be thought of as a notion of size of the $\mathcal{V}$-category, induced from the notion of size of the objects of $\mathcal{V}$.

Suppose that K is a commutative rig (i.e. semiring) with multiplication $\bullet$ and multiplicative identity 1, and that we are given a multiplicative *gauge function*

$$|-|: (\mathsf{ob}(\mathcal{V}), \otimes, I) \longrightarrow (K, \bullet, 1)$$

which sends isomorphic objects to the same element in K. For an object $v \in \mathcal{V}$, we think of $|v|$ as the size of v. (For example, when $\mathcal{V} = \mathsf{FinSet}$, the gauge returns cardinality of the set as an element of $K = \mathbb{Q}$.) In this context, the magnitude of a finite enriched category $\mathbb{A} \in \mathcal{V}\text{-}\mathsf{Cat}$ can be computed (should it exist) according to the following procedure.

Let n be the number of objects in $\mathbb{A}$ and define the $n \times n$ matrix $Z_{\mathbb{A}}$ by

$$(Z_{\mathbb{A}})_{i,j} := |\mathbb{A}(a_i, a_j)|.$$

That is, the $(i,j)^{th}$ entry of $Z_{\mathbb{A}}$ is the size (given by the gauge function) of the mapping object between the i^{th} and j^{th} objects of $\mathbb{A}$. The matrix $Z_{\mathbb{A}}$ is therefore an element of the matrix rig $M_n(K)$ and so can be understood as a linear map on K^n.

A *weighting* for a matrix $A \in M_n(K)$ is a column vector $w \in K^n$ satisfying the *weighting equations* $\sum_{1\leq j\leq n} A_{ij}w_j = 1$ for all $1 \leq i \leq n$. That is, $Aw = (1,1,...,1)^\top$. Similarly, a *coweighting* for A is a row vector c satisfying the *coweighting equations* $\sum_{1\leq i\leq n} c_iA_{ij} = 1$ for all $1 \leq j \leq n$. In other words, $cA = (1,1,...,1)$. Weightings and coweightings need not exist for every A, e.g. $A = 0$. On the other extreme, a matrix has a unique weighting and coweighting if it is invertible. There are also noninvertible matrices with both weightings and coweightings and these need not be unique.[1] Nevertheless, all weightings and coweightings for a matrix A do share an important property in common:

Proposition 1.1.1. *If a matrix $A \in M_n(K)$ has a weighting w and a coweighting c, then:*

$$\sum_{1\leq j\leq n} w_j = \sum_{1\leq i\leq n} c_i.$$

[1] If $A \in M_2(\mathbb{R})$ is given by $A_{ij} = 1$ for all $1 \leq i,j \leq 2$, then weightings and coweightings for A come in a one parameter family.

The short proof can be found in Lemma 1.1.2 of [8].

In the context of the matrix $Z_{\mathbb{A}}$, a weight w for $Z_{\mathbb{A}}$ satisfies the weighting equations

$$\sum_{b\in\mathbb{A}} |\mathbb{A}(a,b)|w_b = 1$$

for each $a \in \mathbb{A}$ and similarly for a coweight c. For simplicity, we define a (co)weight vector for the enriched category $\mathbb{A}$ to be a (co)weight vector for the matrix $Z_{\mathbb{A}}$. We can now state the definition of magnitude.

Definition 1.1.2. Suppose $(\mathcal{V}, \otimes, I)$ is a monoidal category, K a commutative rig, $|-|$ a gauge function, and $\mathbb{A} \in \mathcal{V}\text{-}\mathsf{Cat}$ finite. Then if $\mathbb{A}$ has both a weighting w and coweighting c, we define the *magnitude* of $\mathbb{A}$:

$$|\mathbb{A}| := \sum_{b\in\mathbb{A}} w_b = \sum_{a\in\mathbb{A}} c_a.$$

The magnitude of a finite $\mathcal{V}$-category $\mathbb{A}$ is therefore an element of the rig K associated to $\mathbb{A}$. Magnitude shares a number of properties with the Euler characteristic of topological spaces as we see in the following proposition.

Proposition 1.1.3. *Let $\mathbb{A}$ and $\mathbb{B}$ be finite $\mathcal{V}$-categories. Then we have:*

(1) If both $\mathbb{A}$ and $\mathbb{B}$ have magnitude and there is an enriched functor $X : \mathbb{A} \longrightarrow \mathbb{B}$ with either a left or right adjoint, then $|\mathbb{A}| = |\mathbb{B}|$.

(2) If $\mathbb{A} \simeq \mathbb{B}$, then $\mathbb{A}$ has magnitude if and only if $\mathbb{B}$ does, and in this case $|\mathbb{A}| = |\mathbb{B}|$.

(3) If both $\mathbb{A}$ and $\mathbb{B}$ have magnitude, then so does their monoidal product $\mathbb{A} \otimes \mathbb{B}$ and $|\mathbb{A} \otimes \mathbb{B}| = |\mathbb{A}||\mathbb{B}|$.[2]

(4) If $\mathbb{A}$ and $\mathbb{B}$ have magnitude, then so does their coproduct $\mathbb{A} + \mathbb{B}$ with $|\mathbb{A} + \mathbb{B}| = |\mathbb{A}| + |\mathbb{B}|$.[3]

The proof(s) can be found in section 1.4 of [8].

We can understand the first two items of Proposition 1.1.3 as analogous to the fact that Euler characteristic of topological spaces is a homotopy invariant, where adjoint functors play the role of homotopy equivalence. Magnitude is therefore an adjoint invariant of finite

[2]The monoidal product $\otimes$ of $\mathcal{V}$-categories exists when $\mathcal{V}$ is symmetric.

[3]The coproduct $+$ of $\mathcal{V}$-categories exists when $\mathcal{V}$ has an initial object ∞ with $v \otimes \infty \cong \infty \cong \infty \otimes v$ for all $v \in \mathcal{V}$.

$\mathcal{V}$-categories. The latter two items are analogous to the statements that Euler characteristic is multiplicative over cartesian product and additive over disjoint union. It is also worth noting that (again like Euler characteristic) there are some finite $\mathcal{V}$-categories $\mathbb{A}$ without magnitude (we also may say that the magnitude of such categories is undefined), because the matrix $Z_{\mathbb{A}}$ need not have both a weighting and coweighting.

Magnitude of enriched categories is also a remarkably general concept, specializing to specific invariants for every choice of $\mathcal{V}$ and gauge function. In the three cases of $\mathcal{V} =$ FinSet, $\mathsf{FinVect}$, and an appropriate subcategory of Top, with their respective notions of size, magnitude becomes an invariant of categories, linear categories, and topologically enriched categories respectively. We shall describe more possibilities for $\mathcal{V}$ later on in this paper.

One instance of magnitude which has received a great deal of attention is the case when $\mathcal{V} = \mathsf{FinSet}$ so that $\mathcal{V}$-Cat is the full subcategory of Cat consisting of those categories with finite hom-sets. By *finite category* we mean a category with a finite number of objects and finite hom-sets, that is, a finite FinSet-category. The magnitude of finite categories shares two additional properties with the Euler characteristic of a topological space. Consequently, it also goes by the name of *Euler characteristic of a category* and we will write $\chi(\mathbb{A})$ for the Euler characteristic/magnitude of the finite category $\mathbb{A}$. In the rest of Part 1 of this dissertation, we will rigorously present these two additional properties – multiplicativity over fibrations, and computability via homology – shared by topological and categorical Euler characteristics. We will also show some of the work that has gone into exploring these properties for magnitude of $\mathcal{V}$-categories outside of the context where $\mathcal{V} = \mathsf{FinSet}$.

1.2. **Magnitude and fibrations.** We have already mentioned that the Euler characteristic of topological spaces is multiplicative over topological product: $\chi(X \times Y) = \chi(X)\chi(Y)$ for topological spaces X and Y each with Euler characteristic. This is also the case for very general notions of 'twisted' products. If $p : E \longrightarrow B$ is a $\mathbb{Q}$-orientable (Hurewicz) fibration with fiber F and B path-connected, then if any two of B, E or F have Euler characteristic, then so does the third and $\chi(E) = \chi(F)\chi(B)$. (See Proposition 13.5.1 of [15]) The category of categories Cat also has a distinguished class of maps known as (Grothendieck) fibrations, and we shall see that an analogous result holds for the Euler characteristic of categories.

The total space $\mathcal{E}$ of a fibration of categories $p : \mathcal{E} \longrightarrow \mathcal{B}$ can be considered as resulting from the Grothendieck construction. Given a pseudo-functor $X : \mathcal{B} \longrightarrow \mathsf{Cat}$ we can define a new category EX, the *Grothendieck construction* applied to X:

$$\text{Ob } EX := \{(b,x) \mid b \in \text{Ob } \mathcal{B} \text{ and } x \in \text{Ob } Xb\}.$$

$$EX((b,x),(b',x')) := \{(f,\phi) \mid f : b \longrightarrow b' \text{ and } \phi : (Xf)(x) \longrightarrow x'\}.$$

$$(f',\phi') \circ (f,\phi) := (f' \circ f, \phi' \circ (Xf')(\phi)) : (b,x) \longrightarrow (b',x') \longrightarrow (b'',x'').$$

The category EX comes with a projection $p : EX \longrightarrow \mathcal{B}$ by mapping (b,x) to b and (f,ϕ) to f. Thus the fiber of p over an object $b \in \mathcal{B}$ is the category Xb. This projection is a fibration, and conversely, every total space of a fibration is equivalent to a category EX for some $X : \mathcal{B} \longrightarrow \mathsf{Cat}$.[4]

Proposition 1.2.1. *Let $\mathbb{B}$ be a finite category, $X : \mathbb{B} \longrightarrow \mathsf{Cat}$ a pseudo-functor such that Xb is finite for each $b \in \mathbb{B}$. Suppose $\mathbb{B}$ and Xb for each $b \in \mathbb{B}$ have Euler characteristic. Denoting a weight vector for $\mathbb{B}$ by $w^{\mathbb{B}}$ we then have:*

$$\chi(EX) = \sum_{b \in \mathbb{B}} w_b^{\mathbb{B}} \chi(Xb).$$

This proposition should be understood as the analogue of the multiplicativity of topological Euler characteristic over fibrations. One way of parsing the somewhat more complicated formula for $\chi(EX)$ in the categorical case is that the fibers Xb need not be equivalent (or related by an adjoint pair of functors), whereas the fibers of a topological fibration are each homotopically equivalent, hence have the same Euler characteristic. The proof of Proposition 1.2.1 can be found in [7], but we record it here in full. We will need the following lemma:

Lemma 1.2.2. *Let $\mathbb{B}$ be a finite category, $X : \mathbb{B} \longrightarrow \mathsf{Cat}$ a pseudo-functor such that Xb is finite for each $b \in \mathbb{B}$. Suppose we have weight vectors $w^{\mathbb{B}}$ for $\mathbb{B}$ and w^{Xb} for Xb for each $b \in \mathbb{B}$. Then we can define a weight vector w for EX by:*

$$w_{(b,x)} := w_b^{\mathbb{B}} w_x^{Xb}.$$

Proof. We need only show that the purported definition satisfies the weighting equation for all $(b',x') \in EX$:

[4] See http://ncatlab.org/nlab/show/Grothendieck+fibration.

$$\sum_{(b,x)\in EX} |EX((b',x'),(b,x))| w_{(b,x)} = 1.$$

We have:

$$\begin{aligned}
\sum_{(b,x)\in EX} |EX((b',x'),(b,x))| w_{(b,x)} &= \sum_{b\in\mathbb{B}} \sum_{x\in Xb} |EX((b',x'),(b,x))| w_x^{Xb} w_b^{\mathbb{B}} && (1.2.1)\\
&= \sum_{b\in\mathbb{B}} \sum_{x\in Xb} \left(\sum_{f\in\mathbb{B}(b',b)} |Xb((Xf)(x'),x)| \right) w_x^{Xb} w_b^{\mathbb{B}} && (1.2.2)\\
&= \sum_{b\in\mathbb{B}} \sum_{f\in\mathbb{B}(b',b)} \left(\sum_{x\in Xb} |Xb((Xf)(x'),x)| w_x^{Xb} \right) w_b^{\mathbb{B}} && (1.2.3)\\
&= \sum_{b\in\mathbb{B}} \sum_{f\in\mathbb{B}(b',b)} w_b^{\mathbb{B}} && (1.2.4)\\
&= \sum_{b\in\mathbb{B}} |\mathbb{B}(b',b)| w_b^{\mathbb{B}} && (1.2.5)\\
&= 1. && (1.2.6)
\end{aligned}$$

■

Together with the analogous lemma for coweight vectors, we have that if $\mathbb{B}$ and Xb have Euler characteristic, hence both weight and coweight vectors, then EX has both weight and coweight vectors with formulas given by the above lemma. Under the hypotheses of Proposition 1.2.1, we have therefore:

$$\chi(EX) = \sum_{(b,x)\in EX} w_{(b,x)} = \sum_{b\in\mathbb{B}} \sum_{x\in Xb} w_x^{Xb} w_b^{\mathbb{B}} = \sum_{b\in\mathbb{B}} w_b^{\mathbb{B}} \chi(Xb).$$

This concludes the proof of Proposition 1.2.1.

Recalling that the Euler characteristic of a finite category is just the magnitude of that category viewed as a FinSet-enriched category, one might hope that there is a statement generalizing Proposition 1.2.1 to the context of magnitude for $\mathcal{V}$-enriched categories for any $\mathcal{V}$. The immediate difficulty is trying to make sense of the Grothendieck construction in the context of enriched categories; if $\mathbb{B} \in \mathcal{V}\text{-}\mathsf{Cat}$ is finite, we cannot in general consider enriched functors $X : \mathbb{B} \longrightarrow \mathcal{V}\text{-}\mathsf{Cat}$ because, in general, $\mathcal{V}\text{-}\mathsf{Cat}$ is not itself a $\mathcal{V}$-category (nor can it be made into one in some natural way). Nevertheless, there are two enriching categories besides FinSet where we do have a similar result. The first is when $\mathcal{V} = \mathbb{R}^{\geq 0}$, the non-negative real numbers with monoidal product $+$; see Theorem 2.3.11 of [8]. The second is when $\mathcal{V} = \mathsf{FinCat}$, the 2-category of finite categories with cartesian product $\times$ and unit I, the one object category. In what follows, we will write $2\mathsf{Cat}$ to denote Cat-Cat, the category of strict 2-categories. By a *finite 2-category* we will mean a finite FinCat-Cat, that is, a strict 2-category with finite objects, finite 1-cells, and finite 2-cells.

In order to make sense of magnitude of finite 2-categories, we must provide a rig K and multiplicative gauge function $\mathsf{FinCat} \longrightarrow K$. The upshot of Proposition 1.1.3 is that the Euler characteristic $\chi : \mathsf{FinCat} \longrightarrow \mathbb{Q}$ is a multiplicative isomorphism invariant function, hence can be used as a gauge function. (So $K = \mathbb{Q}$ in this case.) Let us rename this as $\chi_1 : \mathsf{FinCat} \longrightarrow \mathbb{Q}$. Of course, it may be that $\chi_1(\mathbb{A})$ is not defined for some finite category $\mathbb{A}$ – if $\mathbb{A}$ has no weight or coweight vector – but this will not cause problems for us. Now that we have a gauge for $\mathcal{V} = \mathsf{Cat}$, we can follow the procedure to produce magnitude for finite 2-categories, which we will call χ_2. If $\mathbb{B} \in 2\mathsf{Cat}$ is finite such that $\chi_1(\mathbb{B}(b_i, b_j))$ is undefined for some objects b_i and b_j, making the matrix $(Z_{\mathbb{B}})_{i,j}$ undefined, we will simply say $\chi_2(\mathbb{B})$ is undefined as well.[5]

In order to reproduce Proposition 1.2.1 in this context, we must first define a Grothendieck construction E for finite 2-categories.[6]

Definition 1.2.3. Let $\mathbb{B} \in 2\mathsf{Cat}$, and $X : \mathbb{B} \longrightarrow 2\mathsf{Cat}$ be a Cat-enriched functor of strict 2-categories, where the latter is viewed as an object of $2\mathsf{Cat}$ by forgetting the 3-cells. We define the *2-category of elements* $EX \in 2\mathsf{Cat}$ as follows.

$$\mathsf{ob}EX := \{(a \in \mathsf{ob}\mathbb{B}, x \in \mathsf{ob}Xa)\} = \coprod_{a \in \mathsf{ob}\mathbb{B}} Xa.$$

As for the hom-categories, we proceed in the following way. For a pair of objects (a, x) and (b, y), define the functor $X_{axby} : \mathbb{B}(a, b)^{op} \longrightarrow \mathsf{Cat}$ by the composite:

$$\mathbb{B}(a, b) \xrightarrow{X_{a,b}} 2\mathsf{Cat}(Xa, Xb) \xrightarrow{ev_x} Xb \xrightarrow{Xb(-,y)} \mathsf{Cat}^{op}.$$

[5] This procedure can be continued inductively; By Proposition 1.1.3, χ_2 is multiplicative and isomorphism insensitive, and hence can again be used as a gauge function for the finite objects of $2\mathsf{Cat}$, or $\mathsf{Fin2Cat}$. This gives a magnitude for finite 3-categories, χ_3, and so on.

[6] See http://ncatlab.org/nlab/show/n-fibration for a more general (contravariant) version of this construction.

By ev_x we mean the functor taking a Cat-functor $F : Xa \longrightarrow Xb$ to $Fx \in Xb$ and an enriched transformation $H : F \Rightarrow G$ to its component at x, $Hx : Fx \longrightarrow Gx$. Here we view $\mathbb{B}(a, b)$ as a strict 2-category with only identity 2-cells so that all four objects type check as objects of $2\mathsf{Cat}$. Explicitly this composite gives:

$$X_{axby}(f : a \longrightarrow b) := Xb((Xf)(x), y).$$
$$X_{axby}(h : f \Rightarrow g) := Xh^*_x : Xb((Xg)(x), y) \longrightarrow Xb((Xf)(x), y).$$

Here $(Xh)^*_x$ is composition with $(Xh)_x$, hence mapping an object $p \in Xb((Xg)(x), y)$ to the object $(p \circ (Xh)_x) \in Xb((Xf)(x), y)$. Morphisms $s : p \Rightarrow q : (Xg)(x) \longrightarrow y$ are horizontally composed with the identity on $(Xh)_x$, that is, $(Xh)^*_x : s \mapsto s \circ (Xh)_x$.

Finally, for the morphisms of EX, we define the hom-category using the Grothendieck construction a dimension lower:

$$EX(a, x)(b, y) := E(X_{axby}).$$

Thus we have:

$$\begin{aligned} \mathsf{ob}EX(a, x)(b, y) &= \{(f, \eta) | f \in \mathsf{ob}\mathbb{B}(a, b), \eta \in \mathsf{ob}Xb((Xf)(x), y)\} \\ &= \coprod_{f \in \mathsf{ob}\mathbb{B}(a,b)} Xb((Xf)(x), y). \end{aligned}$$

$$\begin{aligned} EX(a, x)(b, y)(f, \eta)(g, \mu) &= \{(h, \varepsilon) | h : f \Rightarrow g \in \mathbb{B}(a, b), \varepsilon : \mu \circ (Xh)_x \Rightarrow \eta\} \\ &= \coprod_{h \in C(a,b)(f,g)} Xb((Xf)(x), y)(\mu \circ (Xh)_x, \eta). \end{aligned}$$

$$\begin{tikzcd}
(Xf)(x) \arrow[rr, "(Xh)_x"] \arrow[dr, "\eta"'] & {} \arrow[d, phantom, "\Leftarrow \varepsilon"] & (Xg)(x) \arrow[dl, "\mu"] \\
& y &
\end{tikzcd}$$

Composition in EX is given by functors $EX(a, x)(b, y) \times EX(b, y)(c, z) \longrightarrow EX(a, x)(c, z)$. Given objects $(f, \eta) \in EX(a, x)(b, y)$ and $(g, \mu) \in EX(b, y)(c, z)$, define the composite

$(g,\mu)\circ(f,\eta) := (g\circ f, \mu\circ(Xg)(\eta))$. Given morphisms $(h,\varepsilon) \in EX(a,x)(b,y)(f,\eta)(f',\eta')$ and $(j,\delta) \in EX(b,y)(c,z)(g,\mu)(g',\mu')$, define the composite $(j,\delta)\circ(h,\varepsilon) := (j\circ h, \delta\circ(Xg)(\varepsilon))$. The non-filled portions of the following diagram commute and show this composite.

$$
\begin{tikzcd}
(Xg)((Xf)(x)) \arrow[rr, "(X(j\circ h))_x"] \arrow[dr, "(Xg)((Xh)_x)"] \arrow[dd, "(Xg)(\eta)"'] & & (Xg')((Xf')(x)) \arrow[dd, "(Xg')(\eta')"] \\
& (Xg)((Xf')(x)) \arrow[ur, "(Xj)_{(Xf')(x)}"'] \arrow[dl, "(Xg)(\eta')"] \arrow[l, Rightarrow, "(Xg)(\varepsilon)"'] & \\
(Xg)(y) \arrow[rr, "(Xj)_y"] \arrow[dr, "\mu"'] \arrow[rr, phantom, "\Leftarrow \delta"', yshift=-4ex] & & (Xg')(y) \arrow[dl, "\mu'"] \\
& z &
\end{tikzcd}
$$

This is an associative assignment and furthermore the object $(1_a, 1_x)$ with its identity form an identity for this composition in the category $EX(a,x)(a,x)$.

Armed with a Grothendieck construction for Cat-functors into $2\mathsf{Cat}$, we can state and prove the analogue to Proposition 1.2.1.

Proposition 1.2.4. *Let $\mathbb{B}$ be a finite 2-category, $X : \mathbb{B} \longrightarrow 2\mathsf{Cat}$ a Cat-functor such that Xb is finite for each $b \in \mathbb{B}$. Suppose $\chi_2(\mathbb{B})$ and $\chi_2(Xb)$ exists for each $b \in \mathbb{B}$. Denoting a weight vector for $\mathbb{B}$ by $w^{\mathbb{B}}$ we then have:*

$$\chi_2(EX) = \sum_{b\in\mathbb{B}} w_b^{\mathbb{B}} \chi_2(Xb).$$

As before, this follows easily from a lemma:

Lemma 1.2.5. *Let $\mathbb{B}$ be a finite 2-category and $X : \mathbb{B} \longrightarrow \mathsf{2Cat}$ a Cat-functor such that Xb is finite for each $b \in \mathbb{B}$. Suppose we have weight vectors $w^{\mathbb{B}}$ for $\mathbb{B}$ and w^{Xb} for Xb for each $b \in \mathbb{B}$. In particular this means that $\chi_1(\mathbb{B}(a,b))$ and $\chi_1(Xb(x,y))$ exists for all $a, b \in \mathbb{B}$ and $x, y \in Xb$. Then we can define a weight vector w for EX by:*

$$w_{(a,x)} := w_a^{\mathbb{B}} w_x^{Xa}.$$

Proof. We simply need to show the claimed weighting satisfies the weight equations.

$$\begin{aligned}
\sum_{(b,y)\in EX} \chi_1(EX(a,x)(b,y)) w_b^{\mathbb{B}} w_y^{Xb} &= \sum_{b\in\mathbb{B}} \sum_{y\in Xb} \chi_1(EX(a,x)(b,y)) w_y^{Xb} w_b^{\mathbb{B}} \\
&= \sum_{b\in\mathbb{B}} \sum_{y\in Xb} \left(\sum_{f\in\mathbb{B}(a,b)} \chi_1(Xb((Xf)(x),y)) w_f^{\mathbb{B}(a,b)} \right) w_y^{Xb} w_b^{\mathbb{B}} \\
&= \sum_{b\in\mathbb{B}} \sum_{f\in\mathbb{B}(a,b)} \left(\sum_{y\in Xb} \chi_1(Xb((Xf)(x),y)) w_y^{Xb} \right) w_f^{\mathbb{B}(a,b)} w_b^{\mathbb{B}} \\
&= \sum_{b\in\mathbb{B}} \sum_{f\in\mathbb{B}(a,b)} w_f^{\mathbb{B}(a,b)} w_b^{\mathbb{B}} \\
&= \sum_{b\in\mathbb{B}} \chi_1(\mathbb{B}(a,b)) w_b^{\mathbb{B}} \\
&= 1.
\end{aligned}$$

■

Proof. (Of Proposition 1.2.4)
The hypotheses together with Lemma 1.2.5 and its analogue for coweight vectors imply both a weight and coweight for EX exist and are given by the formula above. Therefore we can compute:

$$\begin{aligned}
\chi_2(EX) = \sum_{(a,x)\in \mathsf{ob}EX} w_{a,x} = \sum_{a\in\mathsf{ob}\mathbb{B}} \sum_{x\in\mathsf{ob}Xa} w_a^{\mathbb{B}} w_x^{Xa} &= \sum_{a\in\mathsf{ob}\mathbb{B}} \left(w_a^{\mathbb{B}} \sum_{x\in\mathsf{ob}Xa} w_x^{Xa} \right) \\
&= \sum_{a\in\mathsf{ob}\mathbb{B}} w_a^{\mathbb{B}} \chi_2(Xa).
\end{aligned}$$

■

1.3. **Magnitude and homology.** One way to compute the Euler characteristic of a topological space X is to first compute its (singular) homology groups $H_*(X)$ and take the alternating sum of their dimensions:

$$(\star) \qquad \chi(X) = \sum_{i=0} (-1)^i \mathsf{rank}(H_i(X)).$$

Indeed, the modern viewpoint on the topological Euler characteristic is that it is simply a decategorification of the homology groups. It turns out that there is also a homology theory for categories which decategorifies to the categorical Euler characteristic in the same way.

