

How to measure almost anything

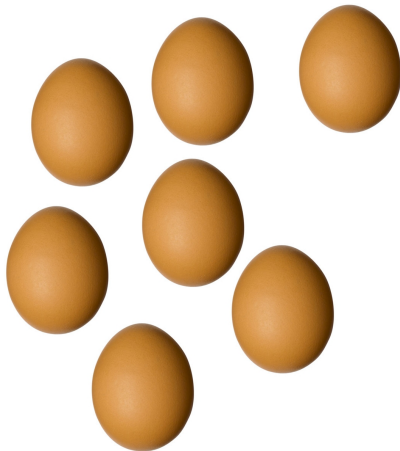
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⋮

How to count eggs



How to count eggs

$$\text{size} \left(\begin{array}{c} \text{[7 brown eggs]} \end{array} \right) = 7$$

How to count eggs

What counts as '1'? Let's **declare**: $\text{size}(\text{🥚}) = 1$.

How to count eggs

What counts as '1'? Let's **declare**: $\text{size}(\text{🥚}) = 1$.

Other rules:

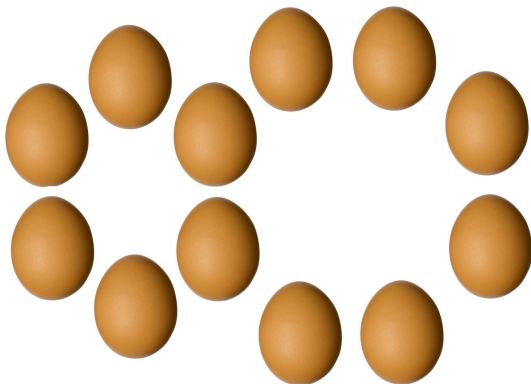
- $\text{size}(\text{ }) = 0$

How to count eggs

What counts as '1'? Let's **declare**: $\text{size}(\text{●}) = 1$.

Other rules:

- $\text{size}(\quad) = 0$

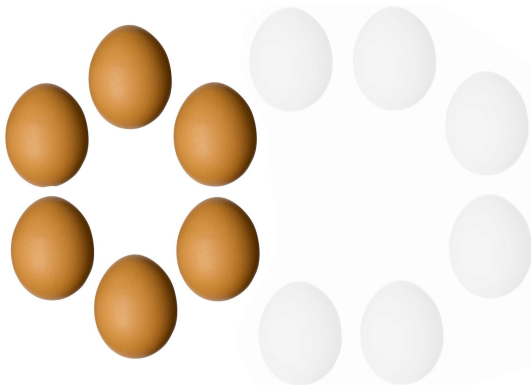


How to count eggs

What counts as '1'? Let's **declare**: $\text{size}(\text{●}) = 1$.

Other rules:

- $\text{size}(\quad) = 0$

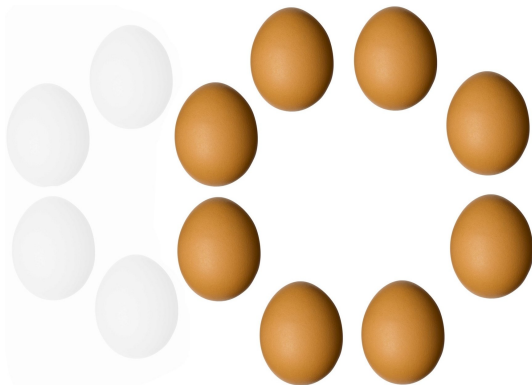


How to count eggs

What counts as '1'? Let's **declare**: $\text{size}(\text{●}) = 1$.

Other rules:

- $\text{size}(\text{ }) = 0$

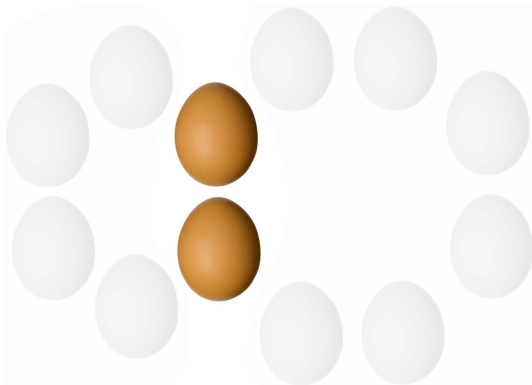


How to count eggs

What counts as '1'? Let's **declare**: $\text{size}(\text{●}) = 1$.

Other rules:

- $\text{size}(\quad) = 0$



How to count eggs

What counts as '1'? Let's **declare**: $\text{size}(\text{🥚}) = 1$.

Other rules:

- $\text{size}(\quad) = 0$

- $\text{size}(\text{🥚🥚🥚🥚🥚🥚🥚🥚🥚🥚})$

How to count eggs

What counts as '1'? Let's **declare**: $\text{size}(\text{🥚}) = 1$.

