Galois Theory Workshop 2

Overview of Galois Theory; group actions, rings and fields; polynomials

There are more questions here than you're likely to have time for in the workshop. I suggest you start from the beginning and do whatever you do in the time, without hurrying, then keep the other ones for practice another day.

In all questions, you're strongly encouraged to use results from the notes. In fact, your life will be much harder if you don't.

1. (i) Can C_6 act faithfully on a 4-element set? Give either an example or a proof that it is impossible.

   (ii) Let G be a finite group acting transitively on a nonempty set X. Prove that $|X|$ divides $|G|$. (Hint: orbit-stabilizer theorem.)

2. (i) Let R be a ring and let $I_0 \subseteq I_1 \subseteq \cdots$ be ideals of R. Prove that $\bigcup_{n=0}^{\infty} I_n$ is an ideal of R.

   (ii) Let R be a principal ideal domain and let $I_0 \subseteq I_1 \subseteq \cdots$ be ideals of R. Prove that there is some $n \geq 0$ such that $I_n = I_{n+1} = I_{n+2} = \cdots$.

   (iii) Let R be an integral domain. Let $r, s \in R$ with $r \neq 0$ and s not a unit. Prove that $\langle rs \rangle$ is a proper subset of $\langle r \rangle$.

   (iv) Let R be a principal ideal domain. Let $r \in R$ be neither 0 nor a unit. Prove that some irreducible divides r.

   (Hint: if not then, writing $r_0 = r$, we have $r_0 = r_1 s_1$ for some non-units r_1 and s_1. Apply the same argument to r_1, and so on forever, then consider the ideals $\langle r_n \rangle$ to get a contradiction.)

   This result is the first step towards proving that in a principal ideal domain, every nonzero element can be expressed as a product of irreducibles in an essentially unique way. We won't need that fact except in rings of polynomials, where we'll use a different proof.

3. Let K be a field. In this question, you'll find all the ring automorphisms of $K[t]$ over K (that is, fixing each element of K).

   (i) For $f \in K[t]$, there is a unique homomorphism $\theta_f \colon K[t] \to K[t]$ such that $\theta_f(t) = f$ and $\theta_f(a) = a$ for all $a \in K$. Which result in the notes guarantees this?

   (ii) For $f, g \in K[t]$, what is $\theta_f(g)$ in explicit terms? (There's a very short answer.) What is its degree?

   (iii) For $f_1, f_2 \in K[t]$, what can you say about the composite homomorphism

   $$K[t] \xrightarrow{\theta_{f_1}} K[t] \xrightarrow{\theta_{f_2}} K[t]?$$

   (iv) Using the previous parts, find all the isomorphisms $K[t] \to K[t]$ over K.

4. Let $f(t) = t^4 + t^3 + t^2 + t + 1$, which has roots $\omega, \omega^2, \omega^3, \omega^4$ where $\omega = e^{2\pi i/5}$. In Example 1.2.7, you were told that one of the elements of $\mathrm{Gal}(f)$ is

   $$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$$

   (i.e. $\omega \mapsto \omega^2 \mapsto \omega^4 \mapsto \omega^3 \mapsto \omega$). We didn't prove this, but for this question you can take it as given.

   Prove that $\mathrm{Gal}(f)$ is generated by σ, and deduce that $\mathrm{Gal}(f) \cong C_4$.

   (This question is maybe a bit harder.)