If $\mathbb{A}$ is a skeletal category which contains no endomorphisms except identities, then by Proposition 2.11 of [7], its Euler characteristic can be computed in terms of the non-degenerate i-simplices of its *nerve*[7] $N(\mathbb{A})_i^{ndg}$ as:

$$(\star\star) \qquad \chi(\mathbb{A}) = \sum_{i=0} (-1)^i |N(\mathbb{A})_i^{ndg}|.$$

We can associate to any simplicial set a chain complex under one direction of the Dold-Kan correspondence and can then compute the homology of the chain complex in the usual way. In this sense the nerve $N(\mathbb{A})_*$ can be thought of as providing a sequence of homology groups $NH(\mathbb{A})_*$ for the category $\mathbb{A}$. Moreover, it is a corollary of the Dold-Kan correspondence that taking the alternating sum of the non-degenerate simplices of $N(\mathbb{A})_*$ – as we have done in $(\star\star)$ – gives the same result as taking the alternating sum of the ranks of the homology groups $NH(\mathbb{A})_*$. Therefore $(\star\star)$ realizes the Euler characteristic of a category χ as the decategorification of a homology theory for categories, at least in some special circumstances. We now turn to the following question: Is the magnitude of enriched categories a decategorification of a homology theory for enriched categories? Specifically, such a *general magnitude homology theory* (GMHT) should not depend on the choice of the enriching category $\mathcal{V}$ and have a number of other properties which we will outline later in this section.

At the time of writing this dissertation, to the best of my understanding, there are no known GMHT's in the literature. Indeed, I myself have tried to produce a GMHT and have not met with much success. My own attempts began with [4]; it produces a homology theory for finite graphs which decategorifies to the magnitude of graphs (viewed as enriched categories in a way we will describe), and I was hopeful their methods may generalize to a

[7]Recall $N(\mathbb{A})_*$ is the simplicial set whose i-simplices consist in i-tuples of composable morphisms in $\mathbb{A}$. A simplex is non-degenerate if it contains no identity morphisms.

full blown GMHT. What follows is a brief account of the relevant portions of that paper, as well as a description of the difficulties in generalizing their methods. Finally, I will present a general homology theory of $\mathcal{V}$-categories which does not depend on $\mathcal{V}$ and which I believe shows great promise, but ultimately has yet to bear fruit in the search for a GMHT.

The key observation which makes a GMHT plausible is a rewrite of ($\star\star$) for enriched categories. If $\mathbb{A} \in \mathcal{V}\text{-}\mathsf{Cat}$ is finite with $|\mathbb{A}(a_i, a_i)| = 1$ for all objects $a_i \in \mathbb{A}$, and the right hand side of the following is finite, by a straightforward adaptation of the proof of Proposition 2.11 of [7] we can compute the magnitude of $\mathbb{A}$ as:

$$(\star\star\star) \qquad |\mathbb{A}| = \sum_{i=0}(-1)^i \sum_{a_0 \neq \ldots \neq a_i} |\mathbb{A}(a_0, a_1) \otimes \ldots \otimes \mathbb{A}(a_{i-1}, a_i)|.$$

It seems therefore that if there is a GMHT, it ought to decategorify to precisely the right hand side of ($\star\star\star$). A first and significant issue is that the summed elements $|\mathbb{A}(a_0, a_1) \otimes \ldots \otimes \mathbb{A}(a_{i-1}, a_i)|$ lie in the rig K, which may a priori be any rig whatsoever, while on the other hand, taking an alternating sum of ranks of homology groups will always produce an integer. This issue is tackled nicely in [4] by creating not a sequence, but a bi-graded collection of homology groups where one of the gradings tracks these K elements. Let us turn now to this homology theory for finite graphs as introduced in [4].

A finite graph[8] can be viewed as a metric space with shortest edge path metric, and as a metric space, can then be viewed as an enriched category as in [5]. The enriching category is $\mathcal{V} = \mathbb{Z}^\infty := (\mathbb{Z}^{\geq 0} \cup \{\infty\})$ with arrows corresponding to $\geq$, $+$ for monoidal product, and 0 for unit. The typical gauge function in this context takes values in a ring of rational functions in one variable, $K = \mathbb{Q}(q)$. The gauge $|-| : \mathbb{Z}^\infty \longrightarrow \mathbb{Q}(q)$ sends a non-negative integer $d \mapsto q^d$ and $\infty \mapsto 0$. The magnitude $|G|$ of a graph G is just its magnitude viewed as a $\mathbb{Z}^\infty$-category. Writing $d(x, y)$ for the hom-object (i.e. distance) between vertices x and y, we can compute this magnitude via ($\star\star\star$)[9] as:

$$|G| = \sum_{i=0}(-1)^i \sum_{a_0 \neq \ldots \neq a_i} q^{d(a_0,a_1)+\ldots+d(a_{i-1},a_i)}.$$

---

[8]By 'graph' we mean undirected graph with no loops or multiple edges.

[9]In the context of graphs, the right hand side is sensible as a formal power series as an element of $\mathbb{Q}(q)$ even if the summation is infinite. This equation therefore holds true for all graphs: See Proposition 3.9 of [9]

The bi-graded homology groups of [4] stem from a sequence of *magnitude chain complexes* $MC_{*,l}(G)$ associated to the graph G, where $l \in \mathbb{Z}^{\infty}$. They define:

$$MC_{i,l}(G) :=$$

$$\mathbb{Z}\{(a_0, a_1, ..., a_i) \in G^{i+1} \mid d(a_0, a_1) + ... + d(a_{i-1}, a_i) = l \text{ and } a_j \neq a_{j+1} \text{ for all } 0 \leq j < i\}.$$

The boundary maps $\partial : MC_{i,l}(G) \longrightarrow MC_{i-1,l}(G)$ are produced as an alternating sum of face-like maps: $\partial := \sum_{j=0}^{i}(-1)^j d_j$ where for $0 \leq j \leq i$ the function $d_j : MC_{i,l}(G) \longrightarrow MC_{i-1,l}(G)$ is given by $d_j(a_0, ..., a_j, ..., a_i) = (a_0, ..., \hat{a_j}, ..., a_i)$ if $d(a_0, a_1) + ... + d(a_{j-1}, a_{j+1}) + ... + d(a_{i-1}, a_i) = l$ and 0 otherwise. One can check that $\partial^2 = 0$, though this is a surprisingly subtle point, as we shall see.

With these definitions, one can rewrite equation (1.3) directly in terms of the magnitude chain complexes:

$$|G| = \sum_{i=0}(-1)^i \sum_{l=0} \mathsf{rank}(MC_{i,l}(G))q^l.$$

Similarly, we can pass to the homology – the *graph magnitude homology* of G – of each chain complex: $MH_{i,l}(G) := H_i(MC_{*,l})$. Once more we have:

$$(\dagger) \qquad |G| = \sum_{i=0}(-1)^i \sum_{l=0} \mathsf{rank}(MH_{i,l}(G))q^l.$$

Remark 1.3.1. *The condition that $a_j \neq a_{j+1}$ for all $0 \leq j < i$ in the definition of $MC_{i,l}(G)$ is not strictly necessary. The chain complexes formed without that condition will be chain homotopic to the complexes $MC_{*,l}(G)$ – again by an aspect of the Dold-Kan correspondence – and hence give rise to the same homology groups $MH_{*,l}(G)$.*

The graph magnitude homology has some very desirable properties beyond its role in equation (†). Firstly, it is a functorial assignment; Given a distance decreasing map of graphs $f : G \longrightarrow H$, there is an induced map $f_\# : MC_{*,l}(G) \longrightarrow MC_{*,l}(H)$ for each l which sends $(a_0, ..., a_i) \mapsto (fa_0, ..., fa_i)$ if the latter is in $MC_{i,l}(H)$ and 0 otherwise. This in turn induces a map on homology $f_\# : MH_{*,l}(G) \longrightarrow MH_{*,l}(H)$ by passing to equivalence classes.

This makes $MH_{*,l}(-)$ a functor taking graphs and distance decreasing maps to groups with group homomorphisms. With this notation in mind, we have:

Proposition 1.3.2. *(See Corollary 16, Theorem 21, and Proposition 17 of* [4]*)*

Let G and H be finite graphs. Then we have:

1. *(1) A distance decreasing map $f : G \longrightarrow H$ is an isomorphism if and only if $f_\# : MH_{*,l}(G) \longrightarrow MH_{*,l}(H)$ is an isomorphism for all l.*
2. *(2) (**Kunneth Theorem**) Let $\times$ denote the cartesian product of graphs. The following is short exact:*

$$0 \longrightarrow MH_{*,*}(G) \otimes MH_{*,*}(H) \longrightarrow MH_{*,*}(G \times H)$$
$$\longrightarrow \mathsf{Tor}(MH_{*-1,*}(G), MH_{*,*}(H)) \longrightarrow 0.$$

   *Here, the second map is induced by the **exterior product**[10].*

3. *(3) (**Additivity**) Writing $G + H$ for the disjoint union of the graphs and e_1, e_2 for the inclusions of G and H into that disjoint union, we have that the induced map below is an isomorphism.*

$$(e_1)_\# \oplus (e_2)_\# : MH_{*,*}(G) \oplus MH_{*,*}(H) \longrightarrow MH_{*,*}(G + H).$$

Remark 1.3.3. *Magnitude homology of graphs also satisfies a form of the Mayer-Vietoris sequence which allows computation of the homology of certain kinds of unions of graphs. See Section 6 of* [4].

Together with equation (†), Proposition 1.3.2 should be viewed as a 1-to-1 categorification of Proposition 1.1.3.[11] Proposition 1.3.2 can be taken as a list of requirements for a full GMHT.

A GMHT ought to be: adjoint functor invariant in the sense that if $\mathbb{A}$ and $\mathbb{B}$ are $\mathcal{V}$-categories connected by a pair of adjoint functors, these functors should induce isomorphisms

[10] See definition 20 of [4]

[11] An adjoint pair of enriched functors between graphs viewed as enriched categories is necessarily an isomorphism of graphs. This explains why item (1) of Proposition 1.3.2 seems much stronger than the corresponding items (1) and (2) of Proposition 1.1.3.

on homology; multiplicative over product in the sense of the Kunneth theorem; additive over coproduct; and of course decategorify to magnitude without depending on the choice of $\mathcal{V}$.

As we have said, no such GMHT is known. We turn now to exploring some of the difficulties in adapting the approach of [4] to a more general context. In what follows, we shall assume that the $\mathcal{V}$-categories under consideration all have the property that $|\mathbb{A}(a,a)| = 1$ for all $a \in \mathbb{A}$ and that the right hand side of equation $(\star\star\star)$ is always sensible (finite, or understandable as a formal power series).

A straightforward adaptation of magnitude homology for graphs to general $\mathcal{V}$-categories would be to define a sequence of chain complexes indexed by objects of $\mathcal{V}$. Let $\mathbb{A} \in \mathcal{V}\text{-}\mathsf{Cat}$ and define the following bigraded collection of free groups:

$$MC_{i,v}(\mathbb{A}) :=$$

$$\mathbb{Z}\{(a_0, ..., a_i) \in \mathsf{ob}\mathbb{A}^{i+1} \mid \mathbb{A}(a_0,a_1) \otimes ... \otimes \mathbb{A}(a_{i-1},a_i) \cong v \text{ and } a_j \neq a_{j+1} \text{ for all } 0 \leq j < i\}.$$

Here the grading is given by $i \in \mathbb{N}$ and $v \in \mathcal{V}$.[12]

With this definition, we can compute the magnitude $|\mathbb{A}|$ of $\mathbb{A}$ by the following rewrite of equation $(\star\star\star)$.

$$|\mathbb{A}| = \sum_{i=0} (-1)^i \sum_{v \in \mathsf{ob}\mathcal{V}} \mathsf{rank}(MC_{i,v}(\mathbb{A})).$$

We can again set $\partial := \sum_{j=0}^{i} (-1)^j d_j$ where by analogy:

$$d_j(a_0, ..., a_j, ..., a_i) := \begin{cases} (a_0, .., \hat{a_j}, ..., a_i) \quad \text{if } \begin{pmatrix} \mathbb{A}(a_0,a_1)\otimes...\otimes\mathbb{A}(a_{j-1},a_{j+1})\otimes... \\ ...\otimes\mathbb{A}(a_{i-1},a_i) \end{pmatrix} \cong v \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise} \end{cases}$$

These definitions *would* induce a GMHT for $\mathcal{V}$-categories as it does for graphs, if not for one key problem: the differential $\partial = \sum_j (-1)^j d_j$ does *not* necessarily satisfy $\partial^2 = 0$ when $\mathcal{V} \neq \mathbb{Z}^\infty$. For graphs, $\partial^2 = 0$ because the face maps d_j commute ($d_k d_j = d_{j-1} d_k$ whenever $k < j$) but this does not happen in general. Suppose $\mathbb{A}$ has four objects $0, 1, 2,$ and 3. Consider the following commutative diagram in $\mathcal{V}$ where each edge is a composition morphism:

[12] Alternatively, and perhaps more naturally, we could require that $|\mathbb{A}(a_0,a_1) \otimes ... \otimes \mathbb{A}(a_{i-1},a_i)| = v$ and take $v \in K$.

$$\begin{tikzcd} & \mathbb{A}(0,1)\otimes\mathbb{A}(1,2)\otimes\mathbb{A}(2,3) \arrow[dl] \arrow[dr] & \\ \mathbb{A}(0,2)\otimes\mathbb{A}(2,3) \arrow[dr] & & \mathbb{A}(0,1)\otimes\mathbb{A}(1,3) \arrow[dl] \\ & \mathbb{A}(0,3) & \end{tikzcd}$$

If $\mathbb{A}(0,1)\otimes\mathbb{A}(1,2)\otimes\mathbb{A}(2,3) \cong \mathbb{A}(0,2)\otimes\mathbb{A}(2,3) \cong \mathbb{A}(0,3) \cong v$, then $d_1d_1(0,1,2,3) = d_1(0,2,3) = (0,3)$. But this situation does not imply in general that three objects on the right hand side are also isomorphic. If the objects $\mathbb{A}(0,1)\otimes\mathbb{A}(1,2)\otimes\mathbb{A}(2,3)$ and $\mathbb{A}(0,1)\otimes\mathbb{A}(1,3)$ are *not* isomorphic, then $\mathbb{A}(0,1)\otimes\mathbb{A}(1,3) \ncong v$ and so $d_2(0,1,2,3) = 0$ and hence $d_1d_2(0,1,2,3) = 0 \neq (0,3)$. This does not happen in the context of graphs because if both edges on the left hand side are equalities, then so must the two edges on the right as can be easily checked.

Therefore the free groups $MC_{*,v}(\mathbb{A})$ do not in general assemble into a chain complex with ∂ as above, and therefore do not provide a homology theory. We may consider different boundary maps ∂ on these sets with the hopes of finding one which makes them into a chain complex, but no interesting examples are known. The alternative is to attempt to tweak the definition of $MC_{*,v}(\mathbb{A})$. In what follows, we drop the condition that $a_j \neq a_{j+1}$ for all $0 \leq j < i$ to simplify our definitions in accordance with Remark 1.3.1.

(1) Define:

$$MC_{i,v}(\mathbb{A}) := \mathbb{Z}\{(a_0,...,a_i) \in \mathsf{ob}\mathbb{A}^{i+1} \mid \exists\phi : v \longrightarrow \mathbb{A}(a_0,a_1)\otimes ... \otimes \mathbb{A}(a_{i-1},a_i)\}.$$

For each $v \in \mathcal{V}$, this is a chain complex with ∂ an alternating sum of $d_j : (a_0,..,a_i) \mapsto (a_0,...,\hat{a_j},...,a_i)$. But the condition that there exists a map from v is no condition at all for many enriching $\mathcal{V}$'s. For example, when $\mathcal{V} = \mathsf{FinSet}$, there exist functions between every pair of non-empty sets. In this sense $MC_{i,v}(\mathbb{A})$ is too large, or, said another way, contains too little information.

(2) Define:

$$MC_{i,v}(\mathbb{A}) := \mathbb{Z}\{(a_0,...,a_i,\phi) \in \mathsf{ob}\mathbb{A}^{i+1} \times \mathsf{mor}\mathcal{V} \mid \phi : v \longrightarrow \mathbb{A}(a_0,a_1)\otimes ... \otimes \mathbb{A}(a_{i-1},a_i).\}$$

This is a chain complex with ∂ an alternating sum of face maps:

$$d_j : (a_0, ..., a_i, \phi) \mapsto (a_0, ..., \hat{a_j}, ..., a_i, \circ_j \circ \phi).$$

Where here we write $\circ_j$ for the composition morphism with domain $\mathbb{A}(a_{j-1}, a_j) \otimes \mathbb{A}(a_j, a_{j+1})$ and codomain $\mathbb{A}(a_{j-1}, a_{j+1})$. But in this case, the alternating sum only goes from $j = 1$ to $j = i - 1$ because there is no natural choice of face maps d_0 or d_i: If we take e.g d_0 to omit a_0, there is no natural way to change ϕ into a map $v \longrightarrow \mathbb{A}(a_1, a_2) \otimes ... \otimes \mathbb{A}(a_{i-1}, a_i)$. While this definition solves the problem of the previous by reintroducing information with the inclusion of ϕ, it turns out that missing d_0 and d_i is a significant problem; for general homological algebra reasons, the associated homology theory is trivial.[13]

(3) Again define:

$$MC_{i,v}(\mathbb{A}) := \mathbb{Z}\{(a_0, ..., a_i, \phi) \in \mathsf{ob}\mathbb{A}^{i+1} \times \mathsf{mor}\mathcal{V} \mid \phi : v \longrightarrow \mathbb{A}(a_0, a_1) \otimes ... \otimes \mathbb{A}(a_{i-1}, a_i)\}.$$

But this time, define d_0 to omit a_0 only if $\mathbb{A}(a_0, a_1) \cong I$ the unit for $\otimes$ and is otherwise 0, and similarly for d_i. We can compose ϕ with the unit isomorphisms λ and ρ of the monoidal category to produce the requisite map $v \longrightarrow \mathbb{A}(a_1, a_2) \otimes ... \otimes \mathbb{A}(a_{i-1}, a_i)$. Unfortunately, these maps d_0 and d_i do not commute with the others as they should, so that ∂ as an alternating sum of the face maps once fails to satisfy $\partial^2 = 0$.

This story continues with a variety of other possible definitions, some more or less exotic and all ultimately unsatisfactory for one reason or the other. We believe, however, the most promising definition is the one we turn to define next. The inspiration for what follows is a combination of the above ideas with classical singular homology of topological spaces. We will construct homology groups for $\mathbb{A} \in \mathcal{V}\text{-}\mathsf{Cat}$ from a simplicial set whose k-simplices consist in the enriched functors from simple 'k-simplex' $\mathcal{V}$-categories to $\mathbb{A}$. Our k-simplex $\mathcal{V}$-categories can be thought of as the topological k-simplices, but with objects of $\mathcal{V}$ labelling the edges. Call a monoidal category $(\mathcal{V}, \otimes, I)$ *magnitudinal* if it is closed, has an initial object ∞, and has the property that if $v, w \in \mathcal{V}$ with $v \otimes w \cong I$, then it must be that

[13]From a simplicial point of view, underlying $MC_{i,v}(\mathbb{A})$ is an augmented simplicial set with extra degeneracies. See e.g. Section 4.5 of [13]

$v \cong I \cong w$. Every enriching category $\mathcal{V}$ mentioned thus far is magnitudinal and we assume in what follows that $\mathcal{V}$ is always magnitudinal.

Given objects $v_1, ..., v_k \in \mathcal{V}$, we can form the free $\mathcal{V}$-category on the directed graph which has $k+1$ vertices $0, ..., k$, and edges from $j-1$ to j labelled by v_j for all $1 < j \leq k$. The resulting $\mathcal{V}$-category – which we shall denote $F_k(v_1, ..., v_k)$ – can be explicitly described: It has $k+1$ objects $0, .., k$; the hom-object from $j-1$ to j is v_j ; the hom-object from j to j is I; the hom-object from j to j' with $j < j'$ is the tensor $v_j \otimes ... \otimes v_{j'-1}$; the hom-object from j' to j with $j < j'$ is ∞. Composition

$$F_k(v_1, ..., v_k)(j, j') \otimes F_k(v_1, ..., v_k)(j', j'') \longrightarrow F_k(v_1, ..., v_k)(j, j'')$$

is the identity if $j < j' < j''$; is λ if $j = j' \leq j''$; is ρ if $j \leq j' = j''$; and is the unique map from the initial object ∞ otherwise.[14] Because $F_k(v_1, ..., v_k)$ is free, to define a functor $\phi : F_k(v_1, ..., v_k) \longrightarrow \mathbb{A}$ where $\mathbb{A} \in \mathcal{V}\text{-}\mathsf{Cat}$, it suffices to choose objects $\phi(0), ..., \phi(k) \in \mathbb{A}$ and maps $\phi_{j-1,j} : v_j \longrightarrow \mathbb{A}(\phi(j-1), \phi(j))$ for all $0 < j \leq k$. As a result, we can consider F_k as a functor $\mathcal{V}^k \longrightarrow \mathcal{V}\text{-}\mathsf{Cat}$.

For a fixed $v \in \mathcal{V}$, we define the category $\mathcal{V}^k_v$ to be the full subcategory of $\mathcal{V}^k$ consisting of those tuples $(v_1, ..., v_k)$ such that $v_1 \otimes ... \otimes v_k \cong v$. The functor F_k restricts to a functor $F_k : \mathcal{V}^k_v \longrightarrow \mathcal{V}\text{-}\mathsf{Cat}$.

Definition 1.3.4. Define the *lax-coslice category* $\mathsf{Cat}//(\mathcal{V}\text{-}\mathsf{Cat})$ to have pairs $(\mathbb{A} \in \mathsf{Cat}, X : \mathbb{A} \longrightarrow \mathcal{V}\text{-}\mathsf{Cat})$ for objects. A morphism $(\mathbb{A}, X) \longrightarrow (\mathbb{B}, Y)$ consists in a functor $\varepsilon_* : \mathbb{B} \longrightarrow \mathbb{A}$ and a natural transformation $\varepsilon^* : X \circ \varepsilon_* \Rightarrow Y$:

$$\begin{tikzcd}
(\mathbb{A}, X) \arrow[d, "{(\varepsilon_*, \varepsilon^*)}"] & \mathbb{B} \arrow[rr, "\varepsilon_*"] \arrow[dr, "Y"'] & & \mathbb{A} \arrow[dl, "X"] \arrow[dl, phantom, shift right=4, "\Leftarrow \varepsilon^*"] \\
(\mathbb{B}, Y) & & \mathcal{V}\text{-}\mathsf{Cat} &
\end{tikzcd}$$

Composition is given in the expected way as: $(\mu_*, \mu^*) \circ (\varepsilon_*, \varepsilon^*) = (\mu_* \circ \varepsilon_*, \varepsilon^* \bullet (\mu^* \circ 1_{\varepsilon_*}))$ where here we write $\bullet$ for vertical composition of natural transformations.

[14]It is a consequence of $\mathcal{V}$ being closed that $v \otimes \infty \cong \infty \cong \infty \otimes v$ for all $v \in \mathcal{V}$. Thus if either $j' < j$ or $j'' < j'$, the source of the composition map is isomorphic to ∞.

Writing Δ for the simplex category, we can define a cosimplicial object

$$F(-,v) : \Delta \longrightarrow \mathsf{Cat}//(\mathcal{V}\text{-}\mathsf{Cat})$$

for every object $v \in \mathcal{V}$ as follows. For an object $[k] \in \Delta$, define

$$F(k,v) := (\mathcal{V}_v^k, F_k : \mathcal{V}_v^k \longrightarrow \mathcal{V}\text{-}\mathsf{Cat}).$$

Writing d^i and s^i for the coface and codgeneracy maps in Δ, we will define the coface maps $F(d^i,v) = (\delta_i, \delta^i) : F(k-1,v) \longrightarrow F(k,v)$ and codegeneracy maps $F(s^i,v) = (\sigma_i, \sigma^i) : F(k+1,v) \longrightarrow F(k,v)$ for $0 \leq i \leq k$ as follows:

$$\delta_i(v_1,...,v_k) := \begin{cases} (v_1,...,v_i \otimes v_{i+1},..,v_k) & \text{if } 1 \leq i \leq k-1 \\ (v_2,...,v_k) & \text{if } i = 0 \text{ and } v_1 \cong I \\ (v_1,...,v_{k-1}) & \text{if } i = k \text{ and } v_k \cong I \end{cases}$$

And define the transformation...

$$\delta^i : F_{k-1} \circ \delta_i \Rightarrow F_k$$

$$\delta^i_{v_1,...,v_k} : F_{k-1}(\delta_i(v_1,...,v_k)) \longrightarrow F_k(v_1,...,v_k)$$

... to be the map defined by sending each object j to $d^i(j)$, and the identity on each mapping object. As for the codegeneracy maps, define:

$$\sigma_i(v_1,...,v_k) := (v_1,...,v_i, I, v_{i+1},...,v_k)$$

And define the transformation...

$$\sigma^i : F_{k+1} \circ \sigma_i \Rightarrow F_k$$

$$\sigma^i_{v_1,...,v_k} : F_{k+1}(\sigma_i(v_1,...,v_k)) \longrightarrow F_k(v_1,...,v_k)$$

... to be the map defined by sending each object j to $s^i(j)$, and again the identity on each mapping object, recalling that $F_k(v_1,...,v_k)(i,i) = I$.

