Galois Theory Assignment 2

Polynomials and field extensions

The deadline for submitting this work is **12 noon on Tuesday 9 February**, through Gradescope. To avoid technical problems, please start uploading by 11.55am. Please report any mistakes to Tom.Leinster@ed.ac.uk.

Take care over communication and presentation. Your answers should be coherent, logical arguments written in full sentences. Marks will be awarded for this.

1. Which of the following polynomials are irreducible over $\mathbb{Q}$?

   (i) $1 + 2t - 5t^3 + 2t^6$

   (ii) $4 - 3t - 2t^2$

   (iii) $4 - 13t - 2t^3$

   (iv) $1 + t + t^2 + t^3 + t^4 + t^5$

   (v) $1 + t + t^2 + t^3 + t^4 + t^5 + t^6$

   (vi) $2.2 + 3.3t - 1.1t^3 + t^7$ (where the dots are decimal points, not products)

   (vii) $1 + t^4$.

2. Let $M : K$ be a field extension and $\alpha, \beta \in M$. Call α and β **indistinguishable** or **conjugate** over K if for all $p \in K[t]$, we have $p(\alpha) = 0 \iff p(\beta) = 0$. (You saw this definition in Chapter 1 for $\mathbb{C} : \mathbb{R}$ and $\mathbb{C} : \mathbb{Q}$.)

   (i) Prove that α and β are indistinguishable over K if and only if *either* both are transcendental *or* both are algebraic and they have the same minimal polynomial.

   (ii) Show that if there exists an irreducible polynomial $p \in K[t]$ such that $p(\alpha) = 0 = p(\beta)$, then α and β are indistinguishable over K.

   (iii) Show that if α and β are indistinguishable over K then $K(\alpha) \cong K(\beta)$ over K.