# Magnitude cohomology of graphs

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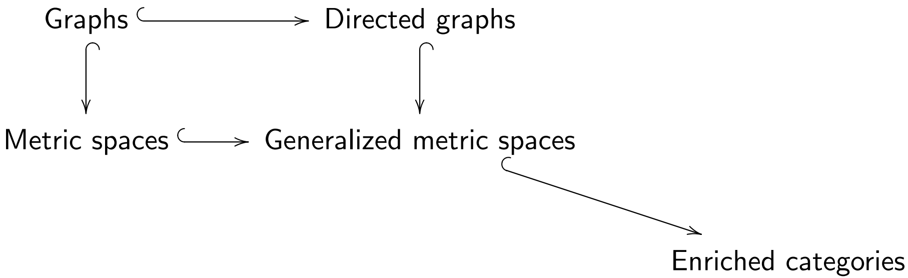
# Plan

1. Definitions
2. Properties of magnitude cohomology
3. Magnitude cohomology is a complete invariant

# *1. Definitions*

# Levels of generality

Magnitude cohomology is defined for enriched categories...




...but today, I'll only discuss it for metric spaces, especially graphs.

Graph means finite undirected graph, seen as a metric space with the shortest path metric.

# Magnitude cohomology

Let $X$ be a metric space.

For integers $n \geq 0$ and real $\ell \geq 0$, put

$$MC_{n,\ell}(X) = \mathbb{Z}\cdot\{(x_0, \ldots, x_n) \; : \; d(x_0, x_1) + \cdots + d(x_{n-1}, x_n) = \ell, \; x_0 \neq \cdots \neq x_n\}$$

(the free abelian group on $\{(n, \ell)\text{-paths}\}$).

Define $\partial$: $MC_{n,\ell}(X) \to MC_{n-1,\ell}(X)$ as $\sum_{0<i<n}(-1)^i \partial_i$, where

$$\partial_i(x_0, \ldots, x_n) = \begin{cases} (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) & \text{if } x_i \text{ is between } x_{i-1} \text{ and } x_{i+1} \\ 0 & \text{otherwise.} \end{cases}$$

This defines an $\mathbb{R}^+$-graded chain complex $MC_{*,*}(X)$.

Let $MC^{*,*}(X)$ be the dual complex:

$$MC^{n,\ell}(X) = \mathrm{Hom}(MC_{n,\ell}(X), \mathbb{Z}).$$

The magnitude cohomology $MH^{*,*}(X)$ is the homology of $MC^{*,*}(X)$.
It is an $(\mathbb{N} \times \mathbb{R}^+)$-graded abelian group.

# Cohomology forms a graded ring

Given

$$\phi \in \left\{\text{functions } \{(n,\ell)\text{-paths}\} \to \mathbb{Z}\right\} \subseteq MC^{n,\ell}(X),$$
$$\psi \in \left\{\text{functions } \{(p,k)\text{-paths}\} \to \mathbb{Z}\right\} \subseteq MC^{p,k}(X),$$

define

$$\phi \cdot \psi \in \left\{\text{functions } \{(n+p,\ell+k)\text{-paths}\} \to \mathbb{Z}\right\} \subseteq MC^{n+p,\ell+k}(X)$$

by

$$(\phi \cdot \psi)(z_0, \dots, z_{n+p}) = \phi(z_0, \dots, z_n)\psi(z_n, \dots, z_{n+p})$$

if

$$d(z_0, z_1) + \cdots + d(z_{n-1}, z_n) = \ell, \quad d(z_n, z_{n+1}) + \cdots + d(z_{n+p-1}, z_{n+p}) = k,$$

or as 0 otherwise.

This induces an $(\mathbb{N} \times \mathbb{R}^+)$-graded ring structure on $MH^{*,*}(X)$.

# *2. Properties of magnitude cohomology*

# The product is noncommutative

For a finite metric space with at least two points, the product on magnitude cohomology is never commutative.

Hepworth shows that when $\mathcal{V}$ is a *cartesian* monoidal category ($\otimes = \times$), the product on magnitude cohomology of $\mathcal{V}$-enriched categories is commutative.

But our $\mathcal{V}$ is $([0, \infty), +, 0)$, which is not cartesian ($+ \neq \max$), and our product is not commutative.

# Magnitude cohomology is trivial when homology is

Hepworth establishes a short exact sequence

$$0 \to \mathrm{Ext}(MH_{n-1,\ell}(X), \mathbb{Z}) \to MH^{n,\ell}(X) \to \mathrm{Hom}(MH_{n,\ell}(X), \mathbb{Z}) \to 0$$

relating magnitude homology $MH_{*,*}$ and magnitude cohomology $MH^{*,*}$.

In particular, if $MH_{n,\ell}(X) = 0$ for all $(n, \ell) \neq (0, 0)$ then the same is true for $MH^{n,\ell}(X)$.