It is straightforward to check that coface maps commute with coface maps and codegeneracy maps commute with codegeneracy maps according to the requirements for a cosimplicial object. We do, however, make use of the property $v \otimes w \cong I$ implies $v \cong I \cong w$ in verifying that outer face maps commute, e.g. that $(\delta_0, \delta^0) \circ (\delta_0, \delta^0) = (\delta_1, \delta^1) \circ (\delta_0, \delta^0)$. There is a slight wrinkle here that $(\sigma_i, \sigma^i) \circ (\delta_i, \delta^i)$ is not the identity on the nose, but instead is an isomorphism making use of ρ. As every monoidal category $\mathcal{V}$ is monoidally equivalent to a strict-monoidal category – one in which in particular ρ and λ are identities – we do not take this to be problematic. We are now ready to define the simplicial set which in turn is used to define the homology theory.

Definition 1.3.5. Let $\mathbb{A} \in \mathcal{V}\text{-}\mathsf{Cat}$ and identify $\mathbb{A}$ with the functor from the terminal category $\mathbb{A} : 1 \longrightarrow \mathcal{V}\text{-}\mathsf{Cat}$ sending the lone object to $\mathbb{A}$. Then for $v \in \mathcal{V}$ define the *singular magnitude simplicial set* $MS_{*,v}(\mathbb{A}) : \Delta^{op} \longrightarrow \mathsf{Set}$ by $MS_{*,v}(\mathbb{A}) = (\mathsf{Cat}//\mathcal{V}\text{-}\mathsf{Cat})(F(*, v), (1, \mathbb{A}))$. An element of $MS_{k,v}(\mathbb{A})$ is thus a pair which we in this circumstance will write (w_*, ϕ^*). Here $w_* : 1 \longrightarrow \mathcal{V}^k$ can be identified with a k-tuple $(w_1, ..., w_k)$, and $\phi^* : F_k \circ w_* \Rightarrow \mathbb{A}$ can be identified with an enriched functor $\phi : F_k(w_1, ..., w_k) \longrightarrow \mathbb{A}$. Therefore, we have:

$$MS_{k,v}(\mathbb{A}) := (\mathsf{Cat}//\mathcal{V}\text{-}\mathsf{Cat})(F(k, v), (1, \mathbb{A})) = \bigcup_{(v_1,...,v_k)\in\mathcal{V}_v^k} \mathcal{V}\text{-}\mathsf{Fun}(F_k(v_1, ..., v_k), \mathbb{A}).$$

The face and degeneracy maps d_i and s_i are given by precomposition with the coface and codegeneracy maps (δ_i, δ^i) and (σ_i, σ^i) respectively. In terms of the identifications above, this amounts to the following:

$$d_i(\phi : F_k(w_1, ..., w_k) \longrightarrow \mathbb{A}) = \phi \circ \delta^i_{w_1,...,w_k} : F_{k-1}(\delta_i(w_1, ..., w_k)) \longrightarrow F_k(w_1, ..., w_k) \longrightarrow \mathbb{A}$$

$$s_i(\phi : F_k(w_1, ..., w_k) \longrightarrow \mathbb{A}) = \phi \circ \sigma^i_{w_1,...,w_k} : F_{k+1}(\sigma_i(w_1, ..., w_k)) \longrightarrow F_k(w_1, ..., w_k) \longrightarrow \mathbb{A}$$

The corresponding homology theory – the *singular magnitude homology* – is denoted: $MSH_{*,v}(\mathbb{A})$.

The singular magnitude simplicial set construction is also functorial,

$$MS_{*,v}(-) : \mathcal{V}\text{-}\mathsf{Cat} \longrightarrow \mathsf{sSet}.$$

A $\mathcal{V}$-functor $F : \mathbb{A} \longrightarrow \mathbb{B}$ induces a map of simplicial sets $F_\# : MS_{*,v}(\mathbb{A}) \longrightarrow MS_{*,v}(\mathbb{B})$ by sending a k-simplex ϕ to $F_\#(\phi) := F \circ \phi$. This map in turn induces a map $F_\# : MSH_{*,v}(\mathbb{A}) \longrightarrow MSH_{*,v}(\mathbb{B})$ by passing to equivalence classes.

There are many promising aspects of singular magnitude homology. The first is that it can be defined for any $\mathcal{V}$-category whatsoever for any magnitudinal $(\mathcal{V}, \otimes, I)$. The second is that, despite this generality, it contains the pertinent information necessary to categorify magnitude in the spirit of [4]; we have indexed the homology groups by objects $v \in \mathcal{V}$, and so will be able to convert the integer valued ranks of the homology groups into values of K by computing $\mathsf{rank}(MSH_{k,v}(\mathbb{A}))|v|$. This was a necessary step in reconstructing the K-valued magnitude of graphs in [4]. Furthermore, the analogy with singular homology of topological spaces gives good reason to think that there may be a proof of a Kunneth theorem based in the topological technique of acyclic models. Finally, the homology theory is invariant under adjoint functors. We will prove this to conclude this chapter of the dissertation.

Unfortunately, singular magnitude homology is not a GMHT, at least in its present form. The largest problem is that there is no known way of decategorifying singular magnitude homology to produce the magnitude of enriched categories in every case.[15] Secondly, while additivity of $MSH_{*,v}(-)$ is easy to prove in the case when the initial object ∞ of $\mathcal{V}$ is not the target of any maps in $\mathcal{V}$ – as when $\mathcal{V} = \mathsf{FinSet}$ or $\mathcal{V} = \mathbb{Z}^\infty$ – we have no proof one way or the other about additivity of $MSH_{*,v}(-)$ when ∞ is the target of maps in $\mathcal{V}$, like when $\mathcal{V} = \mathsf{FinVect}$. Using $\mathsf{FinVect}$ as the enriching category brings a second and seemingly more extreme problem as well; for $\mathbb{A} \in \mathsf{FinVect\text{-}Cat}$, the sets $MS_{k,v}(\mathbb{A})$ are almost always uncountably infinite. Indeed, even the homology groups $MSH_{k,v}(\mathbb{A})$ turn out to be of infinite rank in many examples. Nevertheless, our problems are mostly problems of over-abundance and not deficiency. Hopefully there are ways to cut out excess information to witness this homology theory or a variant of it as a GMHT.

Proposition 1.3.6. *Let $(\mathcal{V}, \otimes, I)$ be a magnitudinal monoidal category, and $F, G : \mathbb{A} \longrightarrow \mathbb{B}$ a pair of $\mathcal{V}$-functors between $\mathbb{A}, \mathbb{B} \in \mathcal{V}$-$\mathsf{Cat}$. Suppose that $H : F \Rightarrow G$ is a $\mathcal{V}$-natural transformation. Then H induces a simplicial homotopy $H_\# : F_\# \simeq G_\# : MS_{*,v}(\mathbb{A}) \longrightarrow MS_{*,v}(\mathbb{B})$ for every $v \in \mathcal{V}$.*

Proof. A simplicial homotopy $H_\# : F_\# \simeq G_\#$ consists in maps $h_i : MS_{k,v}(\mathbb{A}) \longrightarrow MS_{k+1,v}(\mathbb{B})$ for $0 \le i \le k$ such that $d_0 h_0 = G_\#$, $d_{k+1} h_k = F_\#$, and:

$$d_i h_j = \begin{cases} h_{j-1} d_i & \text{if } i < j \\ h_j d_{i-1} & \text{if } i > j \\ d_i h_{i-1} & \text{if } i = j \neq 0 \end{cases} \qquad\qquad s_i h_j = \begin{cases} h_{j+1} s_i & \text{if } i \le j \\ h_j s_{i-1} & \text{if } i > j \end{cases}$$

[15] There are, however, ad-hoc ways of doing so depending on the enriching category $\mathcal{V}$. For example, when $\mathcal{V} = \mathsf{FinSet}$ and writing I for the one point set, the simplicial set $MS_{*,I}(\mathbb{A})$ is precisely the nerve $N(\mathbb{A})$ and hence on its own decategorifies to magnitude.

Let $\phi : F_k(v_1, ..., v_k) \longrightarrow \mathbb{A}$ be an element of $MS_{k,v}(\mathbb{A})$. We will define $h_i(\phi)$ in $MS_{k+1,v}(\mathbb{B})$ as follows. The domain of $h_i(\phi)$ will be $X := F_{k+1}(v_1, ..., v_i, I, v_{i+1}, ..., v_k)$. Thus $v_j = X(j-1, j)$ for all $0 < j \leq i$, $I = X(i, i+1)$, and $v_{j-1} = X(j-1, j)$ for all $i+1 < j \leq k+1$. To define $h_i(\phi) : X \longrightarrow \mathbb{B}$ it suffices to define objects $b_j := h_i(\phi)(j)$ for all $0 \leq j \leq k+1$ and maps $h_i(\phi)_{j-1,j} : X(j-1, j) \longrightarrow \mathbb{B}(b_{j-1}, b_j)$ for all $0 < j \leq k+1$. We define:

$$b_j := \begin{cases} F\phi(j) & \text{if } 0 \leq j \leq i \\ G\phi(j-1) & \text{if } i+1 \leq j \leq k+1 \end{cases}$$

$$h_i(\phi)_{j-1,j} := \begin{cases} F_{\phi(j-1),\phi(j)}\phi_{j-1,j} : v_j \longrightarrow \mathbb{A}(\phi(j-1), \phi(j)) \longrightarrow \mathbb{B}(F\phi(j-1), F\phi(j)) & \\ & \text{if } 0 < j \leq i \\ H_{\phi(i)} : I \longrightarrow \mathbb{B}(F\phi(i), G\phi(i)) & \\ & \text{if } j = i+1 \\ G_{\phi(j-2),\phi(j-1)}\phi_{j-2,j-1} : v_{j-1} \longrightarrow \mathbb{A}(\phi(j-2), \phi(j-1)) & \\ \qquad \longrightarrow \mathbb{B}(G\phi(j-2), G\phi(j-1)) & \\ & \text{if } i < j-1 \leq k \end{cases}$$

Note that as $h_0(\phi)$ has source $F_{k+1}(I, v_1, ..., v_k)$, $d_0 h_0(\phi)$ is simply $h_0(\phi)$ restricted to the objects $1 \leq j \leq k+1$ and is thus precisely $G_\#(\phi)$. Similarly, we see that $d_{k+1} h_k(\phi) = F_\#(\phi)$. Each of these identities can be checked in a straightforward way by definition, though once again some of these equations hold only on the nose if $\mathcal{V}$ were a strict monoidal category. We will show only the sense in which $d_i h_i = d_i h_{i-1}$ here.

Given $\phi : F_k(v_1, ..., v_k) \longrightarrow \mathbb{A}$, $h_i(\phi)$ has domain $F_{k+1}(v_1, ..., v_{i-1}, v_i, I, v_{i+1}, ..., v_k)$ while $h_{i-1}(\phi)$ has domain $F_{k+1}(v_1, ..., v_{i-1}, I, v_i, v_{i+1}, ..., v_k)$. The i^{th} face $d_i h_i(\phi)$ therefore has domain $F_k(v_1, .., v_{i-1}, v_i \otimes I, v_{i+1}, ..., v_k)$, which is not precisely the same as the domain of $d_i h_{i-1}(\phi)$ which is $F_k(v_1, ..., v_{i-1}, I \otimes v_i, v_{i+1}, ..., v_k)$, though the two domains are naturally isomorphic. Nevertheless, the objects j with $0 \leq j \leq i-1$ are mapped to $F(\phi(j))$ and those with $i \leq j \leq k$ are mapped to $G(\phi(j))$ in both cases. Similarly, for j with $0 < j \leq i-1$, the action on mapping objects consists in $F_{\phi(j-1),\phi(j)}\phi_{j-1,j}$ while for j with $i < j \leq k$, the action on mapping objects is $G_{\phi(j),\phi(j-1)}\phi_{j-1,j}$ in both cases. The only possible difference between the enriched functors $d_i h_i(\phi)$ and $d_i h_{i-1}(\phi)$ then is their action on the mapping objects between the object $i-1$ and i. These two maps are given as composites...

$$d_i h_i(\phi)_{i-1,i} : v_i \otimes I \longrightarrow \mathbb{B}(F\phi(i-1), F\phi(i)) \otimes \mathbb{B}(F\phi(i), G\phi(i)) \longrightarrow \mathbb{B}(F\phi(i-1)), G\phi(i))).$$

$$d_i h_{i-1}(\phi)_{i-1,i} : I \otimes v_i \longrightarrow \mathbb{B}(F\phi(i-1), G\phi(i-1)) \otimes \mathbb{B}(G\phi(i-1), G\phi(i)) \longrightarrow \mathbb{B}(F\phi(i-1), G\phi(i)).$$

... where the latter maps are simply given by composition in the enriched category $\mathbb{B}$, and the former maps are given by $F \otimes H$ and $H \otimes G$ respectively. As H is an enriched natural transformation, these two maps would be precisely equal if $\mathcal{V}$ were strict monoidal.

■

Corollary 1.3.7. *If $F : \mathbb{A} \longrightarrow \mathbb{B}$ and $G : \mathbb{B} \longrightarrow \mathbb{A}$ are a pair of adjoint $\mathcal{V}$-functors, then F and G induce isomorphisms between the homology $MSH_{*,v}(\mathbb{A})$ and $MSH_{*,v}(\mathbb{B})$.*

Proof. This amounts to a standard bit of homological algebra. F and G being adjoint mean we have $\mathcal{V}$-natural transformations $1 \Rightarrow GF$ and $FG \Rightarrow 1$ (if F is left adjoint, for example) and so $1_\# \simeq (GF)_\# = G_\# F_\#$ and $F_\# G_\# = (FG)_\# \simeq 1_\#$ as simplicial maps. Homotopies of simplicial maps induce the same maps on homology, hence $1 = G_\# F_\#$ and $F_\# G_\# = 1$.

■

2. The classification of monoidal-type categories

The presence of natural transformations in the 2-category of categories Cat gives a rich vocabulary for describing the ways in which a "tensor product" functor $\otimes : A \times A \longrightarrow A$ for a category A may be associative and unital. We have firstly the *strict* monoidal categories, in which $\otimes$ is *strictly* associative and unital; weakening the definition slightly gives monoidal categories in the sense of [11], in which $\otimes$ is associative and unital only up to coherent natural isomorphism; and one possible further weakening yields *skew* monoidal categories – introduced by Szlachányi in [16] – in which $\otimes$ is associative and unital up to coherent, but not necessarily invertible, natural transformation.

In the latter two cases, it is worth saying something about our use of the word 'coherent'. For a monoidal category, the coherence of the associativity and unitality natural isomorphisms means that they render commutative the so-called pentagon and triangle diagrams.

According to a well known coherence theorem of Maclane's [12], the commutativity of these diagrams implies the commutativity of any well-formed diagram built out of these natural isomorphisms. For skew monoidal categories, the coherence conditions require the commutativity of not two but five diagrams; and by contrast to the monoidal case, the commuativity of these diagrams does not imply the commutativity of all diagrams built from the natural transformations. So where do these five diagrams come from? Why are they the 'right' notion of coherence for skew monoidal categories?

One possible answer to this question is given in *The Catalan simplicial set* [2] ; a key insight is that the theory of *simplicial sets* provides a uniform framework for describing the data and coherence of monoidal-type categories. For example, the data and coherence of a monoidal category consists in: a category, a pair of functors, three natural isomorphisms, and two commutative diagrams thereof. As we go down this list we see that functors mediate between categories, natural transformations mediate between functors, and that commutative diagrams mediate between transformations. Such structure suggests the simplices and face maps of a simplicial set, and indeed this can be formalized in terms of the *pseudo nerve* $N_p(\mathsf{Cat})$ of the monoidal 2-category Cat.

Explicitly, $N_p(\mathsf{Cat})$ is a simplicial set with a single 0-simplex and (small) categories for 1-simplices. The 2-simplices are binary functors $T : A \times B \longrightarrow C$ where A, B, and C are the functor's three faces. The 3-simplices are natural isomorphisms filling in squares of four such functors (the four faces) and higher simplices are commutative diagrams of such natural transformations. All the data and coherence of a monoidal category live within $N_p(\mathsf{Cat})$: Specifically, a 1-simplex, a pair of 2-simplices, three 3-simplices, and a pair of 4-simplices, themselves suitably related by face maps. By changing the strictness of the 3-simplices, we obtain nerves $N_\otimes(\mathsf{Cat})$ and $N_s(\mathsf{Cat})$ suited to the description of strict and skew monoidal categories respectively. For example, the five coherence diagrams in a skew monoidal category can be understood as five 4-simplices in $N_s(\mathsf{Cat})$.

We may wonder whether strict, skew, or plain monoidal categories are shadows of a fixed simplicial set, just as paths in a topological space are shadows of the interval. It is shown in [2] that this is indeed the case. It defines the *Catalan simplicial set* $\mathbb{C}$, recalled in Section 2.1 below, and shows that simplicial maps from $\mathbb{C}$ to $N_p(\mathsf{Cat})$, respectively $N_\otimes(\mathsf{Cat})$ and $N_s(\mathsf{Cat})$, are in one to one correspondence with monoidal, respectively strict and skew monoidal, categories. Now the structure of $\mathbb{C}$ itself provides an *a priori* justification for the five coherence diagrams for skew monoidal categories: they are just what is required for a map $\mathbb{C} \longrightarrow N_s(\mathsf{Cat})$.

Many other kinds of 'monoidal object' contained in a (higher-)categorical structure can be classified by maps out of $\mathbb{C}$ into suitably chosen nerves. As shown in [2] and [1], we may classify each of the following structures in this way:

(1) Monoids in a fixed monoidal category (including the case of strict monoidal categories – which are monoids in Cat).
(2) Monoidal categories.
(3) Skew monoidal categories.
(4) Monads in a bicategory.
(5) Monoidales in a monoidal bicategory.
(6) Skew monoidales in a monoidal bicategory.
(7) Monoidal bicategories.
(8) Skew monoidal bicategories.[16]

As with skew monoidal categories, these classifying results simultaneously justify otherwise complex and ad-hoc definitions. It also suggests that new kinds of monoidal object may be defined directly in terms of maps from $\mathbb{C}$ into any reasonably defined nerve.

The story of this paper begins with a missing item from the above list: is it possible to exhibit monoidal $(\infty, 1)$-categories in the sense of [10] as maps out of $\mathbb{C}$? The idea is as follows. There is a quasi-category qCat whose 0-cells are small quasi-categories, whose 1-cells are quasi-functors, and whose higher cells are suitable quasi-invertible transformations. If we set aside problems of strict associativity, qCat becomes a simplicial monoid under binary product of quasi-categories, and so can be seen as a one-object simplicially enriched category. By taking its homotopy coherent nerve – originally introduced in [3] – we obtain a simplicial set $N_\Delta(\mathsf{qCat})$ that should contain all the data required to define a monoidal $(\infty, 1)$-category, so that these should be definable in terms of maps $\mathbb{C} \longrightarrow N_\Delta(\mathsf{qCat})$. There

[16] The result about skew monoidal bicategories holds only after a mild restriction is placed on the maps out of $\mathbb{C}$; see [1] Section 5.2.

is, however, an analogous construction in ordinary category theory which deserves attention first, and which is the main subject of our present investigation.

Like qCat, the 2-category Cat of categories, functors, and natural transformations can be viewed under the nerve construction as a simplicial set. If we once again ignore problems of strict associativity, this simplicial set is a simplicial monoid under product of categories, and hence as a one-object simplicially enriched category, which we will refer to as $N_2(\mathsf{Cat})$. Applying the homotopy coherent nerve yields a simplical set $N_\Delta(N_2\mathsf{Cat})$, and we can now consider maps $\mathbb{C} \longrightarrow N_\Delta(N_2\mathsf{Cat})$. Far from being a simple warm up for the higher categorical case, these maps hold great interest in their own right: in some sense, they unify an extensive array of monoidal-type categories.

As strict monoidal categories include into monoidal categories, and monoidal categories include into skew monoidal categories [16], so the three corresponding nerves admit parallel inclusions: $N_\otimes(\mathsf{Cat}) \subseteq N_p(\mathsf{Cat}) \subseteq N_s(\mathsf{Cat})$. Thus each strict monoidal, monoidal, or skew monoidal category can be understood as an element of the set $\mathsf{sSet}(\mathbb{C}, N_s(\mathsf{Cat}))$. On the contrary, not *every* sort of monoidal-type category can be understood in this way; there is another common variant of monoidal structure which evades classification by $N_s(\mathsf{Cat})$ and $\mathbb{C}$. The definition of a *lax* monoidal structure on a category A approaches the idea of weakened associativity by introducing n-ary operations $\otimes^n : A^n \longrightarrow A$ for each $n \geq 0$, which we think of as "parenthesis free" n-ary multiplications. As the associativity of a binary operation is given by relationships between higher arity composites of itself, the functors $\otimes^n$ are used to mediate between possible such composites. As before, strict monoidal implies monoidal implies lax monoidal, but there are lax monoidal categories which are not skew and vice versa. Consequently, there are lax monoidal categories which cannot be understood as maps $\mathbb{C} \longrightarrow N_s(\mathsf{Cat})$. This is where $N_\Delta(N_2\mathsf{Cat})$ comes into play.

In Section 2.3 of this paper, we assign a simplicial map $\mathbb{C} \longrightarrow N_\Delta(N_2\mathsf{Cat})$ to each lax monoidal category in such a way that the original lax monoidal category can be completely reconstructed from the assignment. This gives an injective map taking lax monoidal categories into $\mathsf{sSet}(\mathbb{C}, N_\Delta(N_2\mathsf{Cat}))$. Though we do not explicitly characterize the image of our assignment, it shows that $\mathsf{sSet}(\mathbb{C}, N_\Delta(N_2\mathsf{Cat}))$ can classify lax monoidal categories in the spirit of the results of [2] and [1]. In Section 2.4 we show that the nerve $N_s(\mathsf{Cat})$ classifying skew monoidal categories also includes into $N_\Delta(N_2\mathsf{Cat})$. This shows that each kind of monoidal category mentioned thus far can be identified with a map $\mathbb{C} \longrightarrow N_\Delta(N_2\mathsf{Cat})$. In Section 2.5 we examine other kinds of monoidal categories arising as maps in $\mathsf{sSet}(\mathbb{C}, N_\Delta(N_2\mathsf{Cat}))$ and among them find Σ-*monoidal* categories as in [6]. In this sense, the set $\mathsf{sSet}(\mathbb{C}, N_\Delta(N_2\mathsf{Cat}))$ is rich enough to classify an extensive spectrum of monoidal-type structures in category theory. Furthermore, in Sections 2.2 and 2.5 we see that there are maps $\mathbb{C} \longrightarrow N_\Delta(N_2\mathsf{Cat})$ corresponding to yet a weaker type of monoidal-type category not yet defined in the literature.

Future work will include an examination of maps from $\mathbb{C}$ into the homotopy coherent nerve of various higher and enriched categories in the hopes of capturing a full range of possible monoidal objects in many more contexts. This of course includes the original motivation where the category in question is that of quasi-categories.

Acknowledgements: I would like to thank Richard Garner for drawing my attention to this topic and for his illuminating comments, suggestions, and advice throughout the investigation.

2.1. **Definitions and notation: $\mathbb{C}$ and $N_\Delta(N_2\mathsf{Cat})$.** In this section we recall important definitions and introduce some helpful notations and shorthands. We write I for the terminal category, and Δ for the *simplicial category* with finite ordered sets $[n] := \{0 < 1 < ... < n\}$ for objects, and order preserving maps. These morphisms are generated by the *coface* maps $\delta_i : [n-1] \longrightarrow [n]$ and *codegeneracy* maps $\sigma_i : [n+1] \longrightarrow [n]$ given by:

$$\delta_i(p) := \begin{cases} p & \text{if } p < i \\ p+1 & \text{if } p \geq i \end{cases} \qquad \sigma_i(p) := \begin{cases} p & \text{if } p \leq i \\ p-1 & \text{if } p > i \end{cases}$$

We write sSet for the functor category $\mathsf{Fun}(\Delta^{op}, \mathsf{Set})$ and refer to its objects as *simplicial sets*. A simplicial set $X : \Delta^{op} \longrightarrow \mathsf{Set}$ thus gives rise to an indexed sequence of sets $X_n := X[n]$, *face* maps $d_i = X\delta_i$, and *degeneracy* maps $s_i = X\sigma_i$. As X is contravariant, the face and degeneracy maps have now reversed direction: $d_i : X_n \longrightarrow X_{n-1}$ and $s_i : X_n \longrightarrow X_{n+1}$. By slight abuse we may write 'face' ('coface', 'degeneracy', 'codegeneracy') map to refer to a composite of face (coface, degeneracy, codegeneracy) maps. This should always be made clear by the context. At times we may simply use d or s to refer to a non-specific face or degeneracy map, and will always take δ and σ to be the corresponding coface and codegeneracy map.