Other rules:

- $\text{size}(\quad) = 0$

- $\text{size}\left(\begin{array}{c} \text{🥚} \text{🥚} \text{🥚} \text{🥚} \text{🥚} \text{🥚} \text{🥚} \text{🥚} \text{🥚} \end{array}\right) = \text{size}\left(\begin{array}{c} \text{🥚} \text{🥚} \text{🥚} \text{🥚} \text{🥚} \end{array}\right)$

How to count eggs

What counts as '1'? Let's **declare**: $\text{size}(\text{🥚}) = 1$.

Other rules:

- $\text{size}(\text{ }) = 0$

- $\text{size}(\text{🥚🥚🥚🥚🥚🥚🥚🥚}) = \text{size}(\text{🥚🥚🥚🥚}) + \text{size}(\text{🥚🥚🥚🥚🥚🥚})$

How to count eggs

What counts as '1'? Let's **declare**: $\text{size}(\text{●}) = 1$.

Other rules:

- $\text{size}(\quad) = 0$

- $\text{size} \left(\begin{array}{c} \text{●} \text{ ●} \text{ ●} \text{ ●} \text{ ●} \text{ ●} \text{ ●} \text{ ●} \text{ ●} \end{array} \right) = \text{size} \left(\begin{array}{c} \text{●} \text{ ●} \text{ ●} \text{ ●} \end{array} \right) + \text{size} \left(\begin{array}{c} \text{●} \text{ ●} \text{ ●} \text{ ●} \text{ ●} \text{ ●} \end{array} \right) - \text{size} \left(\begin{array}{c} \text{●} \text{ ●} \end{array} \right)$

How to count eggs

What counts as '1'? Let's **declare**: $\text{size}(\text{●}) = 1$.

Other rules:

- $\text{size}(\quad) = 0$

- $\text{size} \left(\begin{array}{c} \text{●} \text{●} \text{●} \text{●} \text{●} \text{●} \text{●} \text{●} \\ \text{●} \text{●} \text{●} \text{●} \text{●} \end{array} \right) = \text{size} \left(\begin{array}{c} \text{●} \text{●} \text{●} \text{●} \\ \text{●} \text{●} \text{●} \end{array} \right) + \text{size} \left(\begin{array}{c} \text{●} \text{●} \text{●} \text{●} \text{●} \text{●} \text{●} \\ \text{●} \text{●} \text{●} \end{array} \right) - \text{size} \left(\begin{array}{c} \text{●} \text{●} \\ \text{●} \end{array} \right)$
 $\text{size}(A \cup B) = \text{size}(A) + \text{size}(B) - \text{size}(A \cap B)$

How to count eggs

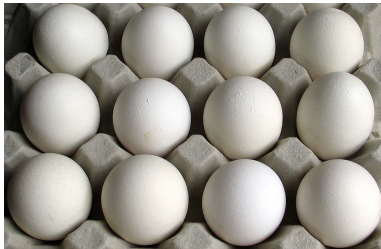
What counts as '1'? Let's **declare**: $\text{size}(\text{●}) = 1$.

Other rules:

- $\text{size}(\quad) = 0$

- $\text{size}\left(\begin{array}{c} \text{●} \text{●} \text{●} \text{●} \text{●} \text{●} \text{●} \text{●} \end{array}\right) = \text{size}\left(\begin{array}{c} \text{●} \text{●} \text{●} \text{●} \end{array}\right) + \text{size}\left(\begin{array}{c} \text{●} \text{●} \text{●} \text{●} \text{●} \text{●} \end{array}\right) - \text{size}\left(\begin{array}{c} \text{●} \text{●} \end{array}\right)$

$$\text{size}(A \cup B) = \text{size}(A) + \text{size}(B) - \text{size}(A \cap B)$$



How to count eggs

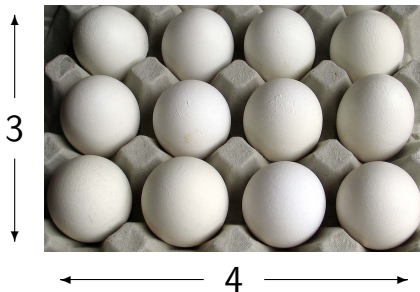
What counts as '1'? Let's **declare**: $\text{size}(\text{egg}) = 1$.

Other rules:

- $\text{size}(\text{empty set}) = 0$

- $\text{size}\left(\begin{array}{c} \text{9 eggs in a ring} \end{array}\right) = \text{size}\left(\begin{array}{c} \text{5 eggs in a cluster} \end{array}\right) + \text{size}\left(\begin{array}{c} \text{4 eggs in a ring} \end{array}\right) - \text{size}\left(\begin{array}{c} \text{2 eggs} \end{array}\right)$

$$\text{size}(A \cup B) = \text{size}(A) + \text{size}(B) - \text{size}(A \cap B)$$



How to count eggs

What counts as '1'? Let's **declare**: $\text{size}(\text{egg}) = 1$.

