Doing without diagrams

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Here's the problem. You're dealing with groups, or rings, or Lie algebras, or Kapranov–Kontsevich–Kuratowski algebras, or whatever. You have a proof that some equation holds in all groups or rings or . . . . Now you want to conclude that it holds in all *internal* groups or rings or . . . . Do you really have to draw lots of huge diagrams? Or is there some general principle telling you that because the equation holds in set-based algebras, it must also hold in internal algebras?

The point of this note is to show that yes: there is such a general principle. There is a magic wand. By uttering the right words (which begin 'ah, but . . . ') you can conclude that your apparently set-based proof is *actually* a proof valid in all internal situations.

Warning There are surely limitations to this principle. I'm not going to attempt to state them.

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1 'Ordinary' elements

The basic observation is this. Fix a one-point set, 1. Then for sets X, there is a natural one-to-one correspondence between elements of X and maps $1 \longrightarrow X$.

This observation is trivial, but central to what follows. I'll now make some further trivial-but-important observations.

Let X be a set. Given an element x of X, write $\overline{x}$ for the corresponding function $1 \longrightarrow X$ (whose image is $\{x\}$). Now suppose that we have an element

x of X and a map $f : X \longrightarrow Y$ of sets. The composite map

$$1 \xrightarrow{\overline{x}} X \xrightarrow{f} Y$$

has image $\{f(x)\}$, so

$$f \circ \overline{x} = \overline{f(x)}.$$

Since there's a natural one-to-one correspondence between elements of X and maps $1 \longrightarrow X$, you might dare to write $\overline{x}$ as x. The equation then reads

$$f \circ x = f(x).$$

Some people write composites $f \circ e$ as simply fe. Some (rather fewer) people write $f(x)$ as simply fx. The equation then reads

$$fx = fx.$$

So these various abbreviations are compatible with each other.

Another way to say this is that under the correspondence between elements and maps from 1, evaluation corresponds to composition.

All of these things are facts, like them or not. But anyone who works with an isomorphism for long enough winds up regarding the isomorphic things as really *the same*. So we're led to the point of view that the elements of a set X *are* the maps $1 \longrightarrow X$. Elements are a special case of maps; evaluation is a special case of composition.

2 Generalized elements

In non-mathematical speech, 'elements' are the basic things, the building blocks, the fundamentals. (Think of Euclid's *Elements*, or elements in chemistry.) The usage of 'generalized element' follows that tradition.

Generalized elements are, of course, going to be more general than ordinary elements, which are also called 'points'. Knowing about the points of a structure is almost never enough—in fact, the only subject where it *is* enough is set theory. (The whole idea of sets is that they're nothing more than collections of points.) If you're studying a group, you don't just want to know about its points, i.e. its (ordinary) elements; you need to know the multiplication too. In projective geometry, the basic things are not only the points but also the lines, planes, etc., and the incidence relations between them; those are what might be called the 'elements' of the subject, in the tradition described above.

Let $\mathcal{A}$ be a category and $X \in \mathcal{A}$. A **generalized element** of X is simply a map in $\mathcal{A}$ with codomain X. To be more specific, we call a map $x : S \longrightarrow X$ an **element of X of shape S** or an **S-element** of X, and write $x \in_S X$.

Examples

- In **Set**, the 1-elements of a set are just its ordinary elements.

- In **Top**, the 1-elements of a space (where 1 is a one-point space) are its points. Write $\Delta^n \in \mathbf{Top}$ for the standard topological n-simplex; then the Δ^n-elements of a space X are the n-simplices in X. Sometimes it's useful to think of S-elements as 'figures of shape S'.

- In **Group**, the 1-elements of a group G (where 1 is the trivial group) are very boring: there's just one of them. The $\mathbb{Z}$-elements of G correspond one-to-one with the ordinary elements of G: consider where $1 \in \mathbb{Z}$ is mapped to. The $(\mathbb{Z}/7\mathbb{Z})$-elements of G are the ordinary elements of G of order 1 or 7, for the same kind of reason.

