Is there an element $\theta \in \mathrm{Gal}_{\mathbb{Q}}(f)$ such that $\theta(\omega)=\omega^2$?

If so, then it must be given by

$$\theta(a+b\omega+c\omega^2+d\omega^3) = a+b\omega^2+c\omega^4+d\omega^6$$

$$= a+b\omega^2+c(-1-\omega-\omega^2-\omega^3)+d\omega$$

$$= [a-c]+[d-c]\omega+[b-c]\omega^2-c\omega^3$$

There's a function $\theta: \mathbb{Q}(\omega) \longrightarrow \mathbb{Q}(\omega)$ over $\mathbb{Q}$

given by

$$\theta(a+b\omega+c\omega^2+d\omega^3) = [a-c]+[d-c]\omega+[b-c]\omega^2-c\omega^3.$$

It preserves $+$. Does it preserve $\cdot$?

After $\boxed{\text{MUCH}}$ mundane work, conclude: yes.

Hence $\theta \in \mathrm{Gal}_{\mathbb{Q}}(f)$.

Hence $\theta\circ\theta$, $\theta\circ\theta\circ\theta$ etc belong to $\mathrm{Gal}_{\mathbb{Q}}(f)$.

It follows that $\mathrm{Gal}_{\mathbb{Q}}(f) = \langle\theta\rangle \cong C_4$.