Other rules:

- $\text{size}(\text{empty set}) = 0$

- $$\text{size}\left(\begin{array}{c} \text{9 eggs in a ring} \end{array}\right) = \text{size}\left(\begin{array}{c} \text{5 eggs in a ring} \end{array}\right) + \text{size}\left(\begin{array}{c} \text{5 eggs in a ring} \end{array}\right) - \text{size}\left(\begin{array}{c} \text{2 eggs in a ring} \end{array}\right)$$
$$\text{size}(A \cup B) = \text{size}(A) + \text{size}(B) - \text{size}(A \cap B)$$

- $\text{size}\left(\begin{array}{c} \text{12 eggs in a 3x4 grid} \end{array}\right)$

How to count eggs

What counts as '1'? Let's **declare**: $\text{size}(\text{●}) = 1$.

Other rules:

- $\text{size}(\quad) = 0$

- $$\text{size}\left(\begin{array}{c} \text{●} \text{ ●} \text{ ●} \text{ ●} \text{ ●} \\ \text{●} \text{ ●} \text{ ●} \text{ ●} \text{ ●} \end{array}\right) = \text{size}\left(\begin{array}{c} \text{●} \text{ ●} \text{ ●} \\ \text{●} \text{ ●} \text{ ●} \end{array}\right) + \text{size}\left(\begin{array}{c} \text{●} \text{ ●} \text{ ●} \text{ ●} \text{ ●} \\ \text{●} \text{ ●} \text{ ●} \text{ ●} \end{array}\right) - \text{size}\left(\begin{array}{c} \text{●} \\ \text{●} \end{array}\right)$$
$$\text{size}(A \cup B) = \text{size}(A) + \text{size}(B) - \text{size}(A \cap B)$$

- $$\text{size}\left(\begin{array}{c} \text{●} \text{ ●} \text{ ●} \text{ ●} \text{ ●} \\ \text{●} \text{ ●} \text{ ●} \text{ ●} \text{ ●} \\ \text{●} \text{ ●} \text{ ●} \text{ ●} \text{ ●} \end{array}\right) = \text{size}(\text{●} \text{ ●} \text{ ●} \text{ ●} \text{ ●})$$

How to count eggs

What counts as '1'? Let's **declare**: $\text{size}(\text{●}) = 1$.

Other rules:

- $\text{size}(\quad) = 0$

- $$\text{size}\left(\begin{array}{c} \text{●} \text{ ●} \text{ ●} \text{ ●} \text{ ●} \\ \text{●} \text{ ●} \text{ ●} \text{ ●} \text{ ●} \end{array}\right) = \text{size}\left(\begin{array}{c} \text{●} \text{ ●} \text{ ●} \\ \text{●} \text{ ●} \text{ ●} \end{array}\right) + \text{size}\left(\begin{array}{c} \text{●} \text{ ●} \text{ ●} \text{ ●} \text{ ●} \\ \text{●} \text{ ●} \text{ ●} \text{ ●} \end{array}\right) - \text{size}\left(\begin{array}{c} \text{●} \\ \text{●} \end{array}\right)$$
$$\text{size}(A \cup B) = \text{size}(A) + \text{size}(B) - \text{size}(A \cap B)$$

- $$\text{size}\left(\begin{array}{c} \text{●} \text{ ●} \text{ ●} \text{ ●} \text{ ●} \\ \text{●} \text{ ●} \text{ ●} \text{ ●} \text{ ●} \\ \text{●} \text{ ●} \text{ ●} \text{ ●} \text{ ●} \end{array}\right) = \text{size}(\text{●} \text{ ●} \text{ ●} \text{ ●} \text{ ●}) \times \text{size}\left(\begin{array}{c} \text{●} \\ \text{●} \\ \text{●} \end{array}\right)$$

How to count eggs

What counts as '1'? Let's **declare**: $\text{size}(\text{●}) = 1$.

Other rules:

- $\text{size}(\quad) = 0$

- $$\text{size} \left(\begin{array}{c} \text{●} \text{●} \text{●} \text{●} \text{●} \\ \text{●} \text{●} \text{●} \text{●} \text{●} \end{array} \right) = \text{size} \left(\begin{array}{c} \text{●} \text{●} \text{●} \\ \text{●} \text{●} \text{●} \end{array} \right) + \text{size} \left(\begin{array}{c} \text{●} \text{●} \text{●} \text{●} \text{●} \\ \text{●} \text{●} \text{●} \text{●} \end{array} \right) - \text{size} \left(\begin{array}{c} \text{●} \\ \text{●} \end{array} \right)$$
$$\text{size}(A \cup B) = \text{size}(A) + \text{size}(B) - \text{size}(A \cap B)$$