- Fix a topological space B and let $\mathbf{VB}_B$ be the category of real vector bundles over B. Write L for the trivial line bundle over B, i.e. $B \times \mathbb{R}$ with its projection to B. Then in $\mathbf{VB}_B$, the L-elements of a vector bundle X are exactly the global sections of X.

In **Set**, a map $X \longrightarrow Y$ can be evaluated at an ordinary element of X, producing an ordinary element of Y. We learned above that when ordinary elements are thought of as generalized elements of shape 1, this evaluation is simply composition of functions.

In any category $\mathcal{A}$ and for any $S \in \mathcal{A}$, a map $f : X \longrightarrow Y$ can be composed with an S-element $x : S \longrightarrow X$ of X, producing an S-element fx of Y. It does no harm to think of fx as 'f evaluated at x', by analogy with the case $\mathcal{A} = \mathbf{Set}$, or to write fx as $f(x)$.

Tautologously, all that matters about a map of sets is what it does to ordinary elements of the domain. In other words, if $f, g : X \longrightarrow Y$ are two maps of sets and $f(x) = g(x)$ for all ordinary elements x of X, then $f = g$. An analogous statement is true for general categories and generalized elements:

Lemma 2.1 *Let $\mathcal{A}$ be a category and $f, g : X \longrightarrow Y$ maps in $\mathcal{A}$. Then*

$$f = g \iff fx = gx \text{ for all generalized elements } x \text{ of } X.$$

Proof '$\Rightarrow$' is immediate. For '$\Leftarrow$', take x to be the identity map $1_X : X \longrightarrow X$. □

Question for class discussion: do objects of a category have elements? In *Categories for the Working Mathematician*, Mac Lane characterizes category theory with some phrase such as 'living without elements'. Maybe that's borne of the observation that in many categories, such as **AbGp** or **Group**, maps from the terminal object 1 are trivial, so the *obvious* generalization of the notion of element isn't useful. Lawvere's generalized elements seem to be a satisfactory substitute appropriate for general category theory. However, they're a big generalization: even in the motivating category, **Set**, generalized elements are more general than ordinary elements.

3 Products

Let's pretend we know almost no category theory, and let's try to figure out what it might mean to take the 'product' of two objects of a category.

We know what it means to take the product of two *sets*. Let X and Y be sets; then an element of $X \times Y$ is an element of X together with an element of Y.

Now let $\mathcal{A}$ be a category and let X and Y be objects of $\mathcal{A}$. Given the observations of the last paragraph, we'd want it to be the case that for any $S \in \mathcal{A}$, an S-element of $X \times Y$ amounts to an S-element of X together with an S-element of Y. This says that for any $S \in \mathcal{A}$, the maps $S \longrightarrow X \times Y$ correspond one-to-one with pairs of maps $(S \longrightarrow X, S \longrightarrow Y)$. And give or take some details, that's the standard definition of product in a category.

If X and Y are sets, x is an element of X, and y is an element of Y, then we write (x, y) for the corresponding element of $X \times Y$. Similarly, let $\mathcal{A}$ be a category and $S \in \mathcal{A}$. If X and Y are objects of $\mathcal{A}$, x is an S-element of X, and y is an S-element of Y, then we write (x, y) for the corresponding element of $X \times Y$. In other words, given maps $x : S \longrightarrow X$ and $y : S \longrightarrow Y$, we write

$$(x, y) : S \longrightarrow X \times Y$$

for the corresponding map into $X \times Y$.

(Don't confuse this with another 'product construction': maps $f : X' \longrightarrow X$ and $g : Y' \longrightarrow Y$ give rise to a map $X' \times Y' \longrightarrow X \times Y$. This is really the observation that product is functorial: you can take the product of *maps* as well as objects. So this map is written $f \times g$. Similarly, if we were doing *tensor* products then we'd write it as $f \otimes g : X' \otimes Y' \longrightarrow X \otimes Y$.)

