Galois Theory Workshop 2

Polynomials and field extensions

There are more questions here than you're likely to have time for in the workshop. I suggest you start from the beginning and do whatever you do in the time, without hurrying, then keep the other ones for practice another day.

In all questions, you're strongly encouraged to use results from the notes. In fact, your life will be much harder if you don't.

1. Which of the following polynomials are irreducible over $\mathbb{Q}$?
   (i) $1 + 2t - 5t^3 + 2t^6$
   (ii) $4 - 3t - 2t^2$
   (iii) $4 - 13t - 2t^3$
   (iv) $1 + t + t^2 + t^3 + t^4 + t^5$
   (v) $1 + t + t^2 + t^3 + t^4 + t^5 + t^6$
   (vi) $2.2 + 3.3t - 1.1t^3 + t^7$ (where the dots are decimal points, not products)
   (vii) $1 + t^4$.

2. Find the minimal polynomial of $(\sqrt{5} + 1)/2$ over $\mathbb{Q}$.

3. Does every field have a nontrivial extension?

4. Let $M : K$ be a field extension and $\alpha, \beta \in M$. Call α and β **indistinguishable** or **conjugate** over K if for all $p \in K[t]$, we have $p(\alpha) = 0 \iff p(\beta) = 0$. (You saw this definition in Chapter 1 for $\mathbb{C} : \mathbb{R}$ and $\mathbb{C} : \mathbb{Q}$.)
   (i) Prove that α and β are indistinguishable over K if and only if *either* both are transcendental *or* both are algebraic and they have the same minimal polynomial.
   (ii) Show that if there exists an irreducible polynomial $p \in K[t]$ such that $p(\alpha) = 0 = p(\beta)$, then α and β are indistinguishable over K.
   (iii) Show that if α and β are indistinguishable over K then $K(\alpha) \cong K(\beta)$ over K.

5. (i) Let $\alpha \in \mathbb{C}$ and let $p \in \mathbb{Q}[t]$ be an irreducible polynomial such that $p(\alpha) = 0$. Prove that $\mathbb{Q}[t]/\langle p \rangle$ is isomorphic to $\mathbb{Q}(\alpha)$ (the smallest subfield of $\mathbb{C}$ containing α).
   (ii) Let p be a prime number and put $\omega = e^{2\pi i/p}$. Prove that $\omega, \ldots, \omega^{p-1}$ are indistinguishable over $\mathbb{Q}$. (This was claimed in Example 1.1.9; now you can prove it.)
   (iii) Prove that $\mathbb{Q}(\pi) \cong \mathbb{Q}(e)$ over $\mathbb{Q}$. (You'll need a bit of mathematical general knowledge to do this.)

6. Find an irreducible polynomial $f \in \mathbb{R}[t]$ such that $\mathbb{R}[t]/\langle f \rangle \cong \mathbb{C}$ (and prove it!).

7. Some of you have asked why we often focus on $\mathbb{Q}$ as opposed to $\mathbb{R}$. One answer is that things over $\mathbb{R}$ tend to be a bit trivial, and this question demonstrates one aspect of that.
   (i) Let $x, y \in \mathbb{R}$. What is the minimal polynomial of $x + iy$ over $\mathbb{R}$?
   (ii) Which polynomials over $\mathbb{R}$ are irreducible? (This might sound like an open-ended question, but you have the tools to give a complete answer.)

8. Let $f(t) \in \mathbb{Z}[t]$. Prove that

$$f \text{ is primitive and irreducible over } \mathbb{Q} \iff f \text{ is irreducible over } \mathbb{Z}.$$

9. Let R be a ring and $r \in R$. Show that there is exactly one homomorphism $\theta\colon \mathbb{Z}[t] \to R$ such that $\theta(t) = r$.

   (Take care over the uniqueness aspect, i.e. proving that there's *only* one such θ. It's easy to slip up logically.)

10. Let K be a field. In this question, you'll find all the ring automorphisms of $K[t]$. (Recall that 'automorphism' means 'isomorphism from a thing to itself'.)

    (i) For $f \in K[t]$, there is a unique homomorphism $\theta_f\colon K[t] \to K[t]$ such that $\theta_f(t) = f$ and $\theta_f(a) = a$ for all $a \in K$. Which result in the notes guarantees this?

    (ii) For $f, g \in K[t]$, what is $\theta_f(g)$ in explicit terms? (There's a very short answer.) What is its degree?

    (iii) For $f_1, f_2 \in K[t]$, what can you say about the composite homomorphism

    $$K[t] \xrightarrow{\theta_{f_1}} K[t] \xrightarrow{\theta_{f_2}} K[t]?$$

    (iv) Using the previous parts, find all the isomorphisms $K[t] \to K[t]$.