- $$\text{size} \left(\begin{array}{ccccc} \text{●} & \text{●} & \text{●} & \text{●} & \text{●} \\ \text{●} & \text{●} & \text{●} & \text{●} & \text{●} \\ \text{●} & \text{●} & \text{●} & \text{●} & \text{●} \end{array} \right) = \text{size}(\text{●} \text{●} \text{●} \text{●} \text{●}) \times \text{size} \left(\begin{array}{c} \text{●} \\ \text{●} \\ \text{●} \end{array} \right)$$
$$\text{size}(A \times B) = \text{size}(A) \times \text{size}(B)$$

How to measure clothing

How to measure clothing

What counts as '1'? Let's **declare**: $\text{size} \left(\text{img} \right) = 1.$

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How to measure clothing

What counts as '1'? Let's **declare**: $\text{size} \left(\text{img} \right) = \text{size} \left(\text{img} \right) = 1.$

Now use the same rules as before! Some consequences:

How to measure clothing

What counts as '1'? Let's **declare**: $\text{size} \left(\text{img1} \right) = \text{size} \left(\text{img2} \right) = 1.$

Now use the same rules as before! Some consequences:

- $\text{size} \left(\text{img3} \text{ } \text{img4} \right)$

How to measure clothing

What counts as '1'? Let's **declare**: $\text{size}(\text{shirt}) = \text{size}(\text{trousers}) = 1$.

Now use the same rules as before! Some consequences:

- $\text{size}(\text{shirt} + \text{trousers}) = \text{size}(\text{shirt}) + \text{size}(\text{trousers}) - \text{size}(\text{empty})$

How to measure clothing

What counts as '1'? Let's **declare**: $\text{size}(\text{shirt}) = \text{size}(\text{pants}) = 1$.

Now use the same rules as before! Some consequences:

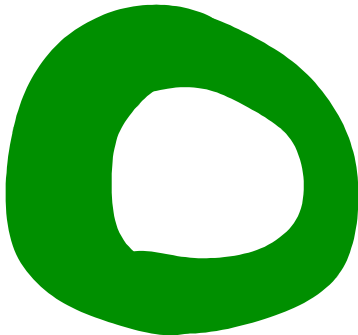
- $$\begin{aligned}\text{size}(\text{shirt} + \text{pants}) &= \text{size}(\text{shirt}) + \text{size}(\text{pants}) - \text{size}(\text{empty}) \\ &= 1 + 1 - 0 = 2\end{aligned}$$

How to measure clothing

What counts as '1'? Let's **declare**: $\text{size}(\text{circle}) = \text{size}(\text{trunk}) = 1$.

Now use the same rules as before! Some consequences:

- $$\begin{aligned}\text{size}(\text{cylinder} + \text{trunk}) &= \text{size}(\text{cylinder}) + \text{size}(\text{trunk}) - \text{size}(\text{circle}) \\ &= 1 + 1 - 0 = 2\end{aligned}$$



How to measure clothing

What counts as '1'? Let's **declare**: $\text{size}(\text{circle}) = \text{size}(\text{trunk}) = 1$.

Now use the same rules as before! Some consequences:

- $\text{size}(\text{cylinder} + \text{trunk}) = \text{size}(\text{cylinder}) + \text{size}(\text{trunk}) - \text{size}(\text{circle})$
 $= 1 + 1 - 0 = 2$



How to measure clothing

What counts as '1'? Let's **declare**: $\text{size}(\text{solid green blob}) = \text{size}(\text{green pants}) = 1$.

Now use the same rules as before! Some consequences:

- $\text{size}(\text{green shirt and green pants}) = \text{size}(\text{green shirt}) + \text{size}(\text{green pants}) - \text{size}(\text{empty})$
 $= 1 + 1 - 0 = 2$



How to measure clothing

What counts as '1'? Let's **declare**: $\text{size}(\text{green blob}) = \text{size}(\text{green pants}) = 1$.

Now use the same rules as before! Some consequences:

- $$\begin{aligned} \text{size}(\text{green blob} + \text{green pants}) &= \text{size}(\text{green blob}) + \text{size}(\text{green pants}) - \text{size}(\text{empty}) \\ &= 1 + 1 - 0 = 2 \end{aligned}$$



How to measure clothing

What counts as '1'? Let's **declare**: $\text{size}(\text{solid blob}) = \text{size}(\text{pair of pants}) = 1.$

Now use the same rules as before! Some consequences:

- $\text{size}(\text{solid blob} + \text{pair of pants}) = \text{size}(\text{solid blob}) + \text{size}(\text{pair of pants}) - \text{size}(\text{empty})$
 $= 1 + 1 - 0 = 2$
- $\text{size}(\text{ring}) = \text{size}(\text{C-shape}) + \text{size}(\text{C-shape}) - \text{size}(\text{two dots})$