An element $x \in X_n$ is an *n-simplex*, and is referred to as *degenerate* whenever x is in the image of a degeneracy map. When $m \leq n$, $x \in X_n$ *is the degeneracy of* $y \in X_m$ when x is the image of y under an $n-m$ fold composition of degeneracy maps $X_m \longrightarrow X_n$. We refer to the $(n-1)$-simplex $d_i(x)$ as the i^{th}*face* of x. More generally, we can identify arbitrary subsets $\mathsf{C} = \{c_0 < c_1 < ... < c_m\} \subseteq [n]$ with the morphism $\delta_{\mathsf{C}} : [m] \longrightarrow [n]$ of Δ which sends i to c_i. For $x \in X_n$, denote $x_{\mathsf{C}} := X(\delta_{\mathsf{C}})(x) = d_{\mathsf{C}}x \in X_m$. We think of and refer to this m-simplex as the C^{th}*-face* of x.[17] For two element subsets $\{p < q\} \subseteq [n]$, we drop the brackets and write simply $x_{p,q}$ or x_{pq} for the $\{p,q\}^{th}$ face of x. In this notation, the *spine* of the simplex x is the collection of successive 1-faces $\mathrm{sp}(x) := \{x_{01}, x_{12}, ..., x_{n-1,n}\}$. It will be convenient to introduce the set $\mathsf{C}^- := \mathsf{C} - \{\max \mathsf{C}\}$ and successor function $s : \mathsf{C}^- \longrightarrow \mathsf{C}$ taking c to the next greatest element of C. In this notation, $\mathrm{sp}(x_{\mathsf{C}}) = \{x_{c,sc} \mid c \in \mathsf{C}^-\}$.

For an n-simplex $x \in X_n$, the *boundary* of x consists in the collection of its faces $\{d_0(x), d_1(x), ..., d_n(x)\}$. Commutativity relations among coface maps δ_i give rise to relations amongst the faces of the boundary : $d_j(d_i(x)) = d_j(d_{i+1}(x))$ for all $0 \leq i \leq j < n$. An *n-boundary* in X is a collection of $(n-1)$-simplices $\{x_0, ..., x_n\}$ satisfying the same relationship: $d_j(x_i) = d_j(x_{i+1})$ for all $0 \leq i \leq j < n$. An n-boundary may in general be the boundary of one, many, or no n-simplices. A simplicial set is called *r-coskeletal* when for each $n > r$ and each n-boundary $\{x_0, ..., x_n\}$, there is a unique n-simplex x with $d_i(x) = x_i$ for $0 \leq i \leq n$. This establishes a bijection between n-boundaries, which are collections of $(n-1)$-simplices, and n-simplices. Thus a definition of an r-coskeletal simplicial set need only specify the simplices up to dimension r, as all higher dimensional simplices are then determined by these bijections. Similarly, a map from a simplicial set Y into an r-coskeletal simplicial set X is given by a map from the r-th truncation of Y to X. In otherwords, such a map is determined by where it sends simplices of dimension $\leq r$.

Definition 2.1.1. The *Catalan simplicial set* $\mathbb{C}$ is the simplicial set defined by the following data:

- A unique 0-simplex: $*$.

[17] With this notation, we might equally well write the face $d_i(x)$ as $x_{[n]-i}$.

- Two 1-simplices: $0 = s_0(*)$, and 1.
- Five 2-simplices:

$$\begin{tikzcd}[column sep=small] & * \arrow[dr, "0"] & \\ * \arrow[ur, "0"] \arrow[rr, "0"'] \arrow[rr, phantom, shift left=3, "s_0(0)"] & & * \end{tikzcd}, \quad \begin{tikzcd}[column sep=small] & * \arrow[dr, "1"] & \\ * \arrow[ur, "0"] \arrow[rr, "1"'] \arrow[rr, phantom, shift left=3, "s_0(1)"] & & * \end{tikzcd}, \quad \begin{tikzcd}[column sep=small] & * \arrow[dr, "0"] & \\ * \arrow[ur, "1"] \arrow[rr, "1"'] \arrow[rr, phantom, shift left=3, "s_1(1)"] & & * \end{tikzcd}, \quad \begin{tikzcd}[column sep=small] & * \arrow[dr, "0"] & \\ * \arrow[ur, "0"] \arrow[rr, "1"'] \arrow[rr, phantom, shift left=3, "u"] & & * \end{tikzcd}, \quad \begin{tikzcd}[column sep=small] & * \arrow[dr, "1"] & \\ * \arrow[ur, "1"] \arrow[rr, "1"'] \arrow[rr, phantom, shift left=3, "m"] & & * \end{tikzcd}$$

  Which we may also write as:

$$s_0(0) : 0 \vee 0 \longrightarrow 0\ ,\ s_0(1) : 0 \vee 1 \longrightarrow 1\ ,\ s_1(1) : 1 \vee 0 \longrightarrow 1\ ,\ u : 0 \vee 0 \longrightarrow 1\ ,\ m : 1 \vee 1 \longrightarrow 1$$

- Higher-dimensional simplices are determined by 2-coskeletality.

Importantly for the proofs to follow, the k-simplices of $\mathbb{C}$ for $k \geq 3$ are determined by their 2-simplex faces by coskeletality, and the 2-simplices are determined by their 1-simplex faces by definition. As a result, all simplices of $\mathbb{C}$ are determined by their collection of 1-simplex faces. Thus we may identify x with the set $\{x_{p,q} \mid 0 \leq p < q \leq n\}$ and x_{C} with $\{x_{c,c'} \mid c < c' \in \mathsf{C}\}$.

The intuition behind the remarkable classifying properties of $\mathbb{C}$ is revealed by thinking of 2-simplices in $\mathbb{C}$ as maps and higher simplices as diagrams of such maps. If we think of the non-degenerate 2-simplex $m : 1 \vee 1 \longrightarrow 1$ as a 'monoidal product' of some kind, then the 3-simplex filling the boundary $\{m, m, m, m\}$ can be thought of as encoding some kind of associativity of m.

$$\begin{tikzcd} 1 \vee 1 \vee 1 \arrow[r, "1 \vee m"] \arrow[d, "m \vee 1"'] & 1 \vee 1 \arrow[d, "m"] \\ 1 \vee 1 \arrow[r, "m"'] & 1 \end{tikzcd}$$

These ideas are made precise in a variety of contexts in [2]. Among their many results are those concerned with the skew monoidal categories of the introduction.

Definition 2.1.2. A *skew monoidal category* $(A, \otimes, \otimes^0, \alpha, \lambda, \rho)$ consists in:

- A category A.
- A functor $\otimes : A \times A \longrightarrow A$.

- A functor $\otimes^0 : I \longrightarrow A$. (i.e an object of A)
- A natural tranformation:

$$\alpha : \otimes \circ (\otimes \times 1_A) \Rightarrow \otimes \circ (1_A \times \otimes)$$

- Natural transformations:

$$\lambda : \otimes \circ (\otimes^0 \times 1_A) \Rightarrow 1_A \quad , \quad \rho : 1_A \Rightarrow \otimes \circ (1_A \times \otimes^0)$$

These natural transformations must additionally commute in five diagrams.

Associativity:

The following pentagon commutes where each edge involves a single application of α:

$$\begin{tikzcd}
 & \otimes \circ (\otimes \times 1) \circ (\otimes \times 1 \times 1) \arrow[dl] \arrow[ddr] & \\
\otimes \circ (\otimes \times 1) \circ (1 \times \otimes \times 1) \arrow[dd] & & \\
 & & \otimes \circ (\otimes \times \otimes) \arrow[ddl] \\
\otimes \circ (1 \times \otimes) \circ (1 \times \otimes \times 1) \arrow[dr] & & \\
 & \otimes \circ (1 \times \otimes) \circ (1 \times 1 \times \otimes) &
\end{tikzcd}$$

Unitality:

Writing $u := \otimes^0(*)$, and denoting $ab := a \otimes b = \otimes(a, b)$, the following must commute for every $a, b \in A$:

$$\begin{tikzcd}
 & uu \arrow[dr, "\lambda_u"] & \\
u \arrow[ur, "\rho_u"] \arrow[rr, "1_u"'] & & u
\end{tikzcd}
\qquad
\begin{tikzcd}
 & ab & \\
(ua)b \arrow[ur, "\lambda_a 1_b"] \arrow[rr, "\alpha_{u,a,b}"'] & & u(ab) \arrow[ul, "\lambda_{ab}"']
\end{tikzcd}$$

$$\begin{tikzcd}
 & ab \arrow[dl, "\rho_{ab}"'] \arrow[dr, "1_a \rho_b"] & \\
(ab)u \arrow[rr, "\alpha_{a,b,u}"'] & & a(bu)
\end{tikzcd}
\qquad
\begin{tikzcd}
(au)b \arrow[r, "\alpha_{a,u,b}"] & a(ub) \arrow[d, "1_a \lambda_b"] \\
ab \arrow[u, "\rho_a 1_b"] \arrow[r, equal] & ab
\end{tikzcd}$$

The authors of [2] show that skew monoidal categories correspond precisely to maps out of $\mathbb{C}$ and into $N_s(\mathsf{Cat})$, the *skew nerve* of Cat.[18] This will be defined formally in Section 2.4.

Proposition 2.1.3. [2] *Skew monoidal categories are in one to one correspondence with simplicial maps* $\mathbb{C} \longrightarrow N_s(\mathsf{Cat})$.[19]

Lax monoidal categories are a second sort of weakened monoidal category characterized by the introduction of n-ary product operations. The higher dimensional simplices of $\mathbb{C}$ capture these higher order products as well, as we will show shortly.

Definition 2.1.4. A *lax monoidal category* $(A, \otimes^n, \gamma, \iota)$ consists in:

- A category A.
- Functors $\otimes^n : A^n \longrightarrow A$ for each $n \in \mathbb{N}$. $(\otimes^0 : I \longrightarrow A)$
- Natural transformations for each $n, k_1, ..., k_n \in \mathbb{N}$:

$$\gamma_{n,k_1,...k_n} : \otimes^n \circ \left(\otimes^{k_1} \times ... \times \otimes^{k_n}\right) \Rightarrow \otimes^{k_1+...+k_n}$$

- A natural transformation $\iota : 1_A \Rightarrow \otimes^1$.

These natural transformations must additionally satisfy two axioms.

Associativity:

For each double sequence $n, k_1, ..., k_n, m_{11}, ..., m_{1k_1}, m_{21}, ..., m_{2k_2}, ..., m_{n1}, ..., m_{nk_n}$, the following square commutes:

[18] It should be mentioned $N_s(\mathsf{Cat})$ is referred to as the *lax nerve* in [2] which we have changed here so as to not suggest a relation to lax monoidal categories.

[19] This can be found in [2] Proposition 4.3.

$$\begin{tikzcd}[column sep=small]
& \otimes^n \circ (\otimes^{k_1} \times ... \times \otimes^{k_n}) \circ ((\otimes^{m_{11}} \times ... \times \otimes^{m_{1k_1}}) \times ... \times (\otimes^{m_{n1}} \times ... \times \otimes^{m_{nk_n}})) \arrow[dl, "\gamma_{n,k_1,...,k_n} \circ 1"'] \arrow[dr, "1 \circ \begin{pmatrix} \gamma_{k_1,m_{11},...,m_{1k_1}} \times ... \\ ... \times \gamma_{m_{n1},...,m_{nk_n}} \end{pmatrix}"] & \\
\otimes^{k_1+...+k_n} \circ \begin{pmatrix} (\otimes^{m_{11}} \times ... \times \otimes^{m_{1k_1}}) \times ... \\ ... \times (\otimes^{m_{n1}} \times ... \times \otimes^{m_{nk_n}}) \end{pmatrix} \arrow[dr, "\gamma_{k_1+...+k_n,m_{11},...,m_{1k_1},...,m_{n1},...,m_{nk_n}}"'] & & \otimes^n \circ \begin{pmatrix} \otimes^{m_{11}+...+m_{1k_1}} \times ... \\ ... \times \otimes^{m_{n1}+...+m_{nk_n}} \end{pmatrix} \arrow[dl, "\gamma_{n,m_{11}+...+m_{1k_1},...,m_{n1}+...+m_{nk_n}}"] \\
& \otimes^{m_{11}+...+m_{1k_1}+...+m_{n1}+...+m_{nk_n}} &
\end{tikzcd}$$

Unitality:

The following two triangles commute:

$$\begin{tikzcd}
& 1_A \circ \otimes^n = \otimes^n = \otimes^n \circ (1_A \times ... \times 1_A) \arrow[dl, "\iota \circ 1"'] \arrow[dr, "1 \circ \iota \times ... \times \iota"] \arrow[dd, "1"'] & \\
\otimes^1 \circ \otimes^n \arrow[dr, "\gamma_{1,n}"'] & & \otimes^n \circ (\otimes^1 \times ... \times \otimes^1) \arrow[dl, "\gamma_{n,1,...,1}"] \\
& \otimes^n &
\end{tikzcd}$$

In particular, the data of a lax monoidal category includes a binary functor $\otimes^2 : A \times A \longrightarrow A$. Though it is not required to be associative or even associative up to natural isomorphism, we are given a pair of natural transformations:

$$\gamma_{2,2,1} \bullet (1 \circ (1 \times \iota)) : \otimes^2 \circ (\otimes^2 \times 1) \Rightarrow \otimes^3.$$

$$\gamma_{2,1,2} \bullet (1 \circ (\iota \times 1)) : \otimes^2 \circ (1 \times \otimes^2) \Rightarrow \otimes^3.$$

where we use '$\bullet$' to denote vertical composition. Writing $\otimes^n(a_1, ..., a_n) = (a_1 \otimes ... \otimes a_n)$, the above natural transformations give us maps:

$$((a \otimes b) \otimes c)) \to (a \otimes b \otimes c) \leftarrow (a \otimes (b \otimes c)).$$

In this way, the n-ary product operations mediate between composites of lower arity operations, and hence say something about a weakened form of associativity.

Whereas skew monoidal categories can be understood as maps $\mathbb{C} \longrightarrow N_s(\mathsf{Cat})$, we will see that lax monoidal categories can be understood as maps $\mathbb{C} \longrightarrow N_\Delta(N_2\mathsf{Cat})$, a nerve containing $N_s(\mathsf{Cat})$ in a way we will make precise in Section 2.4. We turn now to define this nerve.

Definition 2.1.5. The *homotopy coherent nerve* $N_\Delta(U)$ of a simplicially enriched category U is the simplicial set defined on objects by $N_\Delta(U)_n := \mathsf{sSetCat}(S[n], U)$. The enriched category $S[n]$ has as objects the elements $0, 1, ..., n \in [n]$. For each pair of objects $p \leq q$, the simplical mapping object $S[n](p, q)$ is the categorical nerve of the poset whose objects are subsets of $\{p, p+1, ..., q\}$ containing both p and q and ordered by inclusion. (For $q < p$, $S[n](p, q) = \emptyset$.) Here are some examples of simplices in these simplicial mapping objects:

- $(\{0, 3\}) \in S[3](0, 3)_0$.
- $(\{0, 3\} \subset [3]) \in S[3](0, 3)_1$.
- $(\{0, 3\} \subset \{0, 1, 3\} \subset [3]) \in S[3](0, 3)_2$.
- $(\{0, 3\} \subset \{0, 1, 3\} = \{0, 1, 3\}) \in S[3](0, 3)_2$.
- $(\{1, 4, 5\} \subset \{1, 2, 4, 5\}) \in S[9](1, 5)_1$.

We will be careful to reserve the symbol '$\subset$' to mean a *proper* inclusion, and will use '$\subseteq$' otherwise.

Composition maps $S[n](p, r) \times S[n](r, q) \longrightarrow S[n](p, q)$ are given by the union of subsets in dimension 0, and unions of inclusions of subsets in higher dimensions. The identity elements are therefore $\{p\} \in S[n](p, p)_0$. $N_\Delta(U)_n$ then consists of all simplicially enriched functors out of $S[n]$ and into U.

The coface and codegeneracy maps of Δ extend to enriched coface functors $\delta_i : S[n-1] \longrightarrow S[n]$ and codegeneracy functors $\sigma_i : S[n+1] \longrightarrow S[n]$. On objects, these functors match their counterparts in Δ, while on mapping objects, $\delta_i : S[n-1](p, q) \longrightarrow S[n](\delta_i p, \delta_i q)$ sends a subset C in dimension 0 to its direct image $\delta_i\mathsf{C} = \{\delta_i c_0, ..., \delta_i c_m\}$, and sends inclusions to their direct images in higher dimensions. Codegeneracy functors are defined similarly. Finally, the face and degeneracy maps of $N_\Delta(U)$ are defined via precomposition with these enriched functors: $d_i : N_\Delta(U)_n \longrightarrow N_\Delta(U)_{n-1}$ is given by $d_i(L) = L \circ \delta_i$ for $L \in N_\Delta(U)_n$, and similarly for s_i.

In what follows, we will focus specifically on the homotopy coherent nerve of Cat, viewed as a simplicially enriched category in the following way. There is a standard nerve functor $N_2 : \mathsf{2Cat} \longrightarrow \mathsf{sSet}$ which preserves products, thus taking monoids in $\mathsf{2Cat}$ to monoids in sSet. Monoids in sSet can in turn be viewed as one object simplicially enriched categories with the monoid as the lone simplicial mapping object. In summation, we have a nerve $N_2 : \mathsf{Mon}_{\mathsf{2Cat}} \longrightarrow \mathsf{sSetCat}$ which takes a monoid in $\mathsf{2Cat}$ to a one object simplicially enriched category. We would like to apply this nerve to Cat, an object of $\mathsf{2Cat}$. Unfortunately, Cat is not a monoid, strictly speaking. With $\times$ as a binary operation and the terminal category I as unit, Cat is itself a monoidal category, a not-quite-monoid of $\mathsf{2Cat}$: the operation $\times$ is associative and unital merely up to natural isomorphism. However, according to the coherence theorem [12], we know that Cat is monoidally equivalent to another 2-category which is an actual monoid. By $N_2(\mathsf{Cat})$ we mean N_2 applied to this equivalent category. In practice, we consider the following definition.

Definition 2.1.6. The simplicially enriched category $N_2(\mathsf{Cat})$ is defined by the following data:

- A unique object, $*$.

  Its mapping object $N_2(\mathsf{Cat})(*, *)$ is characterized by:
- 0-simplices are Categories B.
- 1-simplices are functors $T : B_1 \longrightarrow B_0$.
- 2-simplices are natural transformations $\eta : T_{12} \circ T_{01} \Rightarrow T_{02}$.
- 3-simplices are commutative diagrams of natural transformations:

$$\begin{tikzcd}
 & T_{23} \circ T_{12} \circ T_{01} \arrow[dl, "\eta_{012} \circ 1"'] \arrow[dr, "1 \circ \eta_{123}"] & \\
T_{23} \circ T_{02} \arrow[dr, "\eta_{023}"'] & & T_{13} \circ T_{01} \arrow[dl, "\eta_{013}"] \\
 & T_{03} &
\end{tikzcd}$$

- Higher-dimensional simplices are given by 3-coskeletality.

Composition in $N_2(\mathsf{Cat})$ is given by $\times$ while the identity map for composition consists in a 0-simplex of $N_2(\mathsf{Cat})(*,*)$: I. Face maps in $N_2(\mathsf{Cat})(*,*)$ are given as suggested by the notation above,
e.g $d_1(\eta : T_{12} \circ T_{01} \longrightarrow T_{02}) = T_{02}$, and degeneracy maps are given by inserting identity maps in the expected ways. We assume that we have the following equalities:

$$(A \times B) \times C = A \times (B \times C).$$

$$A \times I = A = I \times A.$$

It is worth noting that 3-coskeletality of $N_2(\mathsf{Cat})(*,*)$ shows us that every simplex is determined by its 3-faces, but also we see that the 3-simplices are determined by their 3-boundaries: there is at most one 3-simplex with a given 3-boundary. Thus, every simplex of $N_2(\mathsf{Cat})(*,*)$ is determined by its 2-faces.

We can give an informal description of simplices of $N_\Delta(N_2\mathsf{Cat})$. It has a single 0-simplex: the unique map $S[0] \longrightarrow N_2(\mathsf{Cat})$. Its 1-simplices can be thought of as categories A, while its 2-simplices consist in functors $T : A_{01} \times A_{12} \longrightarrow A_{02}$. A 3-simplex $L : S[3] \longrightarrow N_2(\mathsf{Cat})$ is a diagram consisting of five such functors, four of which are the 3-simplex's faces. We can see this diagram most clearly by thinking of the image of L on the simplicial mapping object $S[3](0,3)$. $S[3](0,3)$ has four 0-simplices (subsets of $[3]$ containing $\{0,3\}$), five 1-simplices (inclusions of those subsets), and two 2-simplices (double inclusions). The image therefore consists in four categories, five functors, and two natural transformations. This data fits into the following diagram:

$$
\begin{tikzcd}
\{0,1,2,3\} \arrow[rr] \arrow[d] \arrow[drr] & & \{0,1,3\} \arrow[d] \\
\{0,2,3\} \arrow[rr] & & \{0,3\}
\end{tikzcd}
\qquad
\begin{tikzcd}
A_{01} \times A_{12} \times A_{23} \arrow[rr, "1_{A_{01}} \times T_{123}"] \arrow[d, "T_{012} \times 1_{A_{23}}"'] \arrow[drr] & & A_{01} \times A_{13} \arrow[d, "T_{013}"] \\
A_{02} \times A_{23} \arrow[rr, "T_{023}"'] & & A_{03}
\end{tikzcd}
$$

$S[3](0,3)$ $\qquad$ $L(S[3](0,3))$

Higher dimensional simplices $L : S[n] \longrightarrow N_2(\mathsf{Cat})$ are again helpfully summarized via their image of the simplicial mapping object $S[n](0,n)$. Such a simplex will provide: a category for each subset of $[n]$ containing $\{0,n\}$; a functor between such categories whenever the subset indexing the first contains the subset indexing the second; a natural transformation between a composite of such functors and a third such functor for each double containment; a commutative diagram of such natural transformations for every triple and higher containment of subsets. If $\mathsf{C} \in S[n](0,n)_0$, because $\mathsf{C} = \cup_{c\in\mathsf{C}^-}\{c, sc\}$ and L must send unions to products, we have then that $L(\mathsf{C}) = \prod_{c\in\mathsf{C}^-} L(\{c, sc\})$, where the right hand side comes from L's action on the simplicial mapping objects $S[n](c, sc)$. Functors and natural transformations may be given as products in this way as well. Notably, the images of the simplices of $S[n](0,n)$ of the form $\{0,n\} \subseteq [n]$ and $\{0,n\} \subseteq \mathsf{C} \subseteq [n]$ are of particular importance: the image of every other 1 and 2-simplex of any simplicial mapping object $S[n]$ appears as the data of a proper face of L. For example, if L is a 3-simplex, we can see from the above that the functor $L(\{0,3\} \subset \{0,2,3\})$ is actually the functor associated with the 2-simplex $d_1(L)$. This phenomena continues into the higher dimensions. In what follows, we will study the structure of $N_\Delta(N_2\mathsf{Cat})$ in much greater detail and rigour.

We are now ready to state and prove the three main results. In Section 2.3 of this paper we will, in the spirit of Proposition 2.1.3, prove:

Proposition 2.1.7. *There is an assignment of a map $\alpha : \mathbb{C} \longrightarrow N_\Delta(N_2\mathsf{Cat})$ to each lax monoidal $(A, \otimes^n, \iota, \gamma)$ such that $(A, \otimes^n, \iota, \gamma)$ can be recovered completely from α.*

In Section 2.4 we will extend and unify the previous results concerning skew monoidal categories by showing that:

Proposition 2.1.8. *There is a monomorphism $\beta : N_s(\mathsf{Cat}) \longrightarrow N_\Delta(N_2\mathsf{Cat})$.*

Combined with Proposition 2.1.3, we will then have that skew monoidal categories correspond to maps $\mathbb{C} \longrightarrow N_\Delta(N_2\mathsf{Cat})$ as well.

In Section 2.5 we will define Σ-monoidal categories for a countable signature Σ, and prove:

Proposition 2.1.9. *There is an assignment of a map $\sigma : \mathbb{C} \longrightarrow N_\Delta(N_2\mathsf{Cat})$ to each Σ-monoidal category (A, Σ, γ) such that (A, Σ, γ) can be recovered completely from σ.*

Taken together, our three results along with the crucial Proposition 2.2.6 characterizing maps $\mathbb{C} \longrightarrow N_\Delta(N_2\mathsf{Cat})$ show that maps $\mathbb{C} \longrightarrow N_\Delta(N_2\mathsf{Cat})$ are a natural setting to understand a great many monoidal-type categories, including ones not yet defined in the literature. As a first and significant step, we will examine arbitrary maps $\mathbb{C} \longrightarrow N_\Delta(N_2\mathsf{Cat})$ in detail and identify an alternative way of defining them in terms of a simple list of data subject to some few coherence requirements.

