# Galois Theory Assignment 4

Preparation for the fundamental theorem

- **Please do this work on your own, without consulting others.**
- **Your complete hand-in should be something like '1b 2a 3e ... 10c'. No justification is necessary.**
- ***Email your answers* to Tom.Leinster@ed.ac.uk, from your university account. Your answers should be in the body of the email. No attachments, please.**
- **The deadline is 12:10 on Monday 4 April.**

1. Let $M : K$ be an algebraic field extension, with $M$ algebraically closed. Which of the following statements is true?
   (a) $M : K$ is normal and separable.
   (b) $M : K$ is normal but need not be separable.
   (c) $M : K$ need not be normal but is separable.
   (d) $M : K$ need not be normal and need not be separable.
   (e) The question doesn't make sense.

2. Let $M : K$ be a normal extension. Which of the following statements is true?
   (a) Every polynomial over $K$ splits in $M$.
   (b) Every polynomial over $K$ that has at least one root in $M$ splits in $M$.
   (c) Every irreducible polynomial over $K$ splits in $M$.
   (d) Every irreducible polynomial over $K$ that has at least one root in $M$ splits in $M$.
   (e) None of the other statements is true.

3. Which of the following statements about the extension $\mathbb{Q}(e^{2\pi i/3}) : \mathbb{Q}$ is true?
   (a) It is normal and separable.
   (b) It is normal but not separable.
   (c) It is not normal but is separable.
   (d) It is neither normal nor separable.
   (e) The question doesn't make sense.

4. Let $p$ be a prime number, let $\mathbb{F}_p(u)$ be the field of rational expressions over $\mathbb{F}_p$, and let $M$ be the extension of $\mathbb{F}_p(u)$ by a $p$th root of $u$. Which of the following statements about the extension $M : \mathbb{F}_p(u)$ is true?
   (a) It is normal and separable.
   (b) It is normal but not separable.
   (c) It is not normal but is separable.
   (d) It is neither normal nor separable.
   (e) The question doesn't make sense.

5. Exactly one of the following statements is *false*. Which one?
   (a) Let $M : L : K$ be algebraic extensions, with $L : K$ normal. Then $L$ is a union of conjugacy classes in $M$ over $K$.
   (b) Every splitting field extension is algebraic.
   (c) Every splitting field extension is finitely generated.
   (d) Every splitting field extension is normal.
   (e) For every normal extension $M : K$, there is some $f \in K[t]$ such that $M$ is a splitting field of $f$ over $K$.

6. Let $M : L : K$ be field extensions, with $M : K$ finite and normal. Which of the following statements is true?

   (a) $M : L$ is normal, but $L : K$ need not be.

   (b) $L : K$ is normal, but $M : L$ need not be.

   (c) $M : L$ and $L : K$ are normal.

   (d) Neither $M : L$ nor $L : K$ need be normal.

   (e) The question doesn't make sense.

7. Exactly one of the following statements is true. Which one?

   (a) For every field extension $M : K$, there is a nontrivial natural action of $\mathrm{Gal}(M : K)$ on $K$.

   (b) Let $f$ be a polynomial over a field $K$, and let $\alpha$ and $\alpha'$ be two roots of $f$ in its splitting field. Then $\alpha' = \varphi(\alpha)$ for some $\varphi \in \mathrm{Gal}_K(f)$.

   (c) For all complex roots $\alpha$ and $\beta$ of $f(t) = t^6 - 10t^4 + 15$, there is some element of $\mathrm{Gal}_{\mathbb{Q}}(f)$ that maps $\alpha$ to $\beta$.

   (d) Let $M : L : K$ be field extensions, with $M : K$ finite and normal. Then every element of $\mathrm{Gal}(M : K)$ restricts to an automorphism of $L$.

   (e) None of the other statements is true.

8. Exactly one of the following statements is *false*. Which one?

   (a) Let $f$ be an irreducible polynomial over a field $K$. If $f$ is inseparable then $K$ must be infinite.

   (b) Let $f$ be an irreducible polynomial over a field $K$. If $f$ is inseparable then $K$ must have positive characteristic.

   (c) Formal differentiation of polynomials over an arbitrary field satisfies the product rule (Leibniz rule).

   (d) Let $f \in \mathbb{Q}[t]$. Then $f$ has $\deg(f)$ distinct roots in its splitting field.

   (e) Let $f \in \mathbb{Q}[t]$. If $f$ and its formal derivative $Df$ are both divisible by $t^2 + 1$, then $f$ has a repeated root in its splitting field.

9. Which one of the statements (a)–(d) is *false*? Or if you think more than one is false, choose (e).

   (a) An irreducible polynomial over a field is separable if and only if its formal derivative is the zero polynomial.

   (b) Let $M : L : K$ be field extensions. If any two of $M : L$, $L : K$ and $M : K$ are algebraic, then so is the third.

   (c) Let $M : K$ be a normal separable extension of degree 48. Then $\mathrm{Gal}(M : K)$ is solvable.

   (d) Let $M$ be a normal extension of $\mathbb{F}_3$. If $M$ has 81 elements then $\mathrm{Gal}(M : \mathbb{F}_3)$ is abelian.

   (e) More than one of the other statements is false.

10. Which one of the statements (a)–(d) is *false*? Or if you think more than one is false, choose (e).

   (a) For a field $M$ of characteristic 0, the Galois group $\mathrm{Gal}(M : \mathbb{Q})$ is $\mathrm{Aut}(M)$.

   (b) For a field $M$ of characteristic $p$, the Galois group $\mathrm{Gal}(M : \mathbb{F}_p)$ is $\mathrm{Aut}(M)$.

   (c) There is a subgroup $H$ of $\mathrm{Aut}(\mathbb{C})$ such that $\mathrm{Fix}(H) = \mathbb{Z}$.

   (d) For a field $M$ and a finite subgroup $H$ of $\mathrm{Aut}(M)$, the extension $M : \mathrm{Fix}(H)$ is finite.

   (e) More than one of the other statements is false.