How to measure clothing

What counts as '1'? Let's **declare**: $\text{size}(\text{solid green blob}) = \text{size}(\text{green pants}) = 1.$

Now use the same rules as before! Some consequences:

- $$\begin{aligned}\text{size}(\text{green blob} + \text{green pants}) &= \text{size}(\text{green blob}) + \text{size}(\text{green pants}) - \text{size}(\text{empty}) \\ &= 1 + 1 - 0 = 2\end{aligned}$$
- $$\begin{aligned}\text{size}(\text{green ring}) &= \text{size}(\text{green C}) + \text{size}(\text{green C}) - \text{size}(\text{green colon}) \\ &= 1 + 1 - 2 = 0\end{aligned}$$

How to measure clothing

What counts as '1'? Let's **declare**: $\text{size}(\text{solid disk}) = \text{size}(\text{solid pair of pants}) = 1.$

Now use the same rules as before! Some consequences:

- $$\begin{aligned}\text{size}(\text{solid disk} + \text{solid pair of pants}) &= \text{size}(\text{solid disk}) + \text{size}(\text{solid pair of pants}) - \text{size}(\text{empty set}) \\ &= 1 + 1 - 0 = 2\end{aligned}$$

- $$\begin{aligned}\text{size}(\text{annulus}) &= \text{size}(\text{solid disk}) + \text{size}(\text{solid pair of pants}) - \text{size}(\text{pair of pants with a hole}) \\ &= 1 + 1 - 2 = 0\end{aligned}$$

- $$\text{size}(\text{hollow sphere}) = \text{size}(\text{solid sphere}) + \text{size}(\text{solid pair of pants}) - \text{size}(\text{solid pair of pants})$$

hollow

How to measure clothing

What counts as '1'? Let's **declare**: $\text{size}(\text{solid blob}) = \text{size}(\text{solid pants}) = 1.$

Now use the same rules as before! Some consequences:

- $$\begin{aligned}\text{size}(\text{solid blob} + \text{solid pants}) &= \text{size}(\text{solid blob}) + \text{size}(\text{solid pants}) - \text{size}(\text{empty}) \\ &= 1 + 1 - 0 = 2\end{aligned}$$

- $$\begin{aligned}\text{size}(\text{solid ring}) &= \text{size}(\text{solid C}) + \text{size}(\text{solid C}) - \text{size}(\text{two dots}) \\ &= 1 + 1 - 2 = 0\end{aligned}$$

- $$\begin{aligned}\text{size}(\text{hollow sphere}) &= \text{size}(\text{hollow sphere with line}) + \text{size}(\text{hollow sphere with line}) - \text{size}(\text{hollow ring}) \\ &= \text{size}(\text{solid sphere}) + \text{size}(\text{solid sphere}) - \text{size}(\text{hollow ring})\end{aligned}$$

hollow

How to measure clothing

What counts as '1'? Let's **declare**: $\text{size}(\text{solid blob}) = \text{size}(\text{solid pants}) = 1.$

Now use the same rules as before! Some consequences:

- $$\begin{aligned}\text{size}(\text{solid blob} + \text{solid pants}) &= \text{size}(\text{solid blob}) + \text{size}(\text{solid pants}) - \text{size}(\text{empty}) \\ &= 1 + 1 - 0 = 2\end{aligned}$$

- $$\begin{aligned}\text{size}(\text{solid ring}) &= \text{size}(\text{solid C}) + \text{size}(\text{solid C}) - \text{size}(\text{two dots}) \\ &= 1 + 1 - 2 = 0\end{aligned}$$

- $$\begin{aligned}\text{size}(\text{hollow sphere}) &= \text{size}(\text{hollow sphere with line}) + \text{size}(\text{hollow sphere with line}) - \text{size}(\text{hollow ring}) \\ &= \text{size}(\text{solid sphere}) + \text{size}(\text{solid sphere}) - \text{size}(\text{hollow ring}) \\ &= 1 + 1 - 0 = 2\end{aligned}$$

How to measure clothing

What counts as '1'? Let's **declare**: $\text{size} \left(\text{solid_blob} \right) = \text{size} \left(\text{solid_pants} \right) = 1.$

Now use the same rules as before! Some consequences:

- $$\begin{aligned} \text{size} \left(\text{solid_blob} \text{ } \text{solid_pants} \right) &= \text{size} \left(\text{solid_blob} \right) + \text{size} \left(\text{solid_pants} \right) - \text{size} \left(\text{ } \right) \\ &= 1 + 1 - 0 = 2 \end{aligned}$$

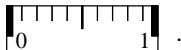
- $$\begin{aligned} \text{size} \left(\text{solid_ring} \right) &= \text{size} \left(\text{solid_C} \right) + \text{size} \left(\text{solid_C} \right) - \text{size} \left(\text{solid_colon} \right) \\ &= 1 + 1 - 2 = 0 \end{aligned}$$