2.2. **Defining maps into $N_\Delta(N_2\mathsf{Cat})$.** Defining maps into $N_\Delta(N_2\mathsf{Cat})$ is not as daunting as it might first appear. In this section, we will explore these maps and develop a succinct way of producing them. In what follows, let X be an arbitrary simplicial set.

Proposition 2.2.1. *A map $\phi : X \longrightarrow N_\Delta(N_2\mathsf{Cat})$ is determined by the values:*

(1) $\phi_1(x)(\{0,1\})$ for each non-degenerate $x \in X_1$.
(2) $\phi_n(x)(\{0,n\} \subset [n])$ for each non-degenerate $x \in X_n$, $n \geq 2$.
(3) $\phi_n(x)(\{0,n\} \subset \mathsf{C} \subset [n])$ for each non-degenerate $x \in X_n$, $n \geq 3$, and non-degenerate $(\{0,n\} \subset \mathsf{C} \subset [n]) \in S[n](0,n)_2$.

Moreover, these values can be explicitly described:

(1) $\phi_1(x)(\{0,1\}) = A^x$, a category.
(2) $\phi_n(x)(\{0,n\} \subset [n]) = T^x : \prod_{i \in [n]^-} A^{x_{i,i+1}} \longrightarrow A^{x_{0,n}}$, a functor.

(3) $\phi_n(x)(\{0,n\} \subset \mathsf{C} \subset [n]) = \eta^x_{\mathsf{C}} : (T^{x_{\mathsf{C}}} \circ \prod_{c \in \mathsf{C}^-} T^{x_{[c,sc]}}) \Rightarrow T^x$*, a natural transformation.*[20]

Proof. Suppose that $\phi : X \longrightarrow N_\Delta(N_2\mathsf{Cat})$ and that we know the values specified in items $(1), (2)$, and (3) above. We will show that all other values of ϕ can be determined from these.

For $x \in X_n$, $\phi_n(x) : S[n] \longrightarrow N_2(\mathsf{Cat})$ is trivial on objects because $N_2(\mathsf{Cat})$ has a single object $*$. We consider then its action on the mapping spaces $S[n](p,q)$ of $S[n]$. For brevity, we call a k-simplex of a mapping space of $S[n]$ a k-simplex of $S[n]$. Note that if $x \in X$ is degenerate, because ϕ commutes with degeneracy maps, its value on x is determined by its value on the unique non-degenerate simplex mapping to x, hence it suffices to consider non-degenerate simplices. In what follows, we suppose x is always non-degenerate.

As $\phi_n(x)$ is an enriched functor, it must respect composition in $S[n]$. For C a 0-simplex of $S[n]$, we have:

$$\phi_n(x)(\mathsf{C}) = \phi_n(x)\left(\bigcup_{c \in \mathsf{C}^-} \{c, sc\}\right) = \prod_{c \in \mathsf{C}^-} \phi_n(x)(\{c, sc\}).$$

This shows that $\phi_n(x)$ is determined on 0-simplices of $S[n]$ by its values on subsets of the form $\{p,q\}$ with $0 \leq p < q \leq n$. For each such $\{p,q\}$ there is the coface map $\delta_{p,q} : S[1] \longrightarrow S[n]$ sending 0 to p and 1 to q. This gives:

$$\phi_n(x)(\{p,q\}) = \phi_n(x)(\delta_{p,q}\{0,1\}) = \phi_n(x) \circ \delta_{p,q}(\{0,1\}) = \phi_1(x_{p,q})(\{0,1\}).$$

The last equality follows from ϕ commuting with face maps. If $x_{p,q}$ is degenerate, then because ϕ commutes with degeneracy maps, we must have $\phi_1(x_{p,q})(\{0,1\}) = I$ while if $x_{p,q}$ is non-degenerate, its value is specified by item (1). Hence $\phi_n(x)$ is specified on all 0-simplices by the data of item (1).

Again because $\phi_n(x)$ respects composition, $\phi_n(x)$ is determined on arbitrary 1-simplices $(\mathsf{C}_0 \subseteq \mathsf{C}_1)$ by 1 simplices of the form $(\{p,q\} \subseteq \mathsf{C})$. Because ϕ commutes with face maps, we have:

[20] Here it may very well be that any of $x_{i,i+1}$, x_{C}, or $x_{[c,sc]}$ is degenerate, or even that $[c,sc] = \{c,sc\}$ so that $x_{c,sc}$ is a 1-simplex. By A^y and T^y we mean always $\phi_n(y)(\{0,1\})$ and $\phi_n(y)(\{0,n\} \subseteq [n])$ respectively, even when y is degenerate or $n = 1$.

$$\phi_n(x)(\{p,q\} \subseteq \mathsf{C}) = \phi_n(x)(\delta_{\mathsf{C}}(\{0,m\} \subseteq [m])) = \phi_m(x_{\mathsf{C}})(\{0,m\} \subseteq [m]).$$

Such data is specified in item (2) in the case x_{C} is non-degenerate. If however x_{C} is the degeneracy of some non-degenerate simplex y of dimension l, $x_{\mathsf{C}} = sy$, we have:

$$\begin{aligned}\phi_m(x_{\mathsf{C}})(\{0,m\} \subseteq [m]) = \phi_m(sy)(\{0,m\} \subseteq [m]) =&\phi_l(y)(\{\sigma 0, \sigma m\} \subseteq \sigma[m]) \\ =&\phi_l(y)(\{0,l\} \subseteq [l]).\end{aligned}$$

Thus this is determined again by the data of item (2).[21].

Similarly, the value of $\phi_n(x)$ on 2-simplices is determined by its values on simplices $(\{0,n\} \subseteq \mathsf{C} \subseteq [n])$ as a result of both respecting composition and commuting with face maps. It is therefore determined by item (3), noting that we may need to commute with degeneracy maps as above. Finally, because $\phi_n(x)(\mathsf{C}_0 \subseteq \mathsf{C}_1 \subseteq ... \subseteq \mathsf{C}_k)$ must be a k-simplex of $N_2(\mathsf{Cat})$, it is hence determined by its 2-faces. Because $\phi_n(x)$ commutes with face and degeneracy maps, these 2-faces are determined by the value of $\phi_n(x)$ on the 2-faces of $(\mathsf{C}_0 \subseteq \mathsf{C}_1 \subseteq ... \subseteq \mathsf{C}_k)$ and hence by the data of item (3) as above.

As for the explicit descriptions, $\phi_1(1)(\{0,1\})$ is a category, $\phi_n(x)(\{0,n\} \subset [n])$, a functor, and $\phi_n(x)(\{0,n\} \subset \mathsf{C} \subset [n])$ a transformation from a composite of functors, all in light of the definition of $N_2(\mathsf{Cat})$. We know that the codomain of the functor T^x is :

$$\begin{aligned}d_0 T^x = d_0\phi_n(x)(\{0,n\} \subset [n]) = \phi_n(x)(d_0(\{0,n\} \subset [n])) = \phi_n(x)(\{0,n\}) =&\phi_1(x_{0,n})(\{0,1\}) \\ =&A^{x_{0,n}}.\end{aligned}$$

Its domain is:

$$\begin{aligned}d_1 T^x = d_1\phi_n(x)(\{0,n\} \subset [n]) = \phi_n(x)(d_1(\{0,n\} \subset [n])) =&\phi_n(x)([n]) \\ =& \prod_{i\in[n]^-} \phi_1(x_{i,i+1})(\{0,1\}) = \prod_{i\in[n]^-} A^{x_{i,i+1}}.\end{aligned}$$

We can see the codomain of η^x_{C} by considering:

[21] In the case that $\sigma 0 = 0$, $\sigma m = 1$, then y is a 1-simplex and $\phi_1(y)(\{0,1\} \subseteq [1]) = 1_{\phi_1(y)(\{0,1\})}$

$$d_1\eta^x_{\mathsf{C}} = d_1\phi_n(x)(\{0,n\} \subset \mathsf{C} \subset [n]) = \phi_n(x)(d_1(\{0,n\} \subset \mathsf{C} \subset [n])) = \phi_n(x)(\{0,n\} \subset [n]) = T^x.$$

And can see its domain by considering both of:

$$d_0\eta^x_{\mathsf{C}} = \phi_n(x)(d_0(\{0,n\} \subset \mathsf{C} \subset [n])) = \phi_n(x)((\{0,n\} \subset \mathsf{C})) = T^{x_{\mathsf{C}}}.$$

$$\begin{aligned} d_2\eta^x_{\mathsf{C}} = \phi_n(x)(d_2(\{0,n\} \subset \mathsf{C} \subset [n])) = \phi_n(x)(\mathsf{C} \subset [n]) &= \prod_{c \in \mathsf{C}^-} \phi_n(x)(\{c, sc\} \subseteq [c, sc]) \\ &= \prod_{c \in \mathsf{C}^-} T^{x_{[c,sc]}}. \end{aligned}$$

■

There is also a converse to the above proposition which tells us exactly the conditions a collection of categories, functors, and natural transformations needs to satisfy in order to extend to a map $X \longrightarrow N_\Delta(N_2\mathsf{Cat})$. We state it properly (Proposition 2.2.6) and prove it below, but will first need some key observations about the homotopy coherent nerve construction. Let U be a simplicially enriched category. We know that n-simplices of $N_\Delta(U)$ are given by simplicially enriched functors $S[n] \longrightarrow U$. The mapping simplicial sets of $S[n]$ are freely generated via union by simplices of the form $\{p,q\}$, $\{p,q\} \subseteq \mathsf{C}_1$, $\{p,q\} \subseteq \mathsf{C}_1 \subseteq \mathsf{C}_2$, and so on, as we have seen in the preceding proof. Therefore a map $L : S[n] \longrightarrow U$ is determined by where it sends the objects $0, 1, ..., n$, and where it sends the generating simplices of the mapping sets just listed. If we are in the situation that the simplicial mapping objects of U are uniformly r-coskeletal for some $r \geq 0$, then $L : S[n] \longrightarrow U$ is determined by where it sends the objects and generating simplices of dimension $\leq r$, that is, generating simplices up to those of the form $\{p,q\} \subseteq \mathsf{C}_1 \subseteq ... \subseteq \mathsf{C}_r$.

The simplicially enriched category $N_2(\mathsf{Cat})$ has a 3-coskeletal simplicial mapping object. Indeed, it is 'nearly' 2-coskeletal, as every 3-boundary is the boundary of at most one 3-simplex. Writing it all out explicitly, we get the following lemma.

Lemma 2.2.2. *To define a simplicially enriched functor $L : S[n] \longrightarrow N_2(\mathsf{Cat})$, it suffices to define*

(1) A category $L(\{p,q\})$ for all $0 \leq p < q \leq n$.

(2) A functor

$$L(\{p,q\}\subseteq \mathsf{C}) : \prod_{c\in \mathsf{C}^-} L(\{c,sc\}) \longrightarrow L(\{p,q\})$$

for each $\{p,q\}\subseteq \mathsf{C}$ *such that* $L(\{p,q\} = \{p,q\}) = 1_{L(\{p,q\})}$.

(3) A transformation

$$L(\{p,q\}\subseteq \mathsf{C}_1\subseteq \mathsf{C}_2) : L(\{p,q\}\subseteq \mathsf{C}_1)\circ \prod_{c\in \mathsf{C}_1^-} L(\{c,sc\}\subseteq \mathsf{C}_2\cap[c,sc]) \Rightarrow L(\{p,q\}\subseteq \mathsf{C}_2)$$

for each $\{p,q\}\subseteq \mathsf{C}_1\subseteq \mathsf{C}_2$ *such that* $L(\{p,q\}\subseteq \mathsf{C}_1\subseteq \mathsf{C}_2) = 1_{L(\{p,q\}\subseteq \mathsf{C}_2)}$ *if* $\mathsf{C}_1 = \{p,q\}$ *or* $\mathsf{C}_1 = \mathsf{C}_2$.

And such that, for every non-degenerate $(\{p,q\}\subset \mathsf{C}_1\subset \mathsf{C}_2\subset \mathsf{C}_3)\in S[n](p,q)_3$*:*

$$(\star)\qquad L(\{p,q\}\subset \mathsf{C}_2\subset \mathsf{C}_3)\bullet(L(\{p,q\}\subset \mathsf{C}_1\subset \mathsf{C}_2)\circ 1) =$$
$$L(\{p,q\}\subset \mathsf{C}_1\subset \mathsf{C}_3)\bullet\left(1\circ \prod_{c\in \mathsf{C}_1^-} L(\{c,sc\}\subseteq \mathsf{C}_2\cap[c,sc]\subseteq \mathsf{C}_3\cap[c,sc])\right).$$

Here the specification of domains and codomains of $L(\{p,q\}\subseteq \mathsf{C})$ and $L(\{p,q\}\subseteq \mathsf{C}_1\subseteq \mathsf{C}_2)$ is precisely what is required for L restricted to simplicial mapping objects of $S[n]$ to commute with face maps. The qualification that, e.g. $L(\{p,q\} = \{p,q\}) = 1_{L(\{p,q\})}$ is what is required for L restricted to simplicial mapping objects to commute with degeneracy maps. Finally, had the mapping object of $N_2(\mathsf{Cat})$ truly been 2-coskeletal, we could have done away with equation $(\star)$. Be that as it may, the 3-boundary consisting of the four natural transformations appearing in equation $(\star)$ is the boundary of a 3-simplex precisely when those transformations commute, i.e, when $(\star)$ is satisfied. Finally, if any of the '$\subset$'s of $(\{p,q\}\subset \mathsf{C}_1\subset \mathsf{C}_2\subset \mathsf{C}_3)$ had been '$=$'s, this commutativity would have been given automatically explaining why we needn't consider such cases.

Proposition 2.2.6 – the converse to Proposition 2.2.1 – takes a bit of work to state correctly. Towards these ends, we will need the following three technical lemmas.

Lemma 2.2.3. *Let* X *be a simplicial set,* $\sigma : [n]\longrightarrow[n-1]$ *an arbitrary codegeneracy map with corresponding degeneracy* s*. Let* $\mathsf{C}\subseteq[n]$ *with* $|\mathsf{C}| = m+1$*. Then for all* $y\in X_{n-1}$*,* $(sy)_{\mathsf{C}} = s'(y_{\sigma\mathsf{C}})$ *where* s' *is a degeneracy map or an identity, and is the latter if and only if* $|\mathsf{C}| = |\sigma\mathsf{C}| = m+1$.

Proof. Let $\delta_{\mathsf{C}} : [m] \longrightarrow [n]$ be the composition of coface maps sending $i \mapsto c_i$. We have that $\sigma\delta_{\mathsf{C}}$ can be rewritten as a surjection followed by an injection. We know that $|\sigma\delta_{\mathsf{C}}[m]| = |\sigma\mathsf{C}| = m+1$ or m, which tells us that the surjection in the rewrite either has target $[m]$ or $[m-1]$, hence is an identity or a codegeneracy map which we denote in either case as σ'. So we have $\sigma\delta_{\mathsf{C}} = \delta_{\sigma\mathsf{C}}\sigma'$. In terms of faces and degeneracies we then have $d_{\mathsf{C}}s = s'd_{\sigma\mathsf{C}}$ and so have:

$$(sy)_{\mathsf{C}} = d_{\mathsf{C}}sy = s'd_{\sigma\mathsf{C}}y = s'(y_{\sigma\mathsf{C}}).$$

■

Lemma 2.2.4. *Let A^x be a category for each $x \in X_1$ such that $A^{sy} = I$ for the degeneracy of a 0-simplex y. For every $n \geq 1$ and $x \in X_n$, define the symbols:*

$$\mathsf{dom}(x) := \prod_{i \in [n]^-} A^{x_{i,i+1}} \quad , \quad \mathsf{cod}(x) := A^{x_{0,n}}.$$

Then we have:

(i) $\mathsf{dom}(x) = A^x = \mathsf{cod}(x)$ *if* $x \in X_1$.

(ii) $\mathsf{dom}(sx) = \mathsf{dom}(x)$ *and* $\mathsf{cod}(sx) = \mathsf{cod}(x)$ *for any degeneracy map* s.

And for every $\{0, n\} \subseteq \mathsf{C} \subseteq [n]$,

(iii) $\mathsf{dom}(x_{\mathsf{C}}) = \prod_{c \in \mathsf{C}^-} A^{x_{c,sc}}$ *and* $\mathsf{cod}(x_{\mathsf{C}}) = A^{x_{c_0,c_m}}$.

Proof. We consider each item in turn.

(i) If $x \in X_1$, then as $[1]^- = \{0\}$, $\mathsf{dom}(x) = A^{x_{0,1}} = A^x = \mathsf{cod}(x)$.

(ii) If sx is the degeneracy of a simplex $x \in X_n$, then by Lemma 2.2.3, $(sx)_{i,i+1} = s'(x_{\sigma i,\sigma i+1})$ where s' is the degeneracy map $X_0 \longrightarrow X_1$ if $\sigma i = \sigma(i+1)$, i.e if $\sigma = \sigma_i$, and is otherwise the identity. Suppose then that $\sigma = \sigma_k$ for some $k \in [n]$. Then we have:

$$
\begin{aligned}
\mathsf{dom}(sx) =& A^{s'(x_{\sigma 0,\sigma 1})} \times A^{s'(x_{\sigma 1,\sigma 2})} \times ... \times A^{s'(x_{\sigma k,\sigma k+1})} \times ... \times A^{s'(x_{\sigma n-1,\sigma n})} \times A^{s'(x_{\sigma n,\sigma n+1})} \\
=& A^{x_{0,1}} \times A^{x_{1,2}} \times ... \times A^{s'(x_{k,k})} \times ... \times A^{x_{n-2,n-1}} \times A^{x_{n-1,n}} \\
=& A^{x_{0,1}} \times A^{x_{1,2}} \times ... \times I \times ... \times A^{x_{n-2,n-1}} \times A^{x_{n-1,n}} \\
=& \mathsf{dom}(x).
\end{aligned}
$$

Similarly, $\mathsf{cod}(sx) = (sx)_{0,n+1} = s'(x_{\sigma 0,\sigma n+1}) = x_{0,n} = \mathsf{cod}(x)$ because s' is always the identity (as $n \geq 1$).

(iii) Given $x \in X_n$ and $\{0, n\} \subseteq \mathsf{C} \subseteq [n]$, we have $x_{c_i,c_{i+1}} = (x_{\mathsf{C}})_{i,i+1}$ simply by composing the face maps δ_{C} and $\delta_{i,i+1}$. The result follows.

■

Lemma 2.2.5. *Let A^x be a category for each $x \in X_1$ such that $A^{sy} = I$ for the degeneracy of a 0-simplex y. For each $n \geq 1$ and $x \in X_n$, let $T^x : \mathsf{dom}(x) \longrightarrow \mathsf{cod}(x)$ be a functor such that $T^x = 1_{A^x}$ if $x \in X_1$ and $T^{sx} = T^x$. For every $n \geq 1$, $x \in X_n$, and $\{0, n\} \subseteq \mathsf{C} \subseteq [n]$, define the symbols:*

$$\mathsf{dom}(x, \mathsf{C}) := T^{x_{\mathsf{C}}} \circ \prod_{c \in \mathsf{C}^-} T^{x_{[c,sc]}} \quad , \quad \mathsf{cod}(x, \mathsf{C}) := T^x.$$ [22]

Then we have:

(i) $\mathsf{dom}(x, \mathsf{C}) = T^x = \mathsf{cod}(x, \mathsf{C})$ *if* $\mathsf{C} = \{0, n\}$ *or* $\mathsf{C} = [n]$.

(ii) $\mathsf{dom}(sx, \mathsf{C}) = \mathsf{dom}(x, \sigma\mathsf{C})$ *and* $\mathsf{cod}(sx, \mathsf{C}) = \mathsf{cod}(x, \sigma\mathsf{C})$ *for any degeneracy map s.*

And for every $\{0, n\} \subseteq \mathsf{A} \subseteq \mathsf{B} \subseteq [n]$,

(iii) $\mathsf{dom}(x_{\mathsf{B}}, \delta_{\mathsf{B}}^{-1}\mathsf{A}) = T^{x_{\mathsf{A}}} \circ \prod_{a \in \mathsf{A}^-} T^{x_{\mathsf{B} \cap [a,sa]}}$ *and* $\mathsf{cod}(x_{\mathsf{B}}, \delta_{\mathsf{B}}^{-1}\mathsf{A}) = T^{x_{\mathsf{B}}}$.

Proof. We consider each item in turn.

(i) If $\{0, n\} \subseteq \mathsf{C} \subseteq [n]$ and $\mathsf{C} = \{0, n\}$, then $x_{\mathsf{C}} = x_{0,n}$ is a 1-simplex, hence $T^{x_{\mathsf{C}}} = 1$ and $\mathsf{dom}(x, \mathsf{C}) = 1 \circ T^{x_{[0,n]}} = T^x$. If $\mathsf{C} = [n]$, then $x_{\mathsf{C}} = x$, and $T^{x_{[c,sc]}} = T^{x_{i,i+1}}$ for some i, and is hence an identity. Thus $\mathsf{dom}(x, \mathsf{C}) = T^x \circ \prod 1 = T^x$.

[22] The composite of $\mathsf{dom}(x, \mathsf{C})$ is well defined by item (iii) of Lemma 2.2.4

(ii) If sx is the degeneracy of a simplex $x \in X_n$ and $\{0,n\} \subseteq \mathsf{C} \subseteq [n]$, then by Lemma 2.2.3, $(sx)_{\mathsf{C}} = s'(x_{\sigma\mathsf{C}})$ and $(sx)_{[c,sc]} = s'(x_{[\sigma c,\sigma sc]})$ so that $T^{(sx)_{\mathsf{C}}} = T^{x_{\sigma\mathsf{C}}}$ and $T^{(sx)_{[c,sc]}} = T^{x_{[\sigma c,\sigma sc]}}$. If $\sigma = \sigma_k$ and not both of $k, k+1 \in \mathsf{C}$, then the product $\prod_{c\in\mathsf{C}^-} T^{x_{[c,sc]}}$ is exactly the product $\prod_{d\in\sigma(\mathsf{C})^-} T^{x_{[\sigma d,\sigma sd]}}$. Otherwise, the two products differ by a factor of $T^{s'(x_{[k,k]})} = 1_{A^{s'(x_{[k,k]})}} = 1_I$, and hence can be ignored. This shows $\mathsf{dom}(sx,\mathsf{C}) = \mathsf{dom}(x,\sigma\mathsf{C})$. We also have $\mathsf{cod}(sx,\mathsf{C}) = T^{sx} = T^x = \mathsf{cod}(x,\sigma\mathsf{C})$ by assumption.

(iii) This last item follows from item (iii) of Lemma 2.2.4 and simply composing face maps.

■

We are now ready to state and prove the converse to Proposition 2.2.1.

Proposition 2.2.6. *Let X be a simplicial set, and suppose we are given:*

(1) A category A^x for each 1-simplex $x \in X_1$ such that:

*(1.a) **:** $A^{sy} = I$ for the degeneracy of a 0-simplex $y \in X_0$.*

(2) A functor

$$T^x : \prod_{i\in[n]^-} A^{x_{i,i+1}} \longrightarrow A^{x_{0,n}}.$$

for each $n \geq 1$ and $x \in X_n$ such that:

*(2.a) **:** $T^x = 1_{A^x}$ when $n = 1$.*

*(2.b) **:** $T^{sx} = T^x$ for any degeneracy map s.*

(3) A natural transformation

$$\eta^x_{\mathsf{C}} : T^{x_{\mathsf{C}}} \circ \prod_{c\in\mathsf{C}^-} T^{x_{[c,sc]}} \Rightarrow T^x.$$

for each $n \geq 1$, $x \in X_n$, and $\{0,n\} \subseteq \mathsf{C} \subseteq [n]$ such that:

*(3.a) **:** $\eta^x_{\mathsf{C}} = 1_{T^x}$ whenever $\mathsf{C} = \{0,n\}$ or $\mathsf{C} = [n]$.*

*(3.b) **:** $\eta^{sx}_{\mathsf{C}} = \eta^x_{\sigma\mathsf{C}}$ for any degeneracy map s.*

Then, there exists a unique map $\phi : X \longrightarrow N_\Delta(N_2\mathsf{Cat})$ extending the above data with...

(1) $\phi_1(x)(\{0,1\}) = A^x$ for each $x \in X_1$.

(2) $\phi_n(x)(\{0,n\} \subseteq [n]) = T^x$ for each $n \geq 1$ and $x \in X_n$.

(3) $\phi_n(x)(\{0,n\} \subseteq \mathsf{C} \subseteq [n]) = \eta^x_{\mathsf{C}}$ for each $n \geq 1$, $x \in X_n$, and $\{0,n\} \subseteq \mathsf{C} \subseteq [n]$.

