

Galois Theory Workshop 1

Overview of Galois theory; rings and fields

1. Let f be a quadratic polynomial over \mathbb{Q} , and let α_1, α_2 be its roots in \mathbb{C} (which may be equal). Show that it is impossible that $\alpha_1 \in \mathbb{Q}$ but $\alpha_2 \notin \mathbb{Q}$.
2. Let R be a ring and let $\varphi: 1 \rightarrow R$ be a homomorphism, where 1 denotes the trivial ring (zero ring). Prove that R is trivial too and that φ is an isomorphism.
3. Let f be a quadratic polynomial over \mathbb{Q} . Using the definition of Galois group in Chapter 1 of the notes, prove that $\text{Gal}(f)$ is S_2 if f has two distinct irrational roots, and trivial otherwise.

(Here ‘irrational’ means not in \mathbb{Q} ; so any non-real complex number is irrational. Hint: use an argument like the first proof of Example 1.1.6, replacing $\sqrt{2}$ by the square root of the discriminant of f .)

4. (i) Let R be a ring and let $I_0 \subseteq I_1 \subseteq \dots$ be ideals of R . Prove that $\bigcup_{n=0}^{\infty} I_n$ is an ideal of R .
- (ii) Let R be a principal ideal domain and let $I_0 \subseteq I_1 \subseteq \dots$ be ideals of R . Prove that there is some $n \geq 0$ such that $I_n = I_{n+1} = I_{n+2} = \dots$.
- (iii) Let R be an integral domain. Let $r, s \in R$ with $r \neq 0$ and s not a unit. Prove that $\langle rs \rangle$ is a proper subset of $\langle r \rangle$.
- (iv) Let R be a principal ideal domain. Let $r \in R$ be neither 0 nor a unit. Prove that some irreducible divides r .

(Hint: if not then, writing $r_0 = r$, we have $r_0 = r_1 s_1$ for some non-units r_1 and s_1 . Apply the same argument to r_1 , and so on forever, then consider the ideals $\langle r_n \rangle$ to get a contradiction.)

This result is the first step towards proving that in a principal ideal domain, every nonzero element can be expressed as a product of irreducibles in an essentially unique way. We won’t need that fact except in rings of polynomials, where we’ll use a different proof.

5. Let $f(t) = t^4 + t^3 + t^2 + t + 1$, which has roots $\omega, \omega^2, \omega^3, \omega^4$ where $\omega = e^{2\pi i/5}$. In the notes, you were told that one of the elements of $\text{Gal}(f)$ is

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$$

(i.e. $\omega \mapsto \omega^2 \mapsto \omega^4 \mapsto \omega^3 \mapsto \omega$). We didn’t prove this, but for this question you can take it as given.

Prove that $\text{Gal}(f)$ is generated by σ , and deduce that $\text{Gal}(f) \cong C_4$.

(This question is maybe a bit harder.)