**Galois Theory Assignment 1**

Overview of Galois theory; rings and fields

The deadline for submitting this work is **12 noon on Tuesday 26 January**, through Gradescope. To avoid technical problems, please start uploading by 11.55am. Please report any mistakes to Tom.Leinster@ed.ac.uk.

**Take care over communication and presentation.** Your answers should be coherent, logical arguments written in full sentences. Marks will be awarded for this.

1. Let $f$ be a quadratic polynomial over $\mathbb{Q}$. Using the definition of Galois group in Chapter 1 of the notes, prove that $\mathrm{Gal}(f)$ is $S_2$ if $f$ has two distinct irrational roots, and trivial otherwise.

   (Here 'irrational' means not in $\mathbb{Q}$; so any non-real complex number is irrational. Hint: use an argument like the first proof of Example 1.1.6, replacing $\sqrt{2}$ by the square root of the discriminant of $f$.)

2. (i) Let $R$ be a ring and let $I_0 \subseteq I_1 \subseteq \cdots$ be ideals of $R$. Prove that $\bigcup_{n=0}^{\infty} I_n$ is an ideal of $R$.

   (ii) Let $R$ be a principal ideal domain and let $I_0 \subseteq I_1 \subseteq \cdots$ be ideals of $R$. Prove that there is some $n \geq 0$ such that $I_n = I_{n+1} = I_{n+2} = \cdots$.

   (iii) Let $R$ be an integral domain. Let $r, s \in R$ with $r \neq 0$ and $s$ not a unit. Prove that $\langle rs \rangle$ is a proper subset of $\langle r \rangle$.

   (iv) Let $R$ be a principal ideal domain. Let $r \in R$ be neither 0 nor a unit. Prove that some irreducible divides $r$.

   (Hint: if not then, writing $r_0 = r$, we have $r_0 = r_1 s_1$ for some non-units $r_1$ and $s_1$. Apply the same argument to $r_1$, and so on forever, then consider the ideals $\langle r_n \rangle$ to get a contradiction.)

   *This result is the first step towards proving that in a principal ideal domain, every nonzero element can be expressed as a product of irreducibles in an essentially unique way. We won't need that fact except in rings of polynomials, where we'll use a different proof.*