*...**if and only if** for every $n \geq 4$ and non-degenerate $x \in X_n$ and non-degenerate $(\{0,n\} \subset \mathsf{A} \subset \mathsf{B} \subset [n]) \in S[n](0,n)_3$, we have:*

$$(\dagger) \qquad \eta^x_{\mathsf{B}} \bullet \left(\eta^{x_{\mathsf{B}}}_{\delta_{\mathsf{B}}^{-1}\mathsf{A}} \circ 1\right) = \eta^x_{\mathsf{A}} \bullet \left(1 \circ \prod_{a \in \mathsf{A}^-} \eta^{x_{[a,sa]}}_{\delta^{-1}_{[a,sa]}\mathsf{B}\cap[a,sa]}\right).$$

Where each η fits into one of the four triangular faces of the diagram below [23]*:*

$$\begin{tikzcd}
\prod_{i\in[n]^-} A^{x_{i,i+1}} \arrow[rr, "\prod_{a\in\mathsf{A}^-} T^{x_{[a,sa]}}"] \arrow[d, "\prod_{b\in\mathsf{B}^-} T^{x_{[b,sb]}}"'] \arrow[drr, bend left, "T^x"] & & \prod_{a\in\mathsf{A}^-} A^{x_{a,sa}} \arrow[d, "T^{x_{\mathsf{A}}}"] \\
\prod_{b\in\mathsf{B}^-} A^{x_{b,sb}} \arrow[rr, "T^{x_{\mathsf{B}}}"'] \arrow[urr, "\prod_{a\in\mathsf{A}^-} T^{x_{\mathsf{B}\cap[a,sa]}}" near start] & & A^{x_{0,n}}
\end{tikzcd}$$

Proof. Given an extension ϕ (which is necessarily unique by Proposition 2.2.1) the equation $(\dagger)$ precisely asserts that $\phi_n(x)(\{0,n\} \subset \mathsf{A} \subset \mathsf{B} \subset [n])$ is indeed a 3-simplex of $N_2(\mathsf{Cat})(*,*)$. That is, it asserts that the diagram of natural transformations associated to the boundary of $\phi_n(x)(\{0,n\} \subset \mathsf{A} \subset \mathsf{B} \subset [n])$ is commutative.

Conversely, given the data of items (1), (2), and (3), such that equation $(\dagger)$ is true, we must assign to each $x \in X_n$ an enriched functor $\phi_n(x) : S[n] \longrightarrow N_2(\mathsf{Cat})$. By Lemma 2.2.2, we must therefore give assignments:

(1) A category $\phi_n(x)(\{p,q\})$ for all $0 \leq p < q \leq n$.

(2) A functor

$$\phi_n(x)(\{p,q\} \subseteq \mathsf{C}) : \prod_{c\in\mathsf{C}^-} \phi_n(x)(\{c,sc\}) \longrightarrow \phi_n(x)(\{p,q\})$$

for each $\{p,q\} \subseteq \mathsf{C}$.

[23] See in particular item (iii) of Lemma 2.2.5

(3) A transformation

$$\phi_n(x)(\{p,q\} \subseteq \mathsf{C}_1 \subseteq \mathsf{C}_2) :$$
$$\phi_n(x)(\{p,q\} \subseteq \mathsf{C}_1) \circ \prod_{c \in \mathsf{C}_1^-} \phi_n(x)(\{c, sc\} \subseteq \mathsf{C}_2 \cap [c, sc]) \Rightarrow \phi_n(x)(\{p,q\} \subseteq \mathsf{C}_2)$$

for each $\{p,q\} \subseteq \mathsf{C}_1 \subseteq \mathsf{C}_2$.

Then for an arbitrary $x \in X_n$, we define:

(1) $\phi_n(x)(\{p,q\}) := A^{x_{p,q}}$.
(2) $\phi_n(x)(\{p,q\} \subseteq \mathsf{C}) := T^{x_\mathsf{C}}$.
(3) $\phi_n(x)(\{p,q\} \subseteq \mathsf{C}_1 \subseteq \mathsf{C}_2) := \eta^{x_{\mathsf{C}_2}}_{\delta^{-1}_{\mathsf{C}_2}\mathsf{C}_1}$.

Note first that $\phi_n(x)(\{p,q\} \subseteq \mathsf{C}) = T^{x_\mathsf{C}}$ should be a functor with domain $\prod_{c \in \mathsf{C}^-} \phi_n(x)(\{c, sc\})$ and codomain $\phi_n(x)(\{p,q\})$, or rather, should be a functor:

$$T^{x_\mathsf{C}} : \prod_{c \in \mathsf{C}^-} A^{x_{c,sc}} \longrightarrow A^{x_{p,q}}.$$

This is so by item (iii) of Lemma 2.2.4, as the stipulations $(2.a)$ and $(2.b)$ of the statement of the proposition verify the hypotheses of that lemma. The natural transformation $\eta^{x_{\mathsf{C}_2}}_{\delta^{-1}_{\mathsf{C}_2}\mathsf{C}_1}$ should be a natural transformation with domain and codomain:

$$\eta^{x_{\mathsf{C}_2}}_{\delta^{-1}_{\mathsf{C}_2}\mathsf{C}_1} : T^{x_{\mathsf{C}_1}} \circ \prod_{c' \in \mathsf{C}_1} T^{x_{\mathsf{C}_2 \cap [c', sc']}} \Rightarrow T^{x_{\mathsf{C}_2}}.$$

This follows from item (iii) of Lemma 2.2.5, as again the additional stipulations $(3.a)$ and $(3.b)$ verify the hypotheses of that lemma.

Note also that to use Lemma 2.2.2, we must have that $\phi_n(x)(\{p,q\} = \{p,q\}) = 1_{\phi_n(x)(\{p,q\})}$. This follows from the stipulation $(2.a)$. Similarly, Lemma 2.2.2 also requires that $\phi_n(x)(\{p,q\} \subseteq \mathsf{C}_1 \subseteq \mathsf{C}_2)$ is $1_{\phi_n(x)(\{p,q\} \subseteq \mathsf{C}_2)}$ if either '$\subseteq$' is an '$=$'. This follows from the stipulation $(3.a)$.

By Lemma 2.2.2, these assignments extend to an enriched functor $\phi_n(x)$ if for every $(\{p,q\} \subset \mathsf{C}_1 \subset \mathsf{C}_2 \subset \mathsf{C}_3) \in S[n](0,n)_3$, we have:

$$\phi_n(x)(\{p,q\} \subset \mathsf{C}_2 \subset \mathsf{C}_3) \bullet (\phi_n(x)(\{p,q\} \subset \mathsf{C}_1 \subset \mathsf{C}_2) \circ 1) =$$

$$\phi_n(x)(\{p,q\} \subset \mathsf{C}_1 \subset \mathsf{C}_3) \bullet \left(1 \circ \prod_{c \in \mathsf{C}_1^-} \phi_n(x)(\{c, sc\} \subseteq \mathsf{C}_2 \cap [c, sc] \subseteq \mathsf{C}_3 \cap [c, sc])\right).$$

Tracing through the definitions and the assignments in Lemma 2.2.2, this is precisely the equation (†) with respect to $\delta_{\mathsf{C}_3}^{-1}(\{p,q\} \subset \mathsf{C}_1 \subset \mathsf{C}_2 \subset \mathsf{C}_3)$, and is hence true by hypothesis.

We have thus far shown that to each $x \in X_n$, we can assign an enriched functor $\phi_n(x) \in N_\Delta(N_2\mathsf{Cat})_n$. We must now check that ϕ commutes with face and degeneracy maps and so is a simplicial map. For each $x \in X_{n+1}$ and $\delta : [n] \longrightarrow [n+1]$ with corresponding face map d, we must show that $\phi_n(dx) = \phi_{n+1}(x) \circ \delta : S[n] \longrightarrow N_2(\mathsf{Cat})$. As such, it suffices to check that their action is the same on generating simplices. This follows more or less by definition, and we show only the 2-dimensional generating simplex case:

$$\begin{aligned}
\phi_n(dx)(\{p,q\} \subseteq \mathsf{C}_1 \subseteq \mathsf{C}_2) =& \eta^{dx_{\mathsf{C}_2}}_{\delta^{-1}_{\mathsf{C}_2}\mathsf{C}_1} \\
=& \eta^{x_{\delta\mathsf{C}_2}}_{\delta^{-1}_{\delta\mathsf{C}_2}\delta\mathsf{C}_1} \\
=& \phi_{n+1}(x)(\{\delta p, \delta q\} \subseteq \delta\mathsf{C}_1 \subseteq \delta\mathsf{C}_2).
\end{aligned}$$

For degeneracy maps we must show that for arbitrary $x \in X_{n-1}$ and $\sigma : [n] \longrightarrow [n-1]$, we have $\phi_n(sx) = \phi_{n-1}(x) \circ \sigma : S[n] \longrightarrow N_2(\mathsf{Cat})$. Again it suffices to check equality on generating simplices. Recalling Lemma 2.2.3, we have:

(i) $\phi_n(sx)(\{p,q\}) = A^{(sx)_{p,q}} = A^{s'(x_{\sigma p, \sigma q})} = A^{x_{\sigma p, \sigma q}} = (\phi_{n-1}(x) \circ \sigma)(\{p,q\})$.

(ii) $\phi_n(sx)(\{p,q\} \subseteq \mathsf{C}) = T^{(sx)_\mathsf{C}} = T^{s'(x_{\sigma\mathsf{C}})} = T^{x_{\sigma\mathsf{C}}} = (\phi_{n-1}(x) \circ \sigma)(\{p,q\} \subseteq \mathsf{C})$.

(iii) $$\begin{aligned}\phi_n(sx)(\{p,q\} \subseteq \mathsf{C}_1 \subseteq \mathsf{C}_2) &= \eta^{(sx)_{\mathsf{C}_2}}_{\delta^{-1}_{\mathsf{C}_2}\mathsf{C}_1} = \eta^{s'(x_{\sigma\mathsf{C}_2})}_{\delta^{-1}_{\mathsf{C}_2}\mathsf{C}_1} \\ &= \eta^{x_{\sigma\mathsf{C}_2}}_{\sigma'\delta^{-1}_{\mathsf{C}_2}\mathsf{C}_1} = \eta^{x_{\sigma\mathsf{C}_2}}_{\delta^{-1}_{\sigma\mathsf{C}_2}\sigma\mathsf{C}_1} = (\phi_{n-1}(x) \circ \sigma)(\{p,q\} \subset \mathsf{C}_1 \subset \mathsf{C}_2).\end{aligned}$$

The fourth equation of item (iii) follows because we have $\delta_{\sigma\mathsf{C}_2}\sigma' = \sigma\delta_{\mathsf{C}_2}$ as in the proof of Lemma 2.2.3.

$\blacksquare$

We will make use of the general version of Proposition 2.2.6, but care most about the specific case when $X = \mathbb{C}$. In this case the proposition simplifies nicely because there are only two 1-simplices, $0, 1 \in \mathbb{C}_1$, and only 1 is non-degenerate. Furthermore, in light of the results to follow, we can think an arbitrary map $\phi : \mathbb{C} \longrightarrow N_\Delta(N_2\mathsf{Cat})$ as a truly general kind of monoidal-type category. When $X = \mathbb{C}$, Proposition 2.2.6 can be taken to be a presentation of a monoidal-type category in the usual dichotomy of data (1), (2), and (3), subject to coherence (†). Writing $\sum \mathrm{sp}(x)$ for the total number of '1's in $\mathrm{sp}(x)$ for a simplex $x \in \mathbb{C}$, we have that an arbitrary map $\phi : \mathbb{C} \longrightarrow N_\Delta(N_2\mathsf{Cat})$ can be defined by:

(1) A category $A = A^1$, and write $A^0 = I$.

(2) A functor

$$T^x : \prod_{i \in [n]^-} A^{x_{i,i+1}} = A^{\sum \mathrm{sp}(x)} \longrightarrow A.$$

for each $n \geq 1$ and $x \in \mathbb{C}_n$ subject to $(2.a)$ and $(2.b)$.

(3) A natural transformation

$$\eta^x_{\mathsf{C}} : (T^{x_{\mathsf{C}}} \circ \prod_{c \in \mathsf{C}^-} T^{x_{[c,sc]}}) \Rightarrow T^x.$$

for each face $n \geq 1$, $x \in \mathbb{C}_n$, and $\{0, n\} \subseteq \mathsf{C} \subseteq [n]$ subject to $(3.a)$ and $(3.b)$.

The natural transformations of item (3) must be *coherent* insofar as the equation (†) is satisfied for every $\{0, n\} \subset \mathsf{A} \subset \mathsf{B} \subset [n]$:

$$\eta^x_{\mathsf{B}} \bullet \left(\eta^{x_{\mathsf{B}}}_{\delta^{-1}_{\mathsf{B}}\mathsf{A}} \circ 1\right) = \eta^x_{\mathsf{A}} \bullet \left(1 \circ \prod_{a \in \mathsf{A}^-} \eta^{x_{[a,sa]}}_{\delta^{-1}_{[a,sa]}\mathsf{B} \cap [a,sa]}\right).$$

2.3. **Classifying lax monoidal categories.** In this section, we will prove Proposition 2.1.7. Given a lax monoidal category $(A, \otimes^n, \iota, \gamma)$, we first define the classifying map α, then verify its well definedness, and lastly show that given α we can reconstruct $(A, \otimes^n, \iota, \gamma)$.

Definition 2.3.1. The Lax Monoidal Category Classifying Map

Let $(A, \otimes^n, \iota, \gamma)$ be a lax monoidal category. We assign to A the map $\alpha : \mathbb{C} \longrightarrow N_\Delta(N_2\mathsf{Cat})$ using Proposition 2.2.6. We define the map as follows:

(1) Let $A^x := A$ if $x = 1$ and I otherwise.

(2) Let $T^x := \otimes^{\sum \mathrm{sp}(x)} : A^{\sum \mathrm{sp}(x)} \longrightarrow A$ if x is non-degenerate with dimension ≥ 2. Otherwise T^x is defined in accord with stipulations $(2.a)$ and $(2.b)$ of Proposition 2.2.6. In particular, $T^x := 1_{A^x}$ if x has dimension 1, or is the degeneracy of a 1-simplex.

(3) For each non-degenerate $x \in \mathbb{C}_n$ with $n \geq 3$ and $\{0,n\} \subset \mathsf{C} \subset [n]$, let

$$\eta^x_{\mathsf{C}} := \gamma \bullet (\bar{\iota} \circ \prod \bar{\iota}) : (T^{x_{\mathsf{C}}} \circ \prod_{c \in \mathsf{C}^-} T^{x_{[c,sc]}}) \Rightarrow T^x.$$

Where γ is short-hand for the appropriately sized associativity transformation $\gamma_{n,k_1,\ldots,k_n}$, and $\bar{\iota} : T^x \longrightarrow \overline{T^x}$ is defined by:

Otherwise η^x_{C} is defined in accord with stipulations $(3.a)$ and $(3.b)$ of Proposition 2.2.6. In particular, we may have $\eta^x_{\mathsf{C}} = 1_{T^x}$ if, for example, $x \in \mathbb{C}_n$ and $\mathsf{C} = [n]$ or $\mathsf{C} = \{0,n\}$.

This transformation $\gamma \bullet (\bar{\iota} \circ \prod \bar{\iota})$ encodes a simple idea: $\bar{\iota} \circ \prod \bar{\iota}$ converts all 1_A's appearing in any factor of either composite of the domain into $\otimes^1$'s, so that by the time γ is applied, only $\otimes^n$ functor factors remain.

This data extends to a map $\alpha : \mathbb{C} \longrightarrow N_\Delta(N_2\mathsf{Cat})$ via Proposition 2.2.6 because we have the following:

Proposition 2.3.2. *Let $(A, \otimes^n, \iota, \gamma)$ be a lax monoidal category, and let $A^x, T^x, \eta^x_{\mathsf{C}}$ be defined as above. Then for every $n \geq 4$, $\{0,n\} \subset \mathsf{A} \subset \mathsf{B} \subset [n]$ and non-degenerate $x \in \mathbb{C}_n$, equation (†) is satisfied:*

$$\eta^x_{\mathsf{B}} \bullet \left(\eta^{x_{\mathsf{B}}}_{\delta^{-1}_{\mathsf{B}}\mathsf{A}} \circ 1\right) = \eta^x_{\mathsf{A}} \bullet \left(1 \circ \prod_{a \in \mathsf{A}^-} \eta^{x_{[a,sa]}}_{\delta^{-1}_{[a,sa]}\mathsf{B} \cap [a,sa]}\right).$$

As $x \in \mathbb{C}_n$ is a fixed non-degenerate simplex throughout the proof, we will simply omit 'x' from our notation. We will write $A_{i,i+1} := A^{x_{i,i+1}}$, $T := T^x$, $T_{\mathsf{C}} := T^{x_{\mathsf{C}}}$ and $\eta_{\mathsf{C}} := \eta^x_{\mathsf{C}}$ for any $\{0,n\} \subseteq \mathsf{C} \subseteq [n]$. We write $\eta^{\mathsf{C}}_{\mathsf{D}} := \eta^{x_{\mathsf{C}}}_{\delta^{-1}_{\mathsf{C}}\mathsf{D}}$ for any $\{0,n\} \subseteq \mathsf{D} \subseteq \mathsf{C} \subseteq [n]$. In this notation, we will prove:

$$\eta_{\mathsf{B}} \bullet (\eta^{\mathsf{B}}_{\mathsf{A}} \circ 1) = \eta_{\mathsf{A}} \bullet \left(1 \circ \prod_{a \in \mathsf{A}^-} \eta^{[a,sa]}_{\mathsf{B} \cap [a,sa]}\right).$$

It will be convenient to recall that these natural transformations each fit into one of the four triangular faces of:

$$\begin{tikzcd}
\prod\limits_{i\in[n]^-} A_{i,i+1} \arrow[rr, "\prod\limits_{a\in\mathsf{A}^-} T_{[a,sa]}"] \arrow[d, "\prod\limits_{b\in\mathsf{B}^-} T_{[b,sb]}"'] \arrow[rrd, bend left, "T"] & & \prod\limits_{a\in\mathsf{A}^-} A_{a,sa} \arrow[d, "T_{\mathsf{A}}"] \\
\prod\limits_{b\in\mathsf{B}^-} A_{b,sb} = \prod\limits_{\substack{a\in\mathsf{A}^- \\ b\in\mathsf{B}\cap[a,sa]^-}} A_{b,sb} \arrow[rru, "\prod\limits_{a\in\mathsf{A}^-} T_{\mathsf{B}\cap[a,sa]}"] \arrow[rr, "T_{\mathsf{B}}"'] & & A_{0,n}
\end{tikzcd}$$

Figure (‡)

Proof. (of Proposition 2.3.2)

Because x is non-degenerate and there are no '='s in $\{0,n\} \subset \mathsf{A} \subset \mathsf{B} \subset [n]$, we know that both η_{A} and η_{B} are of the form $\gamma \bullet \bar{\iota}$ by definition. However, x_{B} or $x_{[a,sa]}$ may be degenerate, and/or we may have '=' in $\{a, sa\} \subseteq \mathsf{B} \cap [a, sa] \subseteq [a, sa]$ for any $a \in \mathsf{A}^-$. As a result, $\eta^{\mathsf{B}}_{\mathsf{A}}$ or $\eta^{[a,sa]}_{[a,sa]\cap\mathsf{B}}$ may be given via stipulations $(3.a)$ and/or $(3.b)$ as identity natural transformations. For example, if x_{B} was the degeneracy of a 2-simplex y, then $\eta^{\mathsf{B}}_{\mathsf{A}} = 1_{T^y}$ as can easily be checked. This shows we will need to think about how all three of γ, ι, and identity natural transformations interact in the proof. Luckily, we need only consider two cases: when $\eta^{\mathsf{B}}_{\mathsf{A}} = \gamma \bullet \bar{\iota}$, and when it is an identity.

Let us first assume that $\eta^{\mathsf{B}}_{\mathsf{A}} = \gamma \bullet \bar{\iota}$. Define the set:

$$\Gamma = \{a \in \mathsf{A}^- \mid \eta^{[a,sa]}_{\mathsf{B}\cap[a,sa]} \text{ is defined by } \gamma \bullet \bar{\iota}\}.$$

So that $a \in (\mathsf{A}^-)\backslash\Gamma$ only if $\eta^{[a,sa]}_{\mathsf{B}\cap[a,sa]} = 1_{T_{[a,sa]}}$. Consider the following diagram:

$$
\begin{tikzcd}[column sep=small]
\begin{array}{c} T_{\mathsf{A}} \circ \prod\limits_{a \in \mathsf{A}^-} T_{\mathsf{B} \cap [a,sa]} \circ \prod\limits_{b \in \mathsf{B}^-} T_{[b,sb]} = \\ T_{\mathsf{A}} \circ \prod\limits_{a \in \mathsf{A}^-} \left(T_{\mathsf{B} \cap [a,sa]} \circ \prod\limits_{b \in (\mathsf{B} \cap [a,sa])^-} T_{[b,sb]} \right) \end{array} \arrow[rr, equal, "1"] \arrow[d, equal, "1"'] \arrow[drr, phantom, "\mathbf{N}"] & & T_{\mathsf{A}} \circ \left(\prod\limits_{a \in \Gamma} \left(T_{\mathsf{B} \cap [a,sa]} \circ \prod\limits_{b \in (\mathsf{B} \cap [a,sa])^-} T_{[b,sb]} \right) \times \prod\limits_{a \notin \Gamma} \left(T_{\mathsf{B} \cap [a,sa]} \circ \prod\limits_{b \in (\mathsf{B} \cap [a,sa])^-} T_{[b,sb]} \right) \right) \arrow[d, "\bar{\iota} \circ ((\bar{\iota} \circ \bar{\iota}) \times (1 \circ 1))"] \\
\begin{array}{c} T_{\mathsf{A}} \circ \left(\prod\limits_{a \in \Gamma} T_{\mathsf{B} \cap [a,sa]} \times \prod\limits_{a \notin \Gamma} T_{\mathsf{B} \cap [a,sa]} \right) \\ \circ \left(\prod\limits_{\substack{b \in (\mathsf{B} \cap [a,sa])^- \\ a \in \Gamma}} T_{[b,sb]} \times \prod\limits_{\substack{b \in (\mathsf{B} \cap [a,sa])^- \\ a \notin \Gamma}} T_{[b,sb]} \right) \end{array} \arrow[dr, "\bar{\iota} \circ (\bar{\iota} \times \bar{\iota}) \circ (\bar{\iota} \times \bar{\iota})"] \arrow[dd, bend right, "\eta^{\mathsf{B}}_{\mathsf{A}} \circ 1"'] & & \overline{T_{\mathsf{A}}} \circ \left(\prod\limits_{a \in \Gamma} \left(\overline{T_{\mathsf{B} \cap [a,sa]}} \circ \prod\limits_{b \in (\mathsf{B} \cap [a,sa])^-} \overline{T_{[b,sb]}} \right) \times \prod\limits_{a \notin \Gamma} \left(T_{\mathsf{B} \cap [a,sa]} \circ \prod\limits_{b \in (\mathsf{B} \cap [a,sa])^-} T_{[b,sb]} \right) \right) \arrow[dl, "1 \circ ((1 \circ 1) \times (\bar{\iota} \circ \bar{\iota}))"'] \arrow[dd, bend left, "1 \circ (\prod \gamma \times 1)"] \\
& \overline{T_{\mathsf{A}}} \circ \left(\prod\limits_{a \in \Gamma} \left(\overline{T_{\mathsf{B} \cap [a,sa]}} \circ \prod\limits_{b \in (\mathsf{B} \cap [a,sa])^-} \overline{T_{[b,sb]}} \right) \times \prod\limits_{a \notin \Gamma} \left(\overline{T_{\mathsf{B} \cap [a,sa]}} \circ \prod\limits_{b \in (\mathsf{B} \cap [a,sa])^-} \overline{T_{[b,sb]}} \right) \right) \arrow[ddl, bend left, "\gamma \circ (1 \times 1)"'] \arrow[ddr, bend right, "1 \circ (\prod \gamma \times \prod \gamma)"] \arrow[dl, phantom, "\mathbf{W}"] \arrow[dr, phantom, "\mathbf{E}"] \arrow[dd, phantom, "\mathbf{S}"] & \\
T_{\mathsf{B}} \circ \left(\prod\limits_{\substack{b \in (\mathsf{B} \cap [a,sa])^- \\ a \in \Gamma}} T_{[b,sb]} \times \prod\limits_{\substack{b \in (\mathsf{B} \cap [a,sa])^- \\ a \notin \Gamma}} T_{[b,sb]} \right) \arrow[d, "1 \circ (\bar{\iota} \times \bar{\iota})"'] & & \overline{T_{\mathsf{A}}} \circ \left(\prod\limits_{a \in \Gamma} \overline{T_{[a,sa]}} \times \prod\limits_{a \notin \Gamma} T_{[a,sa]} \right) \arrow[d, "1 \circ (1 \times \bar{\iota})"] \\
T_{\mathsf{B}} \circ \left(\prod\limits_{\substack{b \in (\mathsf{B} \cap [a,sa])^- \\ a \in \Gamma}} \overline{T_{[b,sb]}} \times \prod\limits_{\substack{b \in (\mathsf{B} \cap [a,sa])^- \\ a \notin \Gamma}} \overline{T_{[b,sb]}} \right) \arrow[r, "\gamma"'] & T & \overline{T_{\mathsf{A}}} \circ \left(\prod\limits_{a \in \Gamma} \overline{T_{[a,sa]}} \times \prod\limits_{a \notin \Gamma} \overline{T_{[a,sa]}} \right) \arrow[l, "\gamma"]
\end{tikzcd}
$$

Due to interchange of horizontal composition with products, the middle node in this diagram can be rewritten:

$$\overline{T_{\mathsf{A}}} \circ \left(\prod_{a \in \Gamma} \left(\overline{T_{\mathsf{B} \cap [a,sa]}} \circ \prod_{b \in (\mathsf{B} \cap [a,sa])^{-}} \overline{T_{[b,sb]}} \right) \times \prod_{a \notin \Gamma} \left(\overline{T_{\mathsf{B} \cap [a,sa]}} \circ \prod_{b \in (\mathsf{B} \cap [a,sa])^{-}} \overline{T_{[b,sb]}} \right) \right) =$$

$$\overline{T_{\mathsf{A}}} \circ \left(\prod_{a \in \Gamma} \overline{T_{\mathsf{B} \cap [a,sa]}} \times \prod_{a \notin \Gamma} \overline{T_{\mathsf{B} \cap [a,sa]}} \right) \circ \left(\prod_{\substack{b \in (\mathsf{B} \cap [a,sa])^{-} \\ a \in \Gamma}} \overline{T_{[b,sb]}} \times \prod_{\substack{b \in (\mathsf{B} \cap [a,sa])^{-} \\ a \notin \Gamma}} \overline{T_{[b,sb]}} \right).$$

The map out of this central node $\gamma \circ (1 \times 1)$ is written in regards to this second expression. The notation for the other map out of this central node, $1 \circ \prod \gamma \times \prod \gamma$ is written with regards to the first.