- By similar calculations, $\text{size} \left(\text{outline_hat} \right) = 1$, $\text{size} \left(\text{outline_skirt} \right) = 0$,

$$\text{size} \left(\text{outline_jacket} \right) = -1, \text{size} \left(\text{outline_sweater} \right) = -2.$$

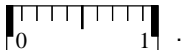
How to measure your paintwork

We'll need a ruler, say of length 1 cm:



How to measure your paintwork

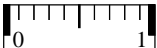
We'll need a ruler, say of length 1 cm:



A mathematician would use a segment of the number line: $0 \text{ --- } 1$.

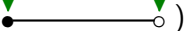
But should we include the endpoints, 0 and 1?

How to measure your paintwork

We'll need a ruler, say of length 1 cm:  .

A mathematician would use a segment of the number line: $0 \text{ --- } 1$.

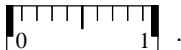
But should we include the endpoints, 0 and 1?

Including just one endpoint () gives a good ruler:

$$\begin{array}{c} \text{1 cm} \\ \bullet \text{ --- } \circ \\ \cup \\ \bullet \text{ --- } \circ \end{array} \quad = \quad \begin{array}{c} \text{2 cm} \\ \bullet \text{ --- } \circ \end{array}$$

How to measure your paintwork

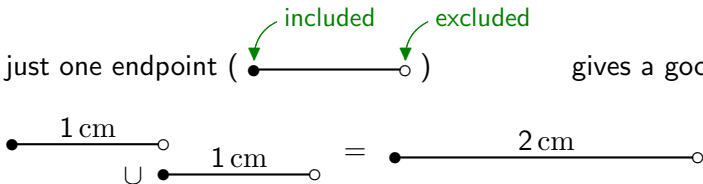
We'll need a ruler, say of length 1 cm:



A mathematician would use a segment of the number line: $0 \text{ --- } 1$.

But should we include the endpoints, 0 and 1?

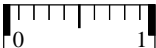
Including just one endpoint ($\bullet \text{ --- } \circ$) gives a good ruler:



So we **declare**: $\text{size}(\bullet \text{ --- } \circ) = 1 \text{ cm}$, and more generally,

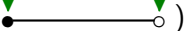
$$\text{size}(\bullet \text{ --- } \circ) = \ell \text{ cm}.$$

How to measure your paintwork

We'll need a ruler, say of length 1 cm:  .

A mathematician would use a segment of the number line: $0 \text{-----} 1$.

But should we include the endpoints, 0 and 1?

Including just one endpoint () gives a good ruler:

$$\begin{array}{c} \bullet \text{-----} 1 \text{ cm} \text{-----} \circ \\ \cup \\ \bullet \text{-----} 1 \text{ cm} \text{-----} \circ \end{array} = \bullet \text{-----} 2 \text{ cm} \text{-----} \circ$$

So we **declare**: $\text{size}(\bullet \text{-----} 1 \text{ cm} \text{-----} \circ) = 1 \text{ cm}$, and more generally,
 $\text{size}(\bullet \text{-----} \ell \text{ cm} \text{-----} \circ) = \ell \text{ cm}$.

We also **declare**: $\text{size}(\bullet) = 1 \text{ point} = 1 \text{ cm}^0 = 1$.

How to measure your paintwork

We've declared: $\text{size}(\bullet \xrightarrow{\ell \text{ cm}} \circ) = \ell \text{ cm}$ and $\text{size}(\bullet) = 1$.

Now let's use the same rules as before, and calculate some sizes.

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- $\text{size} \left(\begin{array}{c} \ell \text{ cm} \\ \text{[rectangle]} \\ k \text{ cm} \end{array} \right)$

boundaries included

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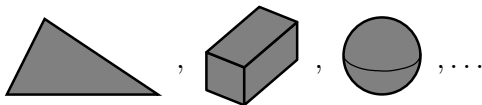
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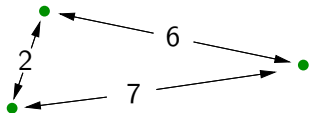
- Similarly, can compute sizes of



How to measure your relationship

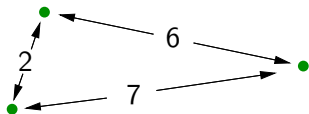
How to measure your relationship

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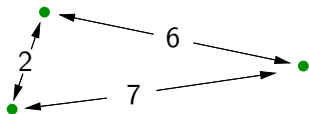
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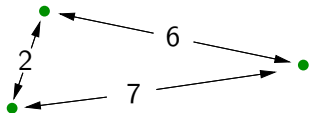
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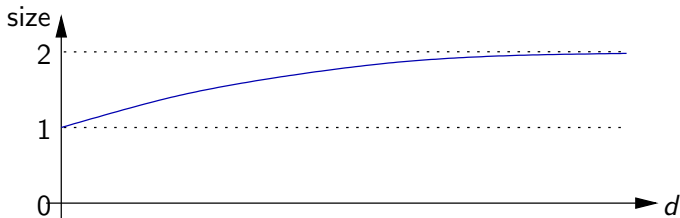


There is a formula saying what the 'size' of any metric space is.

