Chapter 7: Preparation for the fundamental theorem

(1) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

True or false? For an algebraic extension $M : K$, if M is algebraically closed then $M : K$ is normal.

a. True
b. False

(2) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

True or false? Every normal extension is finite.

a. False
b. True

(3) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

True or false? Every finite extension is normal.

a. True
b. False

(4) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

True or false? Every field extension of prime degree is normal.

a. False
b. True

(5) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

True or false? In a normal extension $M : K$, every polynomial over K that has one root in M splits in M.

a. True
b. False

(6) **Preparation for the fundamental theorem** [Multiple choice] [One answer only]

True or false? In a normal extension $M : K$, every irreducible polynomial over K that has one root in M splits in M.

a. True
b. False

(7) **Preparation for the fundamental theorem** [Multiple choice] [One answer only]

Let ξ be the real cube root of 2. Which of the following statements about the extension $\mathbb{Q}(\xi) : \mathbb{Q}$ is true?

a. It is separable but not normal
b. It is not normal or separable
c. It is normal and separable
d. It is normal but inseparable

(8) **Preparation for the fundamental theorem** [Multiple choice] [One answer only]

Which of the following statements about the extension $\mathbb{Q}(e^{2\pi i/3}) : \mathbb{Q}$ is true?

a. It is separable but not normal
b. It is normal and separable
c. It is normal but inseparable
d. It is not normal or separable

(9) **Preparation for the fundamental theorem** [Multiple choice] [One answer only]

True or false? Let $M : K$ be a normal extension and $\alpha \in M$. Then the conjugacy class of α in M has the same cardinality as the conjugacy class of α in the splitting field of its minimal polynomial.

a. False
b. True

(10) **Preparation for the fundamental theorem** [Multiple choice] [One answer only]

Let p be a prime and let $K = \mathbb{F}_p(u)$ be the field of rational expressions over $\mathbb{F}_p$. Extend K by a pth root α of u. Which of the following statements about the extension $K(\alpha) : K$ is true?

a. It is separable but not normal
b. It is normal and separable
c. It is not normal or separable
d. It is normal but inseparable

(11) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

True or false? Let f be a polynomial over a field K. Then the splitting field extension $\mathrm{SF}_K(f) : K$ is finite, normal and separable.

a. False
b. True

(12) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

True or false? In the group of linear automorphisms of $\mathbb{R}^3$, any two rotations by an angle of $\pi/2$ are conjugate to one another.

a. False
b. True

(13) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

True or false? In the group of linear automorphisms of $\mathbb{R}^3$, any two rotations are conjugate to one another.

a. True
b. False

(14) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

True or false? In the group of rotations of a cube, any two rotations about axes through the midpoints of opposite faces are conjugate to one another.

a. False
b. True

(15) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

True or false? In the group of rotations of a cube, any two nontrivial rotations about axes through opposite vertices are conjugate to one another.

a. True
b. False

(16) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

True or false? Let $M : L : K$ be algebraic extensions, with $L : K$ normal. Then L is a union of conjugacy classes in M over K.

a. True
b. False

(17) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

True or false? Every splitting field extension is algebraic.

a. False
b. True

(18) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

True or false? Every splitting field extension is finitely generated.

a. True
b. False

(19) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

True or false? For every normal extension $M : K$, there is some $f \in K[t]$ such that M is a splitting field of f over K.

a. True
b. False

(20) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

True or false? For every finite normal extension $M : K$, there is some $f \in K[t]$ such that M is a splitting field of f over K.

a. False
b. True

(21) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

Let $M : L : K$ be field extensions, with $M : K$ finite and normal. Which of the following is true?

a. $M : L$ and $L : K$ are normal
b. neither $M : L$ nor $L : K$ need be normal
c. $L : K$ is normal, but $M : L$ need not be
d. $M : L$ is normal, but $L : K$ need not be

(22) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

True or false? For a field extension $M : K$, there is a nontrivial natural action of $\mathrm{Gal}(M : K)$ on K.

a. True
b. False

(23) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

True or false? For a field extension $M : K$, there is a natural action of $\mathrm{Gal}(M : K)$ on M.

a. False
b. True

(24) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

True or false? Let f be a polynomial over a field K, and let α and α' be two roots of f in its splitting field. Then $\alpha' = \varphi(\alpha)$ for some φ in $\mathrm{Gal}_K(f)$.