Example Let $\mathcal{A}$ be a category with products, $X \in \mathcal{A}$, and $\mu : X \times X \longrightarrow X$ in $\mathcal{A}$. Let $x, y \in_S X$, for some S. What could the expression '$\mu(x, y)$' mean?

Well, we've just made the convention that (x, y) is the S-element of $X \times X$ corresponding to the elements $x \in_S X$ and $y \in_S Y$. And we saw earlier that a map can be 'evaluated' at any S-element of its domain, producing an S-element of its codomain; thus, $\mu(x, y)$ is the S-element of X obtained by evaluating μ at (x, y). In ordinary categorical language, it's just the composite of the maps μ and (x, y).

Special case: if $\mathcal{A}$ is the category of sets and x, y are ordinary elements of X then $\mu(x, y)$ is the ordinary element of X obtained by evaluating the function μ at the pair $(x, y) \in X \times X$.

When we're working in **Set** and μ is being thought of as a binary operation, everyone agrees that it's harmless to write $x \cdot y$ or xy instead of $\mu(x, y)$. It's equally harmless to do this for generalized elements of an arbitrary category, and later I'll do just that. If that ever gives you vertigo, you can of course just go through and replace every occurrence of 'xy' with '$\mu(x, y)$'.

Example Let $\mathcal{A}$ be a category with products and let $f : X \longrightarrow X'$ and $g : Y \longrightarrow Y'$ be maps in $\mathcal{A}$. As noted above, they give rise to a map

$$X \times Y \xrightarrow{f \times g} X' \times Y'.$$

Let $x \in_S X$ and $y \in_S Y$, for some $S \in \mathcal{A}$. Then

$$(f \times g)(x, y) = (f(x), g(y)) \in_S X' \times Y'$$

—exercise. (To do it, you'll first need to make explicit what $f \times g$ actually *is*.)

Example Again, let $\mathcal{A}$ be a category with products. Let $\mu : X \times X \longrightarrow X$ in $\mathcal{A}$, and consider the composite map

$$\nu = \left(X \times X \times X \xrightarrow{\mu \times 1_X} X \times X \xrightarrow{\mu} X \right).$$

I claim that if $S \in \mathcal{A}$ and $x, y, z \in_S X$ then

$$\nu(x, y, z) = (xy)z.$$

This is clear for ordinary elements.

Proof: by definition, the left-hand side is the composite $\mu \circ (\mu \times 1_X) \circ (x, y, z)$. By the previous example, this is equal to $\mu \circ (\mu(x, y), 1_X(z))$. But since xy is an abbreviation for $\mu(x, y)$ and $1_X(z) = 1_X \circ z = z$, that's the right-hand side. (To apply the previous example, take 'f' to be μ, 'g' to be 1_X, 'x' to be (x, y), and 'y' to be z. I'm pretending that the threefold product $X \times X \times X$ is *equal* to $(X \times X) \times X$; of course it's not, and strictly speaking the expression $((x, y), z)$ should appear in the argument somewhere.)

Example Again, let $\mathcal{A}$ be a category with products and $\mu : X \times X \longrightarrow X$ in $\mathcal{A}$. I claim that the diagram

$$\begin{tikzcd}
X \times X \times X \arrow[r, "1_X \times \mu"] \arrow[d, "\mu \times 1_X"'] & X \times X \arrow[d, "\mu"] \\
X \times X \arrow[r, "\mu"'] & X
\end{tikzcd}$$

commutes if and only if

$$(xy)z = x(yz) \text{ for all } S \in \mathcal{A} \text{ and } x, y, z \in_S X.$$

(Another way of expressing that condition is '$(xy)z = x(yz)$ for all generalized elements x, y, z of X of the same shape'. If you don't say that they're all of the same shape, the statement's meaningless: xy is undefined.)