Examples:

$\text{size}(\emptyset) = 0$ and $\text{size}(\bullet) = 1$.

$$\text{size}(\overset{\leftarrow}{\bullet} \vec{d} \bullet) = \frac{2}{1 + e^{-2d}}:$$



How to measure a datatype

Data in a computer can be arranged in many ways: lists, arrays, trees, These are 'datatypes', or 'types'.

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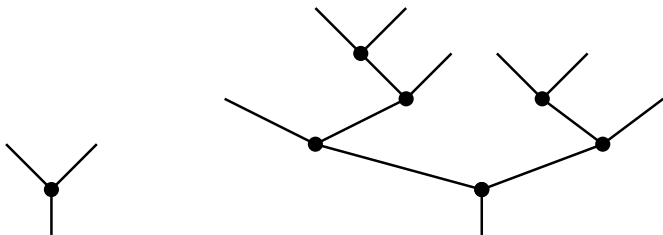
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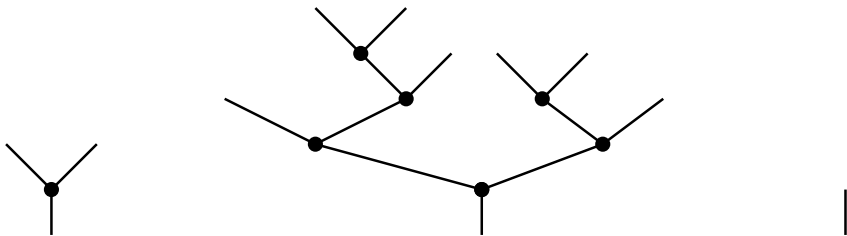
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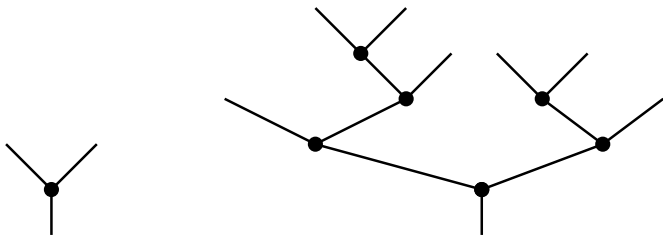
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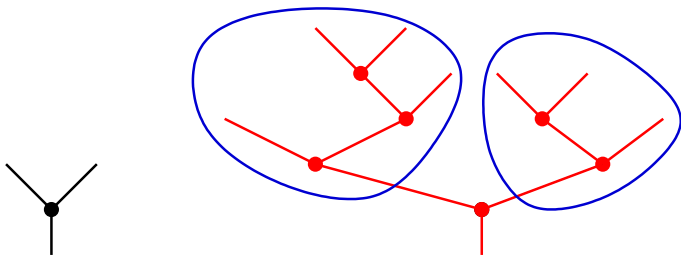
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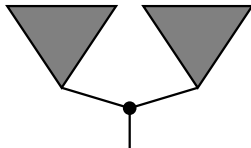
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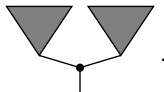
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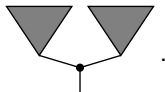
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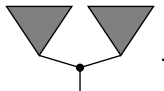
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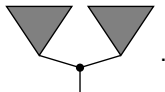


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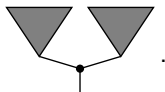
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$$\text{size}(T) = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i = e^{\pm\pi i/3}.$$

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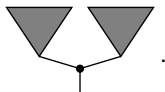
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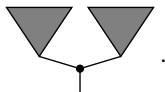
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This might lead one to suspect that $T^7 \cong T$... which really is true!

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We can define the 'diversity' or 'size' of E by

$$\text{size}(E) = p_1^{-p_1} p_2^{-p_2} \dots p_n^{-p_n}.$$

(This is the exponential of the Shannon entropy.)

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The size is greatest when $p_1 = p_2 = \dots = p_n = 1/n$: then $\text{size}(E) = n$.