First we claim that commutativity of the above diagram implies Proposition 2.3. As we have assumed that $\eta_{\mathsf{A}}^{\mathsf{B}}$ is of the form $\gamma \bullet \bar{\iota}$, we have that x_{B} is not the degeneracy of a 2-simplex, and so not the degeneracy of a 1-simplex, and hence that T_{B} is not an identity map. Thus $T_{\mathsf{B}} = \overline{T_{\mathsf{B}}}$. The composition of the left most edges is then:

$$\gamma \bullet (1 \circ (\bar{\iota} \times \bar{\iota})) \bullet \left(\eta_{\mathsf{A}}^{\mathsf{B}} \circ 1 \right) = \eta_{\mathsf{B}} \bullet \left(\eta_{\mathsf{A}}^{\mathsf{B}} \circ 1 \right).$$

The composition of the right most edges is:

$$\gamma \bullet (1 \circ (1 \times \bar{\iota})) \bullet \left(1 \circ \left(\prod \gamma \times 1\right)\right) \bullet (\bar{\iota} \circ ((\bar{\iota} \circ \bar{\iota}) \times (1 \circ 1))) \tag{2.3.1}$$

$$= \gamma \bullet (\bar{\iota} \circ (1 \times \bar{\iota})) \bullet \left(1 \circ \left(\prod \gamma \times 1\right)\right) \bullet (1 \circ ((\bar{\iota} \circ \bar{\iota}) \times (1 \circ 1))) \tag{2.3.2}$$

$$= \gamma \bullet (\bar{\iota} \circ (\bar{\iota} \times \bar{\iota})) \bullet \left(1 \circ \left(\prod \gamma \times 1\right)\right) \bullet (1 \circ ((\bar{\iota} \circ \bar{\iota}) \times (1 \circ 1))) \tag{2.3.3}$$

$$= \eta_{\mathsf{A}} \bullet \left(1 \circ \left(\prod \gamma \times 1\right)\right) \bullet (1 \circ ((\bar{\iota} \circ \bar{\iota}) \times (1 \circ 1))) \tag{2.3.4}$$

$$= \eta_{\mathsf{A}} \bullet \left(1 \circ \left(\left(\prod \gamma \bullet (\bar{\iota} \circ \bar{\iota})\right) \times (1 \bullet (1 \circ 1))\right)\right) \tag{2.3.5}$$

$$= \eta_{\mathsf{A}} \bullet \left(1 \circ \left(\left(\prod_{a \in \Gamma} \eta\right) \times \left(\prod_{a \notin \Gamma} \eta\right)\right)\right) \tag{2.3.6}$$

$$= \eta_{\mathsf{A}} \bullet \left(1 \circ \prod_{a \in \mathsf{A}^-} \eta^{[a,sa]}_{\mathsf{B} \cap [a,sa]}\right). \tag{2.3.7}$$

Equation 2.3.2 follows from commuting $\bar{\iota}$ with the identity functor 1. Equation 2.3.3 follows because for each $a \in \Gamma$, the map...

$$\gamma : \left(\overline{T_{\mathsf{B} \cap [a,sa]}} \circ \prod_{b \in (\mathsf{B} \cap [a,sa])^-} \overline{T_{[b,sb]}}\right) \Rightarrow T_{[a,sa]}.$$

... cannot possibly have as codomain $T_{[a,sa]} = 1_A$, as the output of γ is always a tensor $\otimes$. Hence we have $T_{[a,sa]} = \overline{T_{[a,sa]}}$ for every $a \in \Gamma$. This means:

$$1 = \bar{\iota} : T_{[a,sa]} \Rightarrow \overline{T_{[a,sa]}}.$$

Equation 2.3.4 is then the assumption that η_{A} is of the form $\gamma \bullet \bar{\iota} \circ \prod \bar{\iota}$. Equation 2.3.5 is repeated applications of interchange of vertical composition. Equation 2.3.6 is just the definition of η and Γ: The first grouping corresponds to η for $a \in \Gamma$ which in turn corresponds to all those η of the form $\gamma \bullet \bar{\iota}$; The second grouping corresponds to all of the other η's, which are thus simply 1.

Thus the commutativity of the above diagram implies Proposition 2.3. We verify the commutativity by verifying the commutativity of each sub diagram: **N, W, S, E**.

N: This commutes trivially.

W: This commutes as a result of commuting $\bar{\iota}$ with 1.

$$
\begin{aligned}
&(2.3.8) && (\gamma \circ (1 \times 1)) \bullet (\bar{\iota} \circ (\bar{\iota} \times \bar{\iota}) \circ (\bar{\iota} \times \bar{\iota})) \\
&(2.3.9) && = (\gamma \circ (\bar{\iota} \times \bar{\iota})) \bullet (\bar{\iota} \circ (\bar{\iota} \times \bar{\iota}) \circ (1 \times 1)) \\
&(2.3.10) && = (1 \circ (\bar{\iota} \times \bar{\iota})) \bullet (\gamma \circ (1 \times 1)) \bullet (\bar{\iota} \circ (\bar{\iota} \times \bar{\iota}) \circ (1 \times 1)) \\
&(2.3.11) && = (1 \circ (\bar{\iota} \times \bar{\iota})) \bullet (\gamma \bullet (\bar{\iota} \circ (\bar{\iota} \times \bar{\iota}))) \circ ((1 \times 1) \bullet (1 \times 1)) \\
&(2.3.12) && = (1 \circ (\bar{\iota} \times \bar{\iota})) \bullet \left(\eta_{\mathsf{A}}^{\mathsf{B}} \circ (1 \times 1)\right).
\end{aligned}
$$

S: Note that every vertex in this square consists of only products of $\otimes$'s (and 1_I's, which again, are ignored). The commutativity of this square is simply the associativity axiom for γ coming from the definition of a lax monoidal category.

E: The eastern square E commutes if and only if the following square commutes:

$$\begin{tikzcd}
\prod\limits_{a\in\Gamma}\left(\overline{T_{\mathsf{B}\cap[a,sa]}}\circ\prod\limits_{b\in(\mathsf{B}\cap[a,sa])^-}\overline{T_{[b,sb]}}\right)\times\prod\limits_{a\notin\Gamma}\left(T_{\mathsf{B}\cap[a,sa]}\circ\prod\limits_{b\in(\mathsf{B}\cap[a,sa])^-}T_{[b,sb]}\right) \arrow[dr, "\prod\gamma\times\prod 1"] \arrow[ddd, "\prod 1\times\prod\bar{\iota}\circ\bar{\iota}"'] & \\
& \prod\limits_{a\in\Gamma}\overline{T_{[a,sa]}}\times\prod\limits_{a\in\Gamma}T_{[a,sa]} \arrow[d, "\prod 1\times\prod\bar{\iota}"] \\
& \prod\limits_{a\in\Gamma}\overline{T_{[a,sa]}}\times\prod\limits_{a\in\Gamma}\overline{T_{[a,sa]}} \\
\prod\limits_{a\in\Gamma}\left(\overline{T_{\mathsf{B}\cap[a,sa]}}\circ\prod\limits_{b\in(\mathsf{B}\cap[a,sa])^-}\overline{T_{[b,sb]}}\right)\times\prod\limits_{a\notin\Gamma}\left(\overline{T_{\mathsf{B}\cap[a,sa]}}\circ\prod\limits_{b\in(\mathsf{B}\cap[a,sa])^-}\overline{T_{[b,sb]}}\right) \arrow[ur, "\prod\gamma\times\prod\gamma"'] &
\end{tikzcd}$$

The square commutes in the first component of the binary product trivially:

$$\prod\gamma\bullet\prod 1=\prod 1\bullet\prod\gamma.$$

It suffices then to show that the square commutes in the second component, and decomposing the product $\prod\limits_{a\notin\Gamma}$, we see it suffices to show that for each $a\notin\Gamma$ we have:

$$\begin{tikzcd}
T_{\mathsf{B}\cap[a,sa]} \circ \prod\limits_{b\in(\mathsf{B}\cap[a,sa])^-} T_{[b,sb]} \arrow[r, "1"] \arrow[d, "\bar\iota\circ\bar\iota"'] & T_{[a,sa]} \arrow[d, "\bar\iota"] \\
\overline{T_{\mathsf{B}\cap[a,sa]}} \circ \prod\limits_{b\in(\mathsf{B}\cap[a,sa])^-} \overline{T_{[b,sb]}} \arrow[r, "\gamma"'] & \overline{T_{[a,sa]}}
\end{tikzcd}$$

Now, $T_{[a,sa]}$ may be a tensor $\otimes^m$, 1_A, or 1_I. If $T_{[a,sa]} = \otimes^m$, the map $T_{\mathsf{B}\cap[a,sa]} \circ \prod T_{[b,sb]}$ may be $1_A \circ \otimes^m$ or $\otimes^m \circ \prod 1_A$. The square in question therefore becomes one of:

$$\begin{tikzcd}
\otimes^m = \otimes^m \circ \prod 1_A \arrow[d, "\iota"'] \arrow[dr, "1_{\otimes^m}"] & \\
\otimes^m \circ \prod \otimes^1 \arrow[r, "\gamma"'] & \otimes^m
\end{tikzcd}
\qquad
\begin{tikzcd}
\otimes^m = 1_A \circ \otimes^m \arrow[d, "\iota"'] \arrow[dr, "1_{\otimes^m}"] & \\
\otimes^1 \circ \otimes^m \arrow[r, "\gamma"'] & \otimes^m
\end{tikzcd}$$

These triangles are precisely those appearing in the unitality axiom of the lax monoidal category, hence commute by definition.

In the case when $T_{[a,sa]} = 1_A$, then the square reduces to the following:

$$\begin{tikzcd}
1_A \circ 1_A \arrow[r, "1"] \arrow[d, "\iota\circ\iota"'] & 1_A \arrow[d, "\iota"] \\
\otimes^1 \circ \otimes^1 \arrow[r, "\gamma"'] & \otimes^1
\end{tikzcd}$$

The unitality axiom in the definition of the lax monoidal category gives in particular that the following triangle commutes:

$$\begin{tikzcd}
\otimes^1 = \otimes^1 \circ 1_A \arrow[d, "\iota"'] \arrow[dr, "1_{\otimes^1}"] & \\
\otimes^1 \circ \otimes^1 \arrow[r, "\gamma"'] & \otimes^1
\end{tikzcd}$$

Commutativity of this triangle in turn implies commutativity of the square above, and consequently of the square $\mathbf{E}$ in question.

Finally if $T_{[a,sa]} = 1_I$, then all four vertices of the square in question are 1_I, all four natural transformations are simply 1, and commutativity follows.

We have been operating under the assumption that $\eta^{\mathsf{B}}_{\mathsf{A}}$ was of the form $\gamma \bullet \bar{\iota}$. We must now consider when it is an identity. First, let us suppose that $\eta^{\mathsf{B}}_{\mathsf{A}} = 1_{1_I}$. Thus $T_{\mathsf{B}} = 1_I$ and hence $A_{0,n} = I$. It is a special fact about $\mathbb{C}$ that if $x_{0,n} = 0$, every 1-face of x must also be 0, and so x is a degeneracy of the 1-simplex 0. This contradicts our assumption that x is non-degenerate. We need not worry about this case.

Now suppose that $\eta^{\mathsf{B}}_{\mathsf{A}} = 1_{1_A}$. This immediately implies that $T_{\mathsf{B}} = 1_A$ and $T_{\mathsf{A}} = 1_A$. Consequently, the products comprising their domains – $\prod_{a \in \mathsf{A}^-} A^{x_{a,sa}}$ and $\prod_{b \in \mathsf{B}^-} A^{x_{b,sb}}$ – each contain only a single A, with the rest I. This then means that every product of functors *into* those products must consist entirely of a product of 1_I's, except for the functor which targets the single A. Hence, there exist unique $a' \in \mathsf{A}^-$ and unique $b' \in \mathsf{B}^-$ such that:

$$T_{[a',sa']} \neq 1_I \neq T_{[b',sb']}.$$

Figure (‡) then reduces to:

$$\begin{tikzcd}
\prod\limits_{i \in [n]^-} A^{x_{i,i+1}} \arrow[rr, "T_{[a',sa']}"] \arrow[d, "T_{[b',sb']}"'] \arrow[rrd, bend left, "T = \bigotimes^{\sum \mathrm{sp}(x)}"'] & & A \arrow[d, "1_A"] \\
A \arrow[rr, "1_A"'] \arrow[rru, bend right, "1_A"] & & A
\end{tikzcd}$$

We must then verify:

$$\eta_{\mathsf{A}} \bullet \left(1 \circ \eta^{[a',sa']}_{\mathsf{B} \cap [a',sa']}\right) = \eta_{\mathsf{B}} \bullet (1 \circ 1).$$

Towards these end we note two further cases: $\eta^{[a',sa']}_{\mathsf{B} \cap [a',sa']} = \gamma \bullet \bar{\iota}$ or $\eta^{[a',sa']}_{\mathsf{B} \cap [a',sa']} = 1_{T_{[a',sa']}}$.

In the first case, it must be that $T_{[a',sa']}$ is a tensor given by the number of A's in its domain, hence, $T_{[a',sa']} = \otimes^{\sum \mathrm{sp}(x)}$. We see then that $\eta^{[a',sa']}_{\mathsf{B}\cap[a',sa']} = \eta_{\mathsf{B}}$ and $\eta_{\mathsf{A}} = 1_{\otimes^{\sum \mathrm{sp}(x)}}$ as a result of the unitality axioms of the lax monoidal category. Hence the equation above is verified. In the second case, it must be that $T_{[a',sa']} = 1_A = T_{[b',sb']}$, and we see by inspection that $\eta_{\mathsf{A}} = \eta_{\mathsf{B}}$ once again verifying the equation.

■

Proposition 2.3.3. *The assignment $(A, \otimes^n, \gamma, \iota) \mapsto \alpha : \mathbb{C} \longrightarrow N_\Delta(N_2\mathsf{Cat})$ classifies lax monoidal categories. That is, given a map α, we can recover the data $(A, \otimes^n, \gamma, \iota)$.*

Proof. Given α, we see that $\alpha_1(1)(\{0,1\}) = A^{1_{0,1}} = A^1 = A$, hence we have recovered the category A. For $n \geq 2$, let $\mu \in \mathbb{C}_n$ be the n-simplex with 1-faces $\mu_{p,q} = 1$ for all $p, q \in [n]$. Then we have:

$$\alpha_n(\mu)(\{0,n\} \subset [n]) = T^\mu = \otimes^{\sum \mathrm{sp}(\mu)} = \otimes^n.$$

Hence we have recovered the n-ary operations $\otimes^n$ for $n \geq 2$. We can recover $\otimes^0$ from the non-degenerate 2-simplex $u : 0 \vee 0 \longrightarrow 1$. We have:

$$\alpha_2(u)(\{0,2\} \subset [2]) = T^u = \otimes^{\sum \mathrm{sp}(u)} = \otimes^0.$$

We can recover $\otimes^1$ from the non-degenerate 3-simplex $l \in \mathbb{C}_3$ defined by the following 1-faces:

$$l_{0,1} = 0, l_{1,2} = 0, l_{2,3} = 1, l_{0,2} = 1, l_{1,3} = 1, l_{0,3} = 1.$$

We have:

$$\alpha_3(l)(\{0,3\} \subset [3]) = T^l = \otimes^{\sum \mathrm{sp}(l)} = \otimes^1.$$

This 3 simplex l will also recover the natural transformation ι:

$$\alpha_3(l)(\{0,3\} \subset \{0,1,3\} \subset [3]) = \eta^l_{\{0,1,3\}} : T^l_{013} \circ \left(T^l_{01} \times T^l_{123}\right) \Rightarrow T^l_{[3]}.$$

As $l_{013} = l_{123} = s_1(1)$, $T^{l_{013}} = T^{l_{123}} = 1_A$, and because $l_{01} = 0$ is a degenerate 1-simplex, $T^{l_{01}} = 1_I$. The definition of $\eta^l_{\{0,1,3\}}$ becomes:

$$\eta^l_{\{0,1,3\}} = \gamma_{1,2} \circ (\iota \times \iota) : 1_A \circ 1_A \Rightarrow \otimes^1.$$

By the unitality axiom for the lax monoidal category, this composite is simply ι.

Finally, as for recovering each $\gamma_{n,k_1,\ldots,k_n}$, when $n \geq 2$ consider the simplex x defined as follows. Off its spine, every 1-face $x_{p,q} = 1$. Its spine, $\{x_{i,i+1}\}$, a finite sequence of 0's and 1's , will have two 0's for each $k_i = 0$, two 0's and a 1 for each $k_i = 1$, and k_i 1's for each $k_i \geq 2$, ordered with k_1's digits first and k_n's last. Let $\hat{k_i} \in \mathbb{N}$ stand for the number of digits in the spine of x associated with k_i, i.e $\hat{0} = 2$, $\hat{1} = 3$, $\hat{k_i} = k_i$ for $k_i \geq 2$. Then $x \in \mathbb{C}_k$ with $k := \sum\limits_{1 \leq i \leq n} \hat{k_i}$. Writing $\hat{j} := \sum\limits_{1 \leq i \leq j-1} \hat{k_i}$, we see that the functor $T^{x_{[\hat{j},\hat{j+1}]}}$ corresponding to the face $x_{[\hat{j},\hat{j+1}]}$ is precisely $\otimes^{k_j}$ by the above. We have therefore:

$$\alpha_k(\mu)\Big(\{0,k\} \subseteq \{0, \hat{k_1}, \hat{k_1} + \hat{k_2}, ..., k\} \subseteq [k]\Big) = \gamma_{n,k_1,\ldots,k_n} : \otimes^n \circ (\otimes^{k_1} \times \ldots \times \otimes^{k_n}) \Rightarrow \otimes^{k_1+\ldots+k_n}.$$

The cases when $n = 0$ or $n = 1$ proceed similarly.

■

This concludes the proof of Proposition 2.1.7.

2.4. **Classifying skew monoidal categories.** In this section, we will prove Proposition 2.1.8: that the nerve $N_s(\mathsf{Cat})$ appearing in the classifcation result of [2] embeds into $N_\Delta(N_2\mathsf{Cat})$.

Definition 2.4.1. The *skew nerve* $N_s(\mathsf{Cat})$ of the monoidal bicategory Cat is the simplicial set defined by the following:

- There is a unique 0-simplex, $*$.
- 1-simplices consist in categories B_{01}.
- 2-simplices consist in functors $B_{012} : B_{01} \times B_{12} \longrightarrow B_{02}$.
- 3-simplices are natural transformations $B_{0123} : B_{013} \circ (B_{012} \times 1) \Rightarrow B_{023} \circ (1 \times B_{123})$:

$$\begin{tikzcd}
B_{01}B_{12}B_{23} \arrow[r, "B_{012} \times 1"] \arrow[d, "1 \times B_{123}"'] & B_{02}B_{23} \arrow[d, "B_{023}"] \\
B_{01}B_{13} \arrow[r, "B_{013}"'] \arrow[ur, Rightarrow, "B_{0123}"'] & B_{03}
\end{tikzcd}$$

- A 4-simplex consists in a quintuple of appropriately formed natural transformations making the following pentagon commute:

$$\begin{tikzcd}
 & B_{014} \circ (1 \times B_{124}) \circ (1 \times 1 \times B_{234}) \arrow[dl, "1 \circ (1 \times B_{1234})"'] \arrow[dr, "B_{0124} \circ 1"] & \\
B_{014} \circ (1 \times B_{134}) \circ (1 \times B_{123} \times 1) \arrow[d, "B_{0134} \circ 1"'] & & B_{024} \circ (B_{012} \times 1) \circ (1 \times 1 \times B_{234}) \arrow[d, equal, "1"] \\
B_{034} \circ (B_{013} \times 1) \circ (1 \times B_{123} \times 1) \arrow[dr, "1 \circ (B_{0123} \times 1)"'] & & B_{024} \circ (1 \times B_{234}) \circ (B_{012} \times 1 \times 1) \arrow[dl, "B_{0234} \circ 1"] \\
 & B_{034} \circ (B_{023} \times 1) \circ (B_{012} \times 1 \times 1) &
\end{tikzcd}$$

- Higher-dimensional simplices are determined by 4-coskeletality.

As in the case of the definition of a monoidal category, the pentagon law above gives rise to a coherence theorem which we now explain. The 2-faces of an n-simplex $B \in N_s(\mathsf{Cat})$ consist in functors for every triple of numbers p, q, r with $0 \leq p < r < q \leq n$:

$$B_{prq} : B_{pr} \times B_{rq} \longrightarrow B_{pq}.$$

Given a subset of indices $\mathsf{C} = \{p = c_0 < ... < c_m = q\}$, we then have a number of composite functors formed of these 2 faces:

$$\prod_{c \in \mathsf{C}^-} B_{c,sc} \longrightarrow B_{pq}.$$

Moreover, we have a potential multitude of natural transformation 3-faces mediating between such composites. For example, the pentagon diagram occurring in the definition of $N_s(\mathsf{Cat})$ shows all of the composite 2-face functors from $\prod_{i \in [4]^-} B_{i,i+1} \longrightarrow B_{0,4}$, and shows all the 3-face natural transformations between them.

In the context of a simplex $B \in N_s(\mathsf{Cat})_n$, the coherence theorem says that there is at most one composite of 3-face natural transformations between composites of 2-face functors. This is precisely the content of the commutativity of the pentagon above, and as in the theorem of [12], commutativity of pentagons gives the result in full generality. Note also that every 2-face functor composite $\prod_{c \in \mathsf{C}^-} B_{c,sc} \longrightarrow B_{p,q}$ is the source of a composite of 3-face transformations with target:

$$m(B_{\mathsf{C}}) := B_{c_0 c_{m-1} c_m} \circ (B_{c_0 c_{m-2} c_{m-1}} \times 1) \circ ... \circ (B_{c_0 c_2 c_3} \times 1) \circ (B_{c_0 c_1 c_2} \times 1).$$

In a 4-simplex $B \in N_s(\mathsf{Cat})_4$, this 2-face functor is just the bottom vertex of the pentagon above, $m(B) = B_{034} \circ (B_{023} \times 1) \circ (B_{012} \times 1 \times 1)$. Combined with the coherence theorem we have the following.

Remark 2.4.2. *Let $B \in N_s(\mathsf{Cat})_n$ with $n \geq 2$, and $\mathsf{C} \subseteq [n]$. For each composite of 2-face functors $T : \prod_{c \in \mathsf{C}^-} B_{c,sc} \longrightarrow B_{p,q}$, there exists a unique natural transformation formed of composites of 3-face transformations $T \Rightarrow m(B_{\mathsf{C}})$.*

Finally, we also have that $m(B) = m(s_i(B))$ for any $B \in N_s(\mathsf{Cat})_n$ with $n \geq 2$ and $s_i : N_s(\mathsf{Cat})_n \longrightarrow N_s(\mathsf{Cat})_{n+1}$ any degeneracy map. Again recalling Lemma 2.2.3, we have:

$$\begin{aligned} m(s_i(B)) =& s_i(B)_{0,n,n+1} \circ s_i(B)_{0,n-1,n} \circ ... \circ s_i(B)_{0,i,i+1} \circ ... \circ s_i(B)_{0,1,2} \\ =& B_{\sigma_i(0,n,n+1)} \circ B_{\sigma_i(0,n-1,n)} \circ ... \circ s'(B_{0,i}) \circ ... \circ B_{\sigma_i(0,1,2)} \\ =& B_{0,n-1,n} \circ B_{0,n-2,n-1} \circ ... \circ 1_{B_{0,i}} \circ ... \circ B_{0,1,2} \\ =& m(B). \end{aligned}$$

Definition 2.4.3. The Skew Nerve Embedding

We will again rely on Proposition 2.2.6. As we tend to write simplices in $N_s(\mathsf{Cat})$ with capital letters B, we will recall that proposition in slighty different terminology. To specify a map $\beta : N_s(\mathsf{Cat}) \longrightarrow N_\Delta(N_2\mathsf{Cat})$, we must specify:

(1) A category for each 1-simplex $B \in N_s(\mathsf{Cat})_1$. We will simply assign each 1-simplex category B to itself, as this satisfies $(1.a)$.

