Galois Theory Assignment 3

Degree and splitting fields

The deadline for submitting this work is **12 noon on Tuesday 2 March**, through Gradescope. To avoid technical problems, please start uploading by 11.55am. Please report any mistakes to Tom.Leinster@ed.ac.uk.

Take care over communication and presentation. Your answers should be coherent, logical arguments written in full sentences. Marks will be awarded for this.

1. Let $M : L : K$ be field extensions. Prove that if $M : L$ and $L : K$ are algebraic then so is $M : K$. (You may *not* assume that either extension is finite.)

2. Say whether each of the following statements is true or false, giving one sentence of justification. If you think the statement is false, your sentence should be a counterexample.

   (i) Let $M : K$ be a field extension of degree 10. Then it is not possible to find extensions $M : L_2 : L_1 : K$ that are all nontrivial.

   (ii) Let $f(t) \in K[t]$ be an irreducible polynomial of degree n. Then $[\mathrm{SF}_K(f) : K] \leq n$.

   (iii) Let $M : K$ be a field extension and $\alpha, \beta \in M$. Then $[K(\alpha\beta) : K] \leq [K(\alpha, \beta) : K]$.

   (iv) Let $(x, y) \in \mathbb{R}^2$ and suppose that x and y each have an annihilating polynomial of degree 4 over $\mathbb{Q}$. Then (x, y) is constructible by ruler and compass from $(0, 0)$ and $(1, 0)$.

   (v) For all nontrivial finite field extensions $M : \mathbb{Q}$, the Galois group $\mathrm{Gal}(M : \mathbb{Q})$ is nontrivial.

   (vi) For all finite extensions $M : K$ and $M' : K'$, every isomorphism $\psi\colon K \to K'$ can be extended to a homomorphism $\varphi\colon M \to M'$.

   (vii) A regular 1020-sided polygon can be constructed by ruler and compass, given two points in the plane.

   (viii) Let $f \in \mathbb{Q}[t]$ and let $S = \mathrm{SF}_{\mathbb{Q}}(f)$. Then the splitting field of f over $\mathbb{Q}(\sqrt[3]{2})$ is $S(\sqrt[3]{2})$.

   (ix) Let f be a polynomial over a field K and let $\theta, \varphi \in \mathrm{Gal}_K(f)$. If $\theta(\alpha) = \varphi(\alpha)$ for all roots α of f in the splitting field of f, then $\theta = \varphi$.

   (x) The Galois group of $(t^4 - t^3 + t^2 - t + 1)^2$ over $\mathbb{Q}$ is solvable.