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diversity: low

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diversity: higher

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'Effective number of species'

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‘Effective number of species’

‘Measuring biological diversity’,

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'Effective number of species' = size of the metric space of species

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Notes and references

These slides are available at www.maths.gla.ac.uk/~tl

Eggs: The basic idea of this talk is to take the elementary rules of arithmetic and use them in contexts where they do not obviously apply. This idea has doubtless been explored by many people; I have learned most about it from

John Baez, The mysteries of counting: Euler characteristic versus homotopy cardinality, <http://math.ucr.edu/home/baez/counting>

and

Daniel A. Klain, Gian-Carlo Rota, *Introduction to Geometric Probability*, Lezioni Lincee, Cambridge University Press, 1997

and the papers of Schanuel cited below.

Notes and references

Clothing: Stephen Schanuel seems to have been the first person to have really pushed the thought that Euler characteristic is to topological spaces as cardinality is to sets. Indeed, much of the theory of Euler characteristic (with compact support) follows from the simple axioms on 'size' above. See

Stephen H. Schanuel, Negative sets have Euler characteristic and dimension, *Category Theory (Como, 1990)*, 379–385, Lecture Notes in Mathematics 1488, Springer, 1991

and

Stephen H. Schanuel, What is the length of a potato? An introduction to geometric measure theory, in *Categories in Continuum Physics*, Lecture Notes in Mathematics 1174, Springer, 1986

as well as Klain and Rota (*op. cit.*).

Paintwork: This section draws very heavily on Schanuel's 'What is the length of a potato?' See also Klain and Rota.

Notes and references

Relationship: The slide refers to a notion of the 'size' of a metric space. For mathematicians this seems to be a new concept (but see the notes below on Ecosystems); it has yet to be written up formally. The existing sources are

Tom Leinster, The cardinality of a metric space, post at *The n-Category Café*, 9 February 2008,

http://golem.ph.utexas.edu/category/2008/02/metric_spaces.html

(which is detailed but contains some mistakes) and

Tom Leinster, The cardinality of a metric space, talk at CT08, Calais, www.maths.gla.ac.uk/~tl/calais

First one defines the cardinality (size) of a finite metric space; then, by using an approximating sequence of finite subspaces, one defines the cardinality of a compact metric space. This appears to coincide with the notion of 'size' described in the Paintwork section.

Notes and references

All of the notions of size discussed so far can be derived from the notion of the size (or cardinality, or Euler characteristic) of a category, introduced in

Tom Leinster, The Euler characteristic of a category, *Documenta Mathematica* 13 (2008), 21–49,

www.math.uni-bielefeld.de/documenta/vol-13/02.html

For an overview, see

Tom Leinster, New perspectives on Euler characteristic, talk at British Mathematical Colloquium 2007, Swansea,

www.maths.gla.ac.uk/~tl/swansea

Notes and references

Datatypes: The isomorphism $T^7 \cong T$ was first established in

Andreas Blass, Seven trees in one, *Journal of Pure and Applied Algebra* 103 (1995), 1–21, [arXiv:math.LO/9405205](https://arxiv.org/abs/math.LO/9405205),

following a remark of Lawvere. There is a precise sense in which $T^7 \cong T$ but $T^n \not\cong T$ for any other value of $n > 1$, except of course 13, 19, 25, ...

To state this correctly is a little delicate. Of course, the set of trees is countably infinite, so there is trivially a bijection $T^n \cong T$ for any $n \geq 1$; but that is not what is meant. Blass's result is, roughly, that there is an algorithm giving a one-to-one correspondence between 7-tuples of trees and single trees, and which *only explores each tree to finite depth*.

In this situation, we have $\text{size}(T)^7 = \text{size}(T)$ and, in fact, $T^7 \cong T$. This raises the question: is there some general principle allowing one to deduce the latter from the former? The answer is yes, as shown in

Notes and references

Marcelo Fiore and Tom Leinster, Objects of categories as complex numbers, *Advances in Mathematics* 190 (2005), 264–277, [arXiv:math.CT/0212377](#).

Ecosystem: For this section I am very grateful to [André Joyal](#), who pointed out to me that the exponential of entropy behaves much like cardinality, and to [Christina Cobbold](#), who then suggested that there might be a relation between diversity measures in ecology and the cardinality of metric spaces — as indeed there is.

The Shannon entropy of the system E is $-\sum_i p_i \log(p_i)$. Often the logarithm is taken to base 2 rather than base e .

For an introduction to measures of biodiversity, see, for instance, Chapter 7 of

Notes and references

Russell Lande, Steinar Engen, Bernt-Erik Sæther, *Stochastic Population Dynamics in Ecology and Conservation*, Oxford University Press, 2003,

or

Carlo Ricotta, Laszlo Szeidl, Towards a unifying approach to diversity measures: Bridging the gap between the Shannon entropy and Rao's quadratic index, *Theoretical Population Biology* 70 (2006), 237–243.

The paper of Solow and Polasky cited in the talk introduces the notion of 'effective number of species', which was later rediscovered under the name 'cardinality of a metric space' (see the notes on Relationship). Naturally, they only consider situations in which there is a finite number of species, which corresponds to considering only finite metric spaces.