a. False
b. True

(25) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

True or false? Let f be an irreducible polynomial over a field K, and let α and α' be two roots of f in its splitting field. Then $\alpha' = \varphi(\alpha)$ for some φ in $\mathrm{Gal}_K(f)$.

a. True
b. False

(26) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

True or false? For all complex roots α and β of $f(t) = t^6 - 10t^4 + 15$, there is some element of $\mathrm{Gal}_{\mathbb{Q}}(f)$ that maps α to β.

a. True
b. False

(27) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

True or false? For all complex roots α and β of $f(t) = t^6 - 10t^4 + 15t$, there is some element of $\mathrm{Gal}_{\mathbb{Q}}(f)$ that maps α to β.

a. False
b. True

(28) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

True or false? For every irreducible cubic f over $\mathbb{Q}$, the Galois group $\mathrm{Gal}_{\mathbb{Q}}(f)$ is either A_3 or S_3.

a. False
b. True

(29) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

True or false? Let $M : L : K$ be field extensions where $M : K$ and $L : K$ are both finite and normal. Then every element of $\mathrm{Gal}(M : K)$ restricts to an automorphism of L.

a. False
b. True

(30) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

True or false? Let $M : L : K$ be field extensions where $M : K$ is finite and normal. Then every element of $\mathrm{Gal}(M : K)$ restricts to an automorphism of L.

a. False
b. True

(31) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

True or false? Let $M : L : K$ be field extensions. Then $\mathrm{Gal}(M : L)$ is a subgroup of $\mathrm{Gal}(M : K)$.

a. False
b. True

(32) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

True or false? Let $M : L : K$ be field extensions where $M : K$ and $L : K$ are both finite and normal. Then every automorphism of L over K extends to an automorphism of M over K.

a. False
b. True

(33) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

True or false? Let f be an irreducible polynomial over a field K. Then f has $\deg(f)$ distinct roots in its splitting field.

a. False
b. True

(34) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

True or false? Let f be an irreducible polynomial over $\mathbb{Q}$. Then f has $\deg(f)$ distinct roots in its splitting field.

a. True
b. False

(35) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

True or false? Let f be a polynomial over $\mathbb{Q}$. Then f has $\deg(f)$ distinct roots in its splitting field.

a. False
b. True

(36) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

True or false? Let f be an irreducible polynomial over a field K. If f is inseparable then K must be an infinite field of positive characteristic.

a. False
b. True

(37) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

True or false? Formal differentiation of polynomials over an arbitrary field satisfies the product rule (Leibniz rule).

a. True
b. False

(38) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

True or false? Let $f \in \mathbb{Q}[t]$. If f and its formal derivative Df are both divisible by $t^2 + 1$, then f has a repeated root in its splitting field.

a. True
b. False

(39) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

True or false? An irreducible polynomial over a field is separable if and only if its formal derivative is the zero polynomial.

a. True
b. False

(40) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

True or false? An algebraic extension $M : K$ is separable if and only if every irreducible polynomial over K is separable.

a. True
b. False

(41) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

True or false? Let $M : L : K$ be field extensions. If any two of $M : L$, $L : K$ and $M : K$ are algebraic, then so is the third.

a. False
b. True

(42) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

True or false? Let $M : K$ be a normal separable extension of degree 48. Then $\mathrm{Gal}(M : K)$ is solvable.

a. False
b. True

(43) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

True or false? Let M be a normal extension of $\mathbb{F}_3$. If M has 81 elements then $\mathrm{Gal}(M : \mathbb{F}_3)$ is abelian.

a. True
b. False

(44) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

True or false? Let $M : K$ be a normal separable extension. If $[M : K]$ is a prime number then $\mathrm{Gal}(M : K)$ is cyclic.

a. True
b. False

(45) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

True or false? There is some subgroup H of $\mathrm{Aut}(\mathbb{C})$ such that $\mathrm{Fix}(H) = \mathbb{Z}$.

a. False
b. True

(46) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

True or false? There is some subgroup H of $\mathrm{Aut}(\mathbb{C})$ such that $\mathrm{Fix}(H) = \mathbb{R}$.