Proof: by Lemma 2.1, the diagram commutes if and only if

$$\mu(\mu \times 1_X)t = \mu(1_X \times \mu)t$$

for all generalized elements t of $X \times X \times X$—in other words, for all $S \in \mathcal{A}$ and for all S-elements t of $X \times X \times X$. As observed above, an S-element of $X \times X \times X$ is a triple (x, y, z) of S-elements of X. Hence the diagram commutes if and only if

$$\mu(\mu \times 1_X)(x, y, z) = \mu(1_X \times \mu)(x, y, z)$$

for all $S \in \mathcal{A}$ and $x, y, z \in_S X$. But by the last example, the left-hand side of this equation is $(xy)z$. Similarly, the right-hand side is $x(yz)$.

Definitions without diagrams We've just seen how a diagram can be replaced by an equation. This means that we can write down definitions of internal algebraic structures without using diagrams.

For example, a **group** (or **internal group**) in a category $\mathcal{A}$ with products is an object X together with maps

$$\mu : X \times X \longrightarrow X, \qquad \iota : X \longrightarrow X, \qquad \eta : 1 \longrightarrow X$$

such that

$$(xy)z = x(yz), \qquad ex = x = xe, \qquad x^{-1}x = e = xx^{-1}$$

whenever $S \in \mathcal{A}$ and $x, y, z \in_S X$.

(Here, as usual, xy means $\mu(x, y)$, x^{-1} means $\iota(x)$, and e is the value of η at the unique S-element of the terminal object 1. To put it another way, xy is the composite of μ with $S \xrightarrow{(x,y)} X \times X$, and x^{-1} is the composite of ι with $S \xrightarrow{x} X$, and e is the composite of η with $S \xrightarrow{!} 1$. It's useful to keep in mind that 1 is the product of no things; it's a nullary product, and can be treated in just the same way as binary etc. products.)

Why is that equivalent to the usual definition of group in $\mathcal{A}$? (By the 'usual definition' I mean a definition like that above, but with the equations replaced by certain commutative diagrams.) Well, the last example showed that the associativity equation $(xy)z = x(yz)$ is equivalent to the usual commutative square, and the same kind of thing works for the other equations.

Presenting the definition of internal group in this way allows us to avoid the process of translating the familiar group axioms into diagrams. This process is both automatic and tedious, so it's both possible and desirable to avoid it.

Replacing a diagram by an equation isn't *always* a positive step: diagrams can sometimes be topologically or geometrically suggestive, as in the pentagon/associator story. But there are times when it's useful. One of them is doing *proofs* without diagrams...

4 Proofs without diagrams

Or: 'how to stare at a proof in **Set** and see that it's valid in any category'.

Let's use the theory of groups as an example. A simple equation that holds in all groups is

$$(xy)^{-1} = y^{-1}x^{-1}. \tag{1}$$

This can be proved as follows, writing e for the identity:

$$\begin{aligned}(xy)^{-1} &= (xy)^{-1}e = (xy)^{-1}(xx^{-1}) = (xy)^{-1}((xe)x^{-1}) \\ &= (xy)^{-1}((x(yy^{-1}))x^{-1}) = (xy)^{-1}(((xy)y^{-1})x^{-1}) \\ &= (xy)^{-1}((xy)(y^{-1}x^{-1})) = ((xy)^{-1}(xy))(y^{-1}x^{-1}) \\ &= e(y^{-1}x^{-1}) = y^{-1}x^{-1}.\end{aligned}$$

(Basically the argument is this: $(xy)^{-1} = (xy)^{-1}xyy^{-1}x^{-1} = y^{-1}x^{-1}$. But I've done it in full detail, assuming nothing more than the group axioms. See the remarks below.)

Now let's prove the analogue of (1) for internal groups. This says that if (X, μ, ι, η) is a group in some category $\mathcal{A}$ with products, then the diagram

$$\begin{tikzcd} X \times X \arrow[rr, "\mu"] \arrow[d, "\mathrm{sym}"'] & & X \arrow[d, "\iota"] \\ X \times X \arrow[r, "\iota \times \iota"'] & X \times X \arrow[r, "\mu"'] & X \end{tikzcd}$$

commutes. You might prove this by drawing a huge version of this diagram, dividing it judiciously into many polygons (one for each equality in the proof above), noting that each polygon commutes, and concluding that the whole diagram commutes. Or, you could do the following.

