Galois Theory Workshop 4

Preparation for the fundamental theorem

In all questions, you're strongly encouraged to use results from the notes—and this will make your life much easier!

1. Prove that every field extension of degree 2 is normal.

   This should remind you of the fact that every subgroup of index 2 is normal.

2. Let K be a field and let $f \in K[t]$ be an irreducible polynomial.

   (i) Prove that the order of $\mathrm{Gal}_K(f)$ is divisible by the number of distinct roots of f in its splitting field.

   (ii) Deduce that if $\mathrm{char}\, K = 0$ then $\deg(f)$ divides $|\,\mathrm{Gal}_K(f)|$.

3. Show that any automorphism of a field M is an automorphism over the prime subfield of M.

4. Show by example that for field extensions $M : L : K$,

   $$M : L \text{ and } L : K \text{ normal} \not\Rightarrow M : K \text{ normal.}$$

   Hint: start by trying the simplest possible examples.

5. Let $L : K$ be an algebraic extension. Prove that $L : K$ is normal if and only if it has the following property: for every extension $M : L$, the field L is a union of conjugacy classes in M over K.

   (Conjugacy over K defines an equivalence relation on M, and a 'conjugacy class in M over K' means an equivalence class of this equivalence relation.)

   This should remind you of the fact that a subgroup is normal if and only if it is a union of conjugacy classes in the group-theoretic sense.