Galois Theory Assignment 5

Solvability by radicals and finite fields

The deadline for submitting this work is **12 noon on Tuesday 30 March**, through Gradescope. To avoid technical problems, please start uploading by 11.55am. Please report any mistakes to Tom.Leinster@ed.ac.uk.

Instructions:

- **Please do this work on your own, without consulting others.**
- **No justification is necessary. Your complete hand-in should be something like '1b 2a 3e . . . '.**

1. How many subfields does $\mathbb{F}_{243}$ have?
   (a) 1
   (b) 3
   (c) 5
   (d) 6
   (e) none of the above.

2. Let K and L be fields of order 125. How many isomorphisms $K \to L$ are there?
   (a) 1
   (b) 3
   (c) 5
   (d) impossible to say (depends on what K and L are)
   (e) none of the above.

3. How many homomorphisms $\mathbb{F}_{27} \to \mathbb{F}_{19683}$ are there?
   (a) 0
   (b) 1
   (c) 3
   (d) 9
   (e) none of the above.

4. How many homomorphisms $\mathbb{F}_4 \to \mathbb{F}_8$ are there?
   (a) 0
   (b) 1
   (c) 2
   (d) 3
   (e) none of the above.

5. Which of the following polynomials over $\mathbb{Q}$ is *not* solvable by radicals?
   (a) $t^7 - 2$
   (b) $t^7 + 2t^4 - t^3 - 2$
   (c) $t^7 + t^6 + t^5 + t^4 + t^3 + t^2 + t + 1$
   (d) $t^7 - 10t^5 + 15t + 5$
   (e) they are all solvable by radicals.

6. Let K be a field and $n \geq 1$. Is $\mathrm{Gal}_K(t^n - 1)$ abelian?

   (a) yes, always

   (b) yes when $K = \mathbb{Q}$ or $n \leq 2$, but never otherwise

   (c) yes when $\mathrm{char}\, K = 0$ or $n \leq 2$, but never otherwise

   (d) yes when $\mathrm{char}\, K = 0$ or $n \leq 2$, and sometimes but not always otherwise

   (e) none of the above.

7. Let K be a field of characteristic 0, let $n \geq 1$, and suppose that $t^n - 1$ splits in K. Let $0 \neq a \in K$ and let $\varphi \in \mathrm{Gal}_K(t^n - a)$. Viewing φ as a K-linear operator on $\mathrm{SF}_K(t^n - a)$, how many roots of $t^n - a$ in $\mathrm{SF}_K(t^n - a)$ are eigenvectors of φ? (Hint: Lemma 9.1.8.)

   (a) none

   (b) 1

   (c) n

   (d) it depends on the choice of K, n, a and φ

   (e) none of the above.

8. Up to isomorphism, how many groups G are there such that $G \cong \mathrm{Gal}_{\mathbb{Q}}(f)$ for some polynomial f of degree 5 over $\mathbb{Q}$ that is not solvable by radicals?

   (a) 1

   (b) 2

   (c) 3

   (d) 5

   (e) none of the above.

9. Let α be a generator of the multiplicative group $\mathbb{F}_{65536}^{\times}$. How many roots does the polynomial $t^{256} - \alpha$ have in $\mathbb{F}_{65536}$?

   (a) none

   (b) 2

   (c) 8

   (d) 256

   (e) none of the above.

10. What is the smallest integer $r \geq 0$ with the following property: there exist intermediate fields

$$\mathbb{Q} = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_r = \mathrm{SF}_{\mathbb{Q}}(t^3 - 2)$$

    such that $L_i : L_{i-1}$ is normal and $\mathrm{Gal}(L_i : L_{i-1})$ is abelian for all $i \in \{1, \ldots, r\}$?

    (a) 1

    (b) 2

    (c) 3

    (d) there is no such chain of intermediate fields

    (e) none of the above.