a. False
b. True

(47) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

True or false? For a field M of characteristic p, the Galois group $\mathrm{Gal}(M : \mathbb{F}_p)$ is $\mathrm{Aut}(M)$.

a. True
b. False

(48) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

True or false? For a field M of characteristic 0, the Galois group $\mathrm{Gal}(M : \mathbb{Q})$ is $\mathrm{Aut}(M)$.

a. True
b. False

(49) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

True or false? For a field M and a finite subgroup H of $\mathrm{Aut}(M)$, the extension $M : \mathrm{Fix}(H)$ is also finite.

a. True
b. False

(50) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

Let $M : K$ be an algebraic field extension, with M algebraically closed. Which of the following statements is true?

a. $M : K$ need not be normal and need not be separable.
b. $M : K$ need not be normal but is separable.
c. $M : K$ is normal and separable.
d. $M : K$ is normal but need not be separable.

(51) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

Let $M : K$ be a normal extension. Which of the following statements is true?

a. Every irreducible polynomial over K that has at least one root in M splits in M.
b. Every polynomial over K that has at least one root in M splits in M.
c. Every irreducible polynomial over K splits in M.
d. None of the other statements is true.
e. Every polynomial over K splits in M.

(52) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

Exactly one of the following statements is *false*. Which one?

a. Every splitting field extension is algebraic.

b. Every splitting field extension is normal.
c. Every splitting field extension is finitely generated.
d. Let $M : L : K$ be algebraic extensions, with $L : K$ normal. Then L is a union of conjugacy classes in M over K.
e. For every normal extension $M : K$, there is some $f \in K[t]$ such that M is a splitting field of f over K.

(53) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

Exactly one of the following statements is true. Which one?

a. None of the other statements is true.
b. Let f be a polynomial over a field K, and let α and α' be two roots of f in its splitting field. Then $\alpha' = \phi(\alpha)$ for some $\phi \in \mathrm{Gal}_K(f)$.
c. For every field extension $M : K$, there is a nontrivial natural action of $\mathrm{Gal}(M : K)$ on K.
d. Let $M : L : K$ be field extensions, with $M : K$ finite and normal. Then every element of $\mathrm{Gal}(M : K)$ restricts to an automorphism of L.
e. For all complex roots α and β of $f(t) = t^6 - 10t^4 + 15$, there is some element of $\mathrm{Gal}_{\mathbb{Q}}(f)$ that maps α to β.

(54) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

Exactly one of the following statements is *false*. Which one?

a. Let $f \in \mathbb{Q}[t]$. Then f has $\deg(f)$ distinct roots in its splitting field.
b. Let f be an irreducible polynomial over a field K. If f is inseparable then K must have positive characteristic.
c. Let $f \in \mathbb{Q}[t]$. If f and its formal derivative Df are both divisible by $t^2 + 1$, then f has a repeated root in its splitting field.
d. Formal differentiation of polynomials over an arbitrary field satisfies the product rule (Leibniz rule).
e. Let f be an irreducible polynomial over a field K. If f is inseparable then K must be infinite.

(55) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

Which one of the following statements is *false*?

a. Let $M : L : K$ be field extensions. If any two of $M : L$, $L : K$ and $M : K$ are algebraic, then so is the third.
b. An irreducible polynomial over a field is separable if and only if its formal derivative is the zero polynomial.
c. Let $M : K$ be a normal separable extension of degree 48. Then $\mathrm{Gal}(M : K)$ is solvable.
d. Let M be a normal extension of $\mathbb{F}_3$. If M has 81 elements then $\mathrm{Gal}(M : \mathbb{F}_3)$ is abelian.

(56) **Preparation for the fundamental theorem** MULTIPLE CHOICE One answer only

Which one of the following statements is *false*?

a. There is a subgroup H of $\mathrm{Aut}(\mathbb{C})$ such that $\mathrm{Fix}(H) = \mathbb{Z}$.
b. For a field M and a finite subgroup H of $\mathrm{Aut}(M)$, the extension $M : \mathrm{Fix}(H)$ is finite.
c. For a field M of characteristic 0, the Galois group $\mathrm{Gal}(M : \mathbb{Q})$ is $\mathrm{Aut}(M)$.
d. For a field M of characteristic p, the Galois group $\mathrm{Gal}(M : \mathbb{F}_p)$ is $\mathrm{Aut}(M)$.

Total of marks: 56