THE POWER OF ABSTRACT THINKING

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What does "abstract" mean?

The process of abstraction:

extracting & studying the
essential features of a situation.

"Abstract" $\approx$ "conceptual"

What does "abstract" not mean?

Removed from reality.

(Perspectives vary....)

The march of abstraction

(a history in insults)

SET THEORY:

Kronecker on Cantor —

"charlatan", "corrupter of youth"

ABSTRACT ALGEBRA:

Gordan on Hilbert —

"that is not mathematics — that is theology"

ALGEBRA IN PHYSICS:

Heisenberg's matrix mechanics

The "Gruppenpest"

CATEGORY THEORY:

"General abstract nonsense"

Moore & Seiberg (math. physics):

"We will need to use some very simple notions of category theory, an esoteric subject noted for its difficulty and irrelevance".

The power of abstract thinking: quick examples

- Vector spaces: bring together linear equations & geometry. Clarifies duality ($V \cong V^*$ vs. $V \cong V^{**}$).

- Algebraic geometry: can do number theory geometrically. Harnesses the power of our visual sense.

- Applied maths:
  - increasingly abstract maths applied to physics
  - logic applied to computer science

COMPRESSION: "it's just ..."

WHAT IS

A

SPACE?

Metric spaces

There is the traditional definition.

But ...

- We don't always want the axiom

$$d(x,y)=0 \Rightarrow x=y.$$

E.g. {integrable functions}, $d(f,g)=\int |f(x)-g(x)|\,dx$.

- We don't always want the axiom

$$d(x,y)=d(y,x).$$

E.g. $d(x,y)$ = work done to get from x to y in a mountainous region,

or Hausdorff metric.

Topological spaces

There is the traditional definition.

But for some purposes, it's not suitable.

Some alternatives:

Locales

In many topo spaces, can tell points apart by open sets.

Idea: forget the points & concentrate on open sets!

Given a topo space X, get:

- a set $\mathcal{O}(X) = \{\text{open subsets of } X\}$
- an operation $(U, V) \mapsto U \cap V$ on $\mathcal{O}(X)$
- an operation $(U_i)_{i \in I} \mapsto \bigcup_{i \in I} U_i$ on $\mathcal{O}(X)$,

satisfying axioms.

Abstracting: a locale consists of

- a set $\mathcal{O}$
- an operation $(U, V) \mapsto U \wedge V$ on $\mathcal{O}$
- an operation $(U_i)_{i \in I} \mapsto \bigvee_{i \in I} U_i$ on $\mathcal{O}$,

satisfying axioms.

Topological spaces, continued

Spaces as rings

In many topo spaces, can tell points apart by real-valued functions.

Given topo space X, get comm. ring

$$C(X) = \{\text{continuous functions } X \to \mathbb{R}\}$$

where $(f \cdot g)(x) = f(x) \cdot g(x)$, etc.

For compact Hausdorff spaces, can recover X from $C(X)$! So can regard a compact Hausdorff space as a <u>special ring</u>.

Non-commutative geometry

Imagine some kind of "space" X whose $C(X)$ is a <u>non</u>-commutative ring...

Spaces as algebras

Slogan:

GEOMETRY IS DUAL TO ALGEBRA

The duality: e.g.

given continuous map $X \xrightarrow{f} Y$, get

$$\begin{array}{ccc} \mathcal{O}(X) & \longleftarrow & \mathcal{O}(Y) \\ f^{-1}U & \leftarrow\!\shortmid & U \end{array}$$

and

$$\begin{array}{ccc} C(X) & \longleftarrow & C(Y) \\ \left(\begin{array}{c} X \\ \downarrow{\scriptstyle pf} \\ \mathbb{R} \end{array}\right) & \leftarrow\!\shortmid & \left(\begin{array}{c} Y \\ \downarrow{\scriptstyle p} \\ \mathbb{R} \end{array}\right) \end{array}.$$

Wilder shores

Topology – probe a topo space with continuous real-valued functions

Differential geometry – probe a smooth manifold with differentiable real-valued functions

Algebraic geometry – probe a variety/scheme with polynomial functions.

Now abstract <u>this</u>! Given a space, you can collect together all the ways of probing it ("sheaves").

This forms a structure ....

... topos theory, latest non-comm geometry ...

HUMAN ASPECTS

Innocence

- Re-examining basic things
  (e.g. notions of "space")

- Escaping corruption

- Not noticing that things are supposed
  to be hard

The turn-offs of mathematical culture

- Over-specialization

- Possessiveness

- Dismissiveness

  ("pathological", "technical", "trivial")

Courage

- Pursuing analogies

- Not being afraid to look slow/stupid

HOW TO

COUNT

Counting finite sets

Write $|X|$ for the number of elements of a finite set X.

Then

$$|\emptyset| = 0$$

$$|\{\bullet\}| = 1$$

$$|X \times Y| = |X| \times |Y|$$

disjoint union

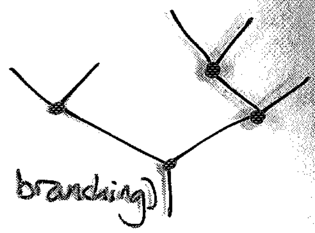
$$|X + Y| = |X| + |Y|$$

$$|X \cup Y| = |X| + |Y| - |X \cap Y|$$

(among other things).

How many trees are there?

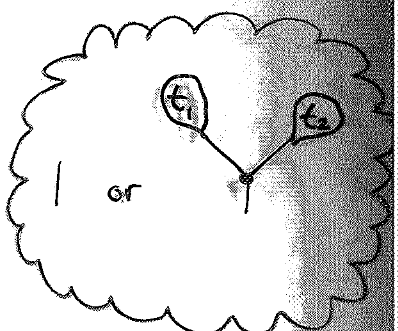
Let $\quad T = \{\text{trees}\}$, $\quad$ e.g.



(binary branching)

Then

$$T \cong 1 + (T \times T).$$



So

$$|T| = 1 + |T|^2$$

so

$$|T| = \tfrac{1}{2}(1 \pm \sqrt{3}\, i) = e^{\pm \pi i/3}.$$

(No. of elements is a complex number!)

Hence $|T|^7 = |T|$, suggesting that $T^7 \cong T$.

This really is true!

How to count the points of a space

E.g.

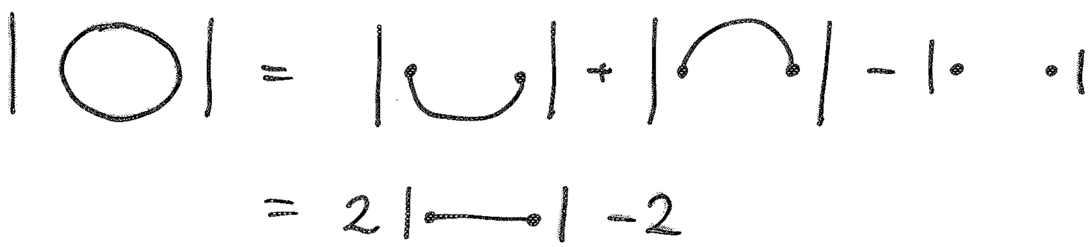
so $\quad |[0,1]| = |[0,1]| + |[0,1]| - |\bullet|$

$\quad = 2|[0,1]| - 1$

so $\quad |[0,1]| = 1.$

So $[0,1]$ "has one point". Huh?

Similarly:



$= 0,$

so the circle "has 0 points".

This "number of points" is the Euler characteristic!
It's an excellent way to "count" a space.

Fundamental questions

- What are the common features?
- What's the right level of generality?
- How do things relate to one another?