First note that, just as in the example in the previous section, this diagram commutes if and only if (1) holds for all $S \in \mathcal{A}$ and S-elements x, y of X. So all we have to do is show that each of the nine equalities in the proof above is valid for S-elements. Take the first one, $(xy)^{-1} = (xy)^{-1}e$. In the last section we listed the axioms for an internal group, and one of them was $xe = e$ for all S-elements x of X. So replacing 'x' by $(xy)^{-1}$, we see that the first step is valid. The other steps are also applications of the group axioms, so hold for analogous reasons.

And that's it! You're done. Proof finished.

I wrote out the proof of (1) in full and excruciating detail, to make the principles clear. Of course, we usually take the associativity axiom for granted and allow ourselves to write expressions like xyz, knowing that it doesn't matter how you bracket it. This step-skipping is just as valid for generalized elements as ordinary elements, and I'll do it from now on.

For another example, let's take a more complicated identity in the theory of groups. Define the commutator $[-,-]$ by $[x, y] = xyx^{-1}y^{-1}$, and conjugation $(-)^{(-)}$ by $x^y = yxy^{-1}$. Then in ordinary groups, we have

$$[xy, z] = [y, z]^x[x, z]. \tag{2}$$

Proof:

$$[xy, z] = xyz(xy)^{-1}z^{-1} = xyzy^{-1}x^{-1}z^{-1} = x[y, z]zx^{-1}z^{-1} = x[y, z]x^{-1}[x, z] = [y, z]^x[x, z].$$

To state the analogue for internal groups, we first have to define commutator and conjugacy. Again, let (X, μ, ι, η) be a group in a category $\mathcal{A}$ with products. We define the commutator $\kappa : X \times X \longrightarrow X$ as the composite

$$X\times X \xrightarrow{\Delta\times\Delta} X\times X\times X\times X \xrightarrow{1_X\times\mathrm{sym}\times 1_X} X\times X\times X\times X \xrightarrow{1_{X\times X}\times\iota\times\iota} X\times X\times X\times X \xrightarrow{\mu\times\mu} X\times X \xrightarrow{\mu} X$$

and conjugation $\gamma : X \times X \longrightarrow X$ as another composite. (There's no harm in writing κ and γ as $[-,-]$ and $(-)^{(-)}$.) We then draw a diagram involving κ and γ, whose commutativity is the analogue of equation (2).

The definition of κ is set up exactly so that $\kappa(x, y) = xyx^{-1}y^{-1}$ for all generalized elements x, y of X of the same shape. That's the acid test: if it weren't the case, we'd have defined κ wrong. Similarly, we define γ in whatever way makes it true that $\gamma(x, y) = yxy^{-1}$ for all generalized elements x, y of X of the same shape. That's why I didn't bother writing down the definition of γ: because as soon as I'd done it, I would have said 'all that matters about γ is that $\gamma(x, y) = yxy^{-1}$ for all $S \in \mathcal{A}$ and $x, y \in_S X$'.

Similarly, I didn't bother drawing the diagram analogous to (2), because as soon I'd drawn it I would have said 'all that matters about this diagram is that its commutativity is equivalent to the statement that (2) holds for all generalized elements x, y, z of X of the same shape'. That's how we know we've got the diagram right.

To prove the commutativity of this undrawn diagram, all we have to do is show that each step of the proof of (2) is valid for generalized elements. So, take $S \in \mathcal{A}$ and $x, y, z \in_S X$. The first step of the proof, $[xy, z] = xyz(xy)^{-1}z^{-1}$, comes from the definition of κ (as in the last-but-one paragraph). The second step comes from the equation $(xy)^{-1}y^{-1}x^{-1}$, which we proved above. And so on. So the diagram commutes; and that's the magic wand.