(2) A functor

$$T(B) : \prod_{i \in [n]^-} B_{i,i+1} \longrightarrow B_{0,n}.$$

For each $n \geq 1$ and $B \in N_s(\mathsf{Cat})_n$. Let $T(B) = 1_B$ when $n = 1$ in accordance with $(2.a)$, and otherwise let $T(B) = m(B)$ when $n \geq 2$, as by the above this is in accordance with $(2.b)$.

(3) A natural transformation

$$\eta^B_{\mathsf{C}} : m(B_{\mathsf{C}}) \circ \prod_{c \in \mathsf{C}^-} m(B_{[c,sc]}) \Rightarrow m(B).$$

For each $n \geq 1$, $B \in N_s(\mathsf{Cat})_n$ and $\{0, n\} \subseteq \mathsf{C} \subseteq [n]$. For $1 \leq n \leq 2$, we take $\eta^B_{\mathsf{C}} := 1_B$ in accordance with $(3.a)$. Otherwise, take η^B_{C} to be the unique such transformation formed of composite of 3-faces of B in all cases given by Remark 2.4.2. It is easy to check that this assignment is in accordance with $(3.b)$ given that $m(s_i(B)) = m(B)$.

This assignment gives rise to a map $N_s(\mathsf{Cat}) \longrightarrow N_\Delta(N_2\mathsf{Cat})$ if equation $(\dagger)$ is satisfied for every $n \geq 4$ and $(\{0, n\} \subset \mathsf{A} \subset \mathsf{B} \subset [n])$. This is again the case by the uniqueness clause of 2.4.2.

Therefore by 2.2.6 we have a map $\beta : N_s(\mathsf{Cat}) \longrightarrow N_\Delta(N_2\mathsf{Cat})$ such that:

$$\beta(B)(\{0,1\}) = B_{0,1},\ \beta(B)(\{0,n\} \subseteq [n]) = m(B), \text{ and } \beta(B)(\{0,n\} \subseteq \mathsf{C} \subseteq [n]) = \eta^B_{\mathsf{C}}.$$

Proposition 2.4.4. *The map $\beta_n : N_s(\mathsf{Cat})_n \longrightarrow N_\Delta(N_2\mathsf{Cat})_n$ is injective for each $n \geq 0$. Hence β is a faithful embedding.*

Proof. Let $B, D \in N_s(\mathsf{Cat})_n$ and $\beta_n(B) = \beta_n(D)$. For each subset $\{p \leq r \leq q\} \subset [n]$ we have $B_{pq} = \beta_n(B)(\{p,q\})$ and $B_{prq} = m(B_{prq}) = \beta_n(B)(\{p,q\} \subseteq \{p,r,q\})$, hence B and D have precisely the same 1 and 2-faces.

Note that for $\mathsf{C} = \{c_0 < c_1 < c_2 < c_3\}$, we have that:

$$\beta_n(B)(\{c_0, c_3\} \subset \mathsf{C}) = m(B_{\mathsf{C}}) = B_{c_0c_2c_3} \circ (B_{c_0c_1c_2} \times 1).$$

because β commutes with face maps. We therefore have:

$$\begin{aligned}\beta_n(B)(\{c_0, c_3\} \subset \{c_0, c_1, c_3\} \subset \mathsf{C}) : m(B_{c_0c_1c_3}) \circ (m(B_{c_0c_1}) \times m(B_{c_1c_2c_3})) \Rightarrow m(B_{\mathsf{C}}) \\ = B_{c_0c_1c_2c_3} : B_{c_0c_1c_3} \circ (1 \times B_{c_1c_2c_3}) \Rightarrow B_{c_0c_2c_3} \circ (B_{c_0c_1c_2} \times 1).\end{aligned}$$

This implies that B and D have the same 3-faces. As a result, commutative pentagons of such transformations occur in B exactly when they occur in D, and so the two have the same 4-faces. They have the same k-faces for $k > 4$ by $4-$coskeletality. ■

This concludes the proof of Proposition 2.1.8.

2.5. **Classifying Σ-monoidal categories and general maps in $\mathsf{sSet}(\mathbb{C}, N_\Delta(N_2\mathsf{Cat}))$.** We have seen now that both lax monoidal categories (Section 2.3), skew monoidal categores (Section 2.4), and hence both monoidal and strict monoidal categories, all can be understood as maps in $\mathsf{sSet}(\mathbb{C}, N_\Delta(N_2\mathsf{Cat}))$. The question we turn now to explore is then: what other monoidal-type categories do we find in this simplicial set? We begin with a simple comparison of those β, α, and $\phi : \mathbb{C} \longrightarrow N_\Delta(N_2\mathsf{Cat})$ corresponding to skew monoidal categories, lax monoidal categories, and arbitrary maps respectively. Recalling Propositions 2.2.1 and 2.2.6, we have that these three maps determine and are determined by three types of data:

(1) A category $A^1 = A$, where $A^0 = I$.

(2) A functor

$$T^x : \prod_{i \in [n]^-} A^{x_{i,i+1}} = A^{\sum \mathrm{sp}(x)} \longrightarrow A.$$

for each $n \geq 1$ and $x \in \mathbb{C}_n$ subject to stipulations $(2.a)$ and $(2.b)$ of Proposition 2.2.6.

(3) A natural transformation

$$\eta^x_{\mathsf{C}} : (T^{x_{\mathsf{C}}} \circ \prod_{c \in \mathsf{C}^-} T^{x_{[c,sc]}}) \Rightarrow T^x.$$

for each face $n \geq 1$, $x \in \mathbb{C}_n$, and $\{0, n\} \subseteq \mathsf{C} \subseteq [n]$ subject to (3.a) and (3.b) of Proposition 2.2.6.

And, this data is equivalent to defining a map $\mathbb{C} \longrightarrow N_\Delta(N_2\mathsf{Cat})$ whenever the natural transformations of item (3) satisfy equation (†) of 2.2.6 for every $\{0, n\} \subset \mathsf{A} \subset \mathsf{B} \subset [n]$. Let us then explore these three types of data associated to β, α, and ϕ. Recall the five 2-simplices of $\mathbb{C}$: We have $s_0(0) : 0 \vee 0 \longrightarrow 0$, $s_0(1) : 0 \vee 1 \longrightarrow 1$, $s_1(1) : 1 \vee 0 \longrightarrow 1$, $u : 0 \vee 0 \longrightarrow 1$, and $m : 1 \vee 1 \longrightarrow 1$. Here we write the simplex with 1-faces $x_{0,1}$, $x_{1,2}$, and $x_{0,2}$ as $x_{0,1} \vee x_{1,2} \longrightarrow x_{0,2}$.

(1) The category data associated to each of β, α, and ϕ is just the monoidal-type category associated to the map.

(2) The functor data associated to the skew monoidal category map β is defined by a pair of functors, $\otimes^2 := T^m$, and $\otimes^0 := T^u$. By the stipulations (2.a) and (2.b), this alone gives T^x for every simplex $x \in \mathbb{C}_2$. For an arbitrary $x \in \mathbb{C}_n$, T^x is defined in terms of the image of its 2-faces:

$$T^x := T^{x_{\{0,1,n\}}} \circ (1 \times T^{x_{\{1,2,n\}}}) \circ ... \circ (1 \times T^{x_{\{n-2,n-1,n\}}}).$$

In contrast, the functor data associated to a lax monoidal category map α is defined by functors $\otimes^n := T^\mu$ for every $n \geq 0$ where $\mu \in \mathbb{C}_n$ is given by $\mu_{p,q} = 1$ for all $0 \leq p < q \leq n$. For arbitrary $x \in \mathbb{C}$, $T^x := \otimes^{\sum \mathrm{sp}(x)}$.

Finally, ϕ may specify apriori unrelated functors $T^x : A^{\sum \mathrm{sp}(x)} \longrightarrow A$ for every non-degenerate x. Note that there are a countable infinite number of non-degenerate simplices x with $\sum \mathrm{sp}(x) = n$ for every $n \geq 0$,[24] and thus, ϕ may include the data of an arbitrary (countable) number of n-ary functors for each n.

(3) Key differences in the natural transformation data associated to β, α, and ϕ, can already be seen when considering only 3-simplices $x \in \mathbb{C}_3$. Let us represent such a 3-simplex with 1-faces $x_{p,q}$, $0 \leq p < q \leq 3$ as the diagram:

[24]Take a simplex with $x_{i,i+1} = 1$ for $i \leq n-1$, $x_{i,i+1} = 0$ for $i \geq n$, and $x_{p,q} = 1$ for $q - p \geq 2$. There is one simplex of this form in every dimension $\geq n$, it is non-degenerate, and $\sum \mathrm{sp}(x) = n$.

$$\begin{tikzcd}
x_{0,1}\vee x_{1,2}\vee x_{2,3} \arrow[r] \arrow[d] \arrow[dr, phantom, "x"] & x_{0,1}\vee x_{1,3} \arrow[d] \\
x_{0,2}\vee x_{2,3} \arrow[r] & x_{0,3}
\end{tikzcd}$$

Consider then the pair of simplices l and $r \in \mathbb{C}_3$:

$$\begin{tikzcd}
0\vee 0\vee 1 \arrow[r] \arrow[d] \arrow[dr, phantom, "l"] & 0\vee 1 \arrow[d] \\
1\vee 1 \arrow[r] & 1
\end{tikzcd}
\qquad
\begin{tikzcd}
1\vee 0\vee 0 \arrow[r] \arrow[d] \arrow[dr, phantom, "r"] & 1\vee 1 \arrow[d] \\
1\vee 0 \arrow[r] & 1
\end{tikzcd}$$

Associated to each of β, α, and ϕ is a functor $T^l : A^{\Sigma\,\mathrm{sp}(l)} = IIA = A \longrightarrow A$, as well as functors $T^{l_{\{0,1,2\}}} = T^u$, $T^{l_{\{0,1,3\}}} = T^{s_0(1)} = 1_A$, $T^{l_{\{0,2,3\}}} = T^m$, and $T^{l_{\{1,2,3\}}} = T^{s_0(1)} = 1_A$ corresponding to the four arrows making up the edges of the above squares. There are also the pair of natural transformations:

$$\eta^l_{\{0,2,3\}} : T^{l_{\{0,2,3\}}} \circ \left(T^{l_{\{0,1,2\}}} \times T^{l_{\{2,3\}}}\right) \Rightarrow T^l.$$

$$\eta^l_{\{0,1,3\}} : T^{l_{\{0,1,3\}}} \circ \left(T^{l_{\{0,1\}}} \times T^{l_{\{1,2,3\}}}\right) \Rightarrow T^l.$$

We get similar data for r, and we can represent all of it succinctly in the following two diagrams:

$$\begin{tikzcd}[row sep=huge, column sep=huge]
IIA \arrow[r, "1_I\times T^{l_{\{1,2,3\}}}"] \arrow[d] \arrow[dr, "T^l" description] & IA \arrow[d, "T^{l_{\{0,1,3\}}}"] \arrow[dl, phantom, pos=0.3, "\Swarrow\,\eta^l_{\{0,1,3\}}"] \\
AA \arrow[r, "T^{l_{\{0,2,3\}}}"'] \arrow[ur, phantom, pos=0.3, "\eta^l_{\{0,2,3\}}\,\Nearrow"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=huge, column sep=huge]
AII \arrow[r, "1_A\times T^{r_{\{1,2,3\}}}"] \arrow[d, "T^{r_{\{0,1,2\}}}\times 1_I"'] \arrow[dr, "T^r" description] & AA \arrow[d] \arrow[dl, phantom, pos=0.3, "\Swarrow\,\eta^r_{\{0,1,3\}}"] \\
AI \arrow[r, "T^{r_{\{0,2,3\}}}"'] \arrow[ur, phantom, pos=0.3, "\eta^r_{\{0,2,3\}}\,\Nearrow"] & A
\end{tikzcd}$$

The remaining functors associated to the skew monoidal category map β are defined $T^l := 1_A$ and $T^r := \otimes^2 \circ (1_A \times \otimes^0)$. Naming $\lambda := \eta^l_{0,2,3}$ and $\rho := \eta^r_{0,2,3}$, we get the following diagrams:

$$
\begin{tikzcd}
IIA \arrow[r, "1_I \times 1_A"] \arrow[d, "\otimes^0 \times 1_A"'] \arrow[dr] & IA \arrow[d, "1_A"] \arrow[dl, phantom, "1"] \\
AA \arrow[r, "\otimes^2"'] \arrow[ur, phantom, "\lambda"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
AII \arrow[r, "1_A \times \otimes^0"] \arrow[d, "1_A \times 1_I"'] \arrow[dr] & AA \arrow[d, "\otimes^2"] \arrow[dl, phantom, "1"] \\
AI \arrow[r, "1_A"'] \arrow[ur, phantom, "\rho"] & A
\end{tikzcd}
$$

We see then λ and ρ arise naturally in this context. Associated to the lax monoidal category map α are instead $T^l = \otimes^1 = T^r$ and the following diagrams:

$$
\begin{tikzcd}
IIA \arrow[r, "1_I \times 1_A"] \arrow[d, "\otimes^0 \times 1_A"'] \arrow[dr, "\otimes^1"'] & IA \arrow[d, "1_A"] \arrow[dl, phantom, "\iota"] \\
AA \arrow[r, "\otimes^2"'] \arrow[ur, phantom, "\gamma_{2,0,1}"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
AII \arrow[r, "1_A \times \otimes^0"] \arrow[d, "1_A \times 1_I"'] \arrow[dr, "\otimes^1"'] & AA \arrow[d, "\otimes^2"] \arrow[dl, phantom, "\gamma_{2,1,0}"] \\
AI \arrow[r, "1_A"'] \arrow[ur, phantom, "\iota"] & A
\end{tikzcd}
$$

So we see the transformation ι and two of the many γ transformations arise naturally in this context as well. Finally, for the general map ϕ, there are, apriori no relationships between these various functors and natural transformations, aside from those relationships contributed by the other data of ϕ.

We have seen in particular that arbitrary maps ϕ may include the data of up to a countable number of distinct n-ary functors for each n. Such a monoidal-type category is reminiscent of the following definition of [6] which we present only informally here:

Definition 2.5.1. A Σ-*monoidal category* (A, Σ, γ) consists in a category A, a countable[25] set Σ_k of k-ary functors $A^k \longrightarrow A$ for each $k \geq 0$, and natural isomorphisms γ between each

[25]The definition in [6] considers aribtrary cardinality.

possible composite of functors of the same total arity. Each composition of these natural isomorphisms with the same domain and codomain must be equal.

There is no canonical way of associating a map $\sigma \in \mathsf{sSet}(\mathbb{C}, N_\Delta(N_2\mathsf{Cat}))$ to a Σ-monoidal category (A, Σ, γ) because the definition presents all k-ary functors in Σ_k as morally indistinguishable, whereas the k-ary functors arising in the image of a map $\mathbb{C} \longrightarrow N_\Delta(N_2\mathsf{Cat})$ can be distinguished in many ways, for example, by the dimension of the simplex x mapping to that functor. However, given a surjective function

$$h_k : \{x \mid x \text{ is nondegenerate with dimension } \geq 2 \text{ and } \sum \mathrm{sp}(x) = k\} \longrightarrow \Sigma_k.$$

for each $k \geq 0$, we can define $\sigma : \mathbb{C} \longrightarrow N_\Delta(N_2\mathsf{Cat})$ by Proposition 2.2.6.

Definition 2.5.2. The Σ-Monoidal Classifying Map

(1) Let $A^x := A$ if $x = 1$ and I otherwise.
(2) Let $T^x := h_{\sum \mathrm{sp}(x)}(x) : A^{\sum \mathrm{sp}(x)} \longrightarrow A$ if x is non-degenerate with dimension ≥ 2. Otherwise T^x is defined in accordance with stipulations $(2.a)$ and $(2.b)$.
(3) Let $\eta^x_{\mathsf{C}} : T^{x_{\mathsf{C}}} \circ \prod_{c \in \mathsf{C}^-} T^{x_{[c,sc]}} \Rightarrow T^x$ be the unique γ natural isomorphism guaranteed by the definition of Σ-monoidal category if $n \geq 3$, and 1_{T^x} otherwise, as this is in accordance with $(3.a)$ and $(3.b)$.

The commutativity of equation (†) is implied directly by the commutativity of the isomorphisms γ given in the definition, and hence this data gives rise to a map $\sigma : \mathbb{C} \longrightarrow N_\Delta(N_2\mathsf{Cat})$.

This assignment also classifies Σ-monoidal categories.

Proposition 2.5.3. *The assignment $(A, \Sigma, \gamma) \mapsto \sigma : \mathbb{C} \longrightarrow N_\Delta(N_2\mathsf{Cat})$ classifies Σ-monoidal categories. That is, given the map σ, we can recover the data (A, Σ, γ).*

We get A from item (1), and each map from each Σ_k from item (2), using the fact that h_k is assumed to be surjective. We must therefore only show that *every* natural isomorphism implied by the definition of Σ-monoidal categories can be generated – via horizontal composition, vertical composition, and by product – by the transformations η^x_{C} of item (3). We need a few lemmas which will incidently reveal some additional structure of general maps ϕ.

Lemma 2.5.4. *Let $\phi : \mathbb{C} \longrightarrow N_\Delta(N_2\mathsf{Cat})$ be a general map. Suppose $x \in \mathbb{C}_m$ such that $\sum \mathrm{sp}(x) = 1$. Then there is a transformation $E : 1_A \Rightarrow T^x$ generated by transformations η^y_{C} in the image of ϕ.*

Proof. Let $x \in \mathbb{C}_m$ with $\sum \mathrm{sp}(x) = 1$. Let $x_{i-1,i} = 1$ be the unique such 1-face in $\mathrm{sp}(x)$. If $i \neq 1$, then:

$$\eta^x_{0,1,\ldots,i-2,i,i+1,\ldots,m} : T^{x_{0,1,\ldots,i-2,i,i+1,\ldots,m}} \circ (1_I \times ... \times 1_I \times 1_A \times 1_I \times ... \times 1_I) \Rightarrow T^x.$$

Here, we get $T^{x_{i-2,i-1,i}} = 1_A$ because $x_{i-2,i-1,i}$ is necessarily $s_0(1)$, and hence degenerate. So we get a transformation $T^{x_{0,1,\ldots,i-2,i,i+1,\ldots,m}} \Rightarrow T^x$. On the other hand if $i = 1$ we have:

$$\eta^x_{0,2,\ldots,m} : T^{x_{0,2,\ldots,m}} \circ (1_A \times 1_I \times ... \times 1_I) \Rightarrow T^x.$$

This time $T^{x_{0,1,2}} = 1_A$ because $x_{0,1,2} = s_1(1)$. In each case, the face $x_{0,1,\ldots,i-2,i,i+1,\ldots,m}$ and $x_{0,2,\ldots,m}$ both have a single 1 on the spine, and one fewer 0. By repeating the argument on these faces inductively, we get a composite:

$$E : T^{x_{0,m}} = 1_A \Rightarrow T^x.$$

■

Lemma 2.5.5. *Let $\phi : \mathbb{C} \longrightarrow N_\Delta(N_2\mathsf{Cat})$ be a general map. Let $x \in \mathbb{C}_m$ with $\sum \mathrm{sp}(x) = 0$ and x not a degeneracy of 0. Then there is a transformation $E : T^u \Rightarrow T^x$ generated by transformations η^y_{C} in the image of ϕ.*

Proof. We consider two cases. If $x_{0,1,2} = s_0(0)$ then $T^{x_{0,2,3,\ldots,n}} = T^x$, and we are finished by induction. If on the other hand $x_{0,1,2} = u$, then $\sum \mathrm{sp}(x_{0,2,3,\ldots,n}) = 1$ and by the previous lemma, we have a transformation $F : 1_A \Rightarrow T^{x_{0,2,3,\ldots,n}}$. We have:

$$\eta^x_{0,2,3,\ldots,n} : T^{x_{0,2,3,\ldots,n}} \circ (T^u \times 1_I \times ... \times 1_I) \Rightarrow T^x$$
$$E := \eta^x_{0,2,3,\ldots,n} \bullet (E \circ 1) : T^u \Rightarrow T^x.$$

■

Lemma 2.5.6. *Let $\phi : \mathbb{C} \longrightarrow N_\Delta(N_2\mathsf{Cat})$ be a general map. For $n \geq 2$, let $\mu \in \mathbb{C}_n$ be given by $\mu_{p,q} = 1$ for all $0 \leq p < q \leq n$, and let T^μ be the associated n-ary functor. Let $x \in \mathbb{C}_m$ with $\sum \mathrm{sp}(x) = n$. Then there is a transformation $E : T^x \Rightarrow T^\mu$ generated by transformations η^y_{C} in the image of ϕ.*

Proof. Now, let $\mu \in \mathbb{C}_n$ as above, and let $x \in \mathbb{C}_m$ with $m \geq n \geq 2$ with $\sum \mathrm{sp}(x) = n$. Suppose that $x_{i-1,i} = 1$ if and only if $i \in I = \{i_1, ..., i_n\}$. Then the faces $\sum \mathrm{sp}(x_{0,...,i_1}) = \sum \mathrm{sp}(x_{i_1,...,i_2,}) = ... = \sum \mathrm{sp}(x_{i_{n-1},...,i_n-1,i_n,...,m}) = 1$. Thus we have:

$$E_1 \times ... \times E_n : 1 \Rightarrow T^{x_{0,...,i_1}} \times ... \times T^{x_{i_{n-1},...,m}}.$$

We also have that $x_{0,i_1,...,i_{n-1},m} = \mu$ because μ is the unique n-simplex with $\sum \mathrm{sp}(\mu) = n$. We have then:

$$\eta^x_{0,i_1,i_2,...,i_{n-1},m} : T^{x_{0,i_1,...,i_{n-1},m}} \circ (T^{x_{0,...,i_1}} \times ... \times T^{x_{i_{n-1},...,m}}) \Rightarrow T^x.$$

$$E := \eta^x_{0,i_1,i_2,...,i_{n-1},m} \bullet (1 \circ E_1 \times ... \times E_n) : (T^\mu \circ 1) \Rightarrow (T^\mu \circ T^{x_{0,...,i_1}} \times ... \times T^{x_{i_{n-1},...,m}}) \Rightarrow T^x.$$

■

Proof. (of Proposition 2.5.3) Given a map σ in the image of the assignment, we must show that every natural isomorphism implied by the definition of Σ-monoidal categories can be generated by transformations η^x_{C} in the image of σ. As σ assigns to each η^x_{C} a natural isomorphism, the above three lemmas imply that for every $k \geq 0$, each pair of elements $f, f' \in \Sigma_k$ are isomorophic to one another via isomorphisms in the image of σ. Let $\mu^n \in \mathbb{C}_n$ denote the unique n-simplex with $\sum \mathrm{sp}(x) = n$, i.e $\mu^n_{p,q} = 1$ for all $0 \leq p < q \leq n$. Now, given $n = n_1 + ... + n_k$, $f_i \in \Sigma_{n_i}, f \in \Sigma_k$, and $g \in \Sigma_n$, we must show that the isomorphism $\gamma : f \circ (f_1 \times ... \times f_k) \Rightarrow g$ can be generated by η^x_{C}'s. Without loss of generality we can assume that $f_i = T^{\mu^{n_i}}$ if $n_i \geq 2$, $f_i = 1_A$ if $n_i = 1$, and $f_i = T^u$ if $n_i = 0$ and similarly for f and g. Let x be the simplex with two consecutive 0's on its spine for each $n_i = 0$, with

n_i consecutive 1's on its spine for each $n_i \geq 1$, and all other 1-faces $x_{p,q} = 1$. Let the total dimension of x be m. Then choosing C with $\{0, m\} \subseteq \mathsf{C} \subseteq [m]$ so that consecutive indicies of C correspond to the part of $\mathrm{sp}(x)$ corresponding to each n_i, we get the transformation:

$$\eta^x_{\mathsf{C}} : f \circ (f_{n_1} \times ... \times f_{n_k}) \Rightarrow T^x.$$

Note that $x_{\mathsf{C}} = \mu^k$ as it has only 1's on its spine, and so $T^{x_{\mathsf{C}}} = f$. Finally, as $\sum \mathrm{sp}(x) = n$, we get a transformation $E : T^x \Rightarrow g = T^{\mu^n}$. Together these show the isomorphism between these n-ary functors to be generated by σ.

■

This concludes the proof of Propostion 2.1.9

We might therefore think of general maps $\phi : \mathbb{C} \longrightarrow N_\Delta(N_2\mathsf{Cat})$ as specifying a category along with an arbitrary (countable) number of n-ary functors for every n which are not necessarily isomorphic. They are, however, related by an intricate web of natural transformations which mirror aspects of the structure of $\mathbb{C}$. Perhaps one might therefore think of the data associated to an abitrary map $\mathbb{C} \longrightarrow N_\Delta(N_2\mathsf{Cat})$ along with the equation (†) as indicating the necessary structure and coherence needed to go about weakening the definition of Σ-monoidal category to not require natural isomorphisms between all functors with the same total arity.

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