We know that $MH_{*,*}(X)$ is trivial when $X$ is a convex subset of $\mathbb{R}^N$.

So the same is true of $MH^{*,*}(X)$: all convex sets have trivial magnitude cohomology.

## *3. Magnitude cohomology is a complete invariant*

# The recovery theorem

Hepworth shows that for finite metric spaces $X$ (and slightly more generally), the graded ring $MH^{*,*}(X)$ determines $X$ completely — up to isometry.

Warning This is false for general metric spaces: consider convex sets.

So, magnitude cohomology defines an embedding

$$MH^{*,*}\colon \{\text{iso classes of finite metric spaces}\} \hookrightarrow \{\text{iso classes of graded rings}\}.$$

General question Is viewing finite metric spaces as special graded rings a helpful viewpoint?

E.g. is it helpful for graphs?

# The recovery theorem: digging deeper

In fact, to recover the space $X$ from the graded ring $MH^{*,*}(X)$, you only need the part of the ring in homological degrees $n = 0, 1$.

Observation For an $\mathbb{N}$-graded ring $R = (R^n)_{n \in \mathbb{N}}$:

- $R^0$ is a ring
- $R^1$ is an $(R^0, R^0)$-bimodule, i.e. has compatible left and right actions by $R^0$.

So taking the degree 0 and degree 1 parts gives a forgetful functor

$$\textbf{GrRing} = (\mathbb{N}\text{-graded rings}) \to \textbf{Bimod},$$

where an object of **Bimod** is a ring $A$ together with an $(A, A)$-bimodule.

# The recovery theorem for graphs

Let $G$ be a graph (undirected and finite), seen as a metric space: points are vertices, distances are shortest path lengths.

Write $V(G)$ for the set of vertices and $E(G) \subseteq V(G) \times V(G)$ for the set of *directed* edges.

A short calculation shows:

$$MH^{0,\ell}(G) = \begin{cases} \mathbb{Z}^{V(G)} & \text{if } \ell = 0 \\ 0 & \text{otherwise,} \end{cases}$$

$$MH^{1,\ell}(G) = \begin{cases} \mathbb{Z}^{E(G)} & \text{if } \ell = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then:

- $MH^{0,0}(G)$ is a ring by pointwise multiplication
- $MH^{1,1}(G)$ is an $(MH^{0,0}(G), MH^{0,0}(G))$-bimodule via

$$(\alpha \cdot \phi \cdot \beta)(x, y) = \alpha(x)\phi(x, y)\beta(y)$$

$(\alpha, \beta \in MH^{0,0}(G),\ \phi \in MH^{1,1}(G),\ (x, y) \in E(G))$.

# How to recover a graph from its cohomology ring

- From the ring $MH^{0,0}(G) = \mathbb{Z}^{V(G)}$, we can extract $V(G)$ as the set of primitive idempotents.
- From the bimodule $MH^{1,1}(G) = \mathbb{Z}^{E(G)}$, we can extract $E(G)$: for primitive idempotents (vertices) $x$ and $y$, the set

$$x \cdot MH^{1,1}(G) \cdot y \subseteq MH^{1,1}(G)$$

  is nontrivial if and only if $(x, y) \in E(G)$.

# The recovery theorem for graphs: functorial version

For graphs, Hepworth's theorem states that the functor

$$MH^{*,*}\colon \mathbf{Graph}^{\mathrm{op}} \to \mathbf{GrRing}$$

is injective on isomorphism classes of objects.

He also notes (essentially) that even after composing with the forgetful functor **GrRing** $\to$ **Bimod**, the composite

$$MH^{*,*}\colon \mathbf{Graph}^{\mathrm{op}} \to \mathbf{Bimod}$$

is still injective on iso classes of objects.

In fact, we can go even further and show that the functor

$$MH^{*,*}\colon \mathbf{Graph}^{\mathrm{op}} \to \mathbf{Bimod}$$

is full and faithful, which implies injectivity on iso classes. So:

> Magnitude homology sets up a dual equivalence between **Graph** and a full subcategory of **Bimod**.

# Where next?

Hepworth proves substantial, detailed results on presentations of the magnitude cohomology ring, for various classes of graph.

But one can also look in different directions. For instance:

- Does magnitude cohomology similarly set up a dual equivalence between the category of finite metric spaces and some category of algebras?
- When graphs are seen as bimodules, how should we understand the higher magnitude (co)homology groups?
- When **Graph** is embedded into $\mathbf{Bimod}^{\mathrm{op}}$ or $\mathbf{GrRing}^{\mathrm{op}}$, what becomes of the various graph products $(\times, \square, \boxtimes, \ldots)$?
- Are any network-inspired operations on graphs usefully viewed in terms of their cohomology bimodules or graded rings?
- What is the magnitude cohomology of a random graph?