The functoriality of the reflexive completion

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1. What is the reflexive completion?

Isbell conjugacy for small categories

Let $\mathcal{A}$ be a small category.

Given $X \colon \mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Set}$, define $X^{\vee} \colon \mathcal{A} \longrightarrow \mathbf{Set}$ by

$$X^{\vee}(a) = [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}](X, \mathcal{A}(-, a)).$$

Note! X is contravariant but $X^{\vee}$ is covariant.

Substitute $\mathcal{A}^{\mathrm{op}}$ for $\mathcal{A}$: then from $Y \colon \mathcal{A} \longrightarrow \mathbf{Set}$, we get $Y^{\vee} \colon \mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Set}$, given by

$$Y^{\vee}(a) = [\mathcal{A}, \mathbf{Set}](Y, \mathcal{A}(a, -)).$$

E.g. $\mathcal{A}(-, b)^{\vee} \cong \mathcal{A}(b, -)$ and $\mathcal{A}(b, -)^{\vee} \cong \mathcal{A}(-, b)$.

Reflexive functors on small categories

For $X\colon \mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ and $Y\colon \mathcal{A} \longrightarrow \mathbf{Set}$,

$$\mathrm{Hom}(X, Y^{\vee}) \cong \mathrm{Hom}(Y, X^{\vee}).$$

So there is an adjunction

$$[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}] \underset{\vee^{\mathrm{op}}}{\overset{\vee}{\rightleftarrows}} [\mathcal{A}, \mathbf{Set}]^{\mathrm{op}}$$

whose unit and counit maps look like

$$\eta_X\colon X \longrightarrow X^{\vee\vee}, \quad \eta_Y\colon Y \longrightarrow Y^{\vee\vee}.$$

The functor X is reflexive if η_X is an isomorphism. Then $X \cong X^{\vee\vee}$.

Similarly for Y.

The reflexive completion of a small category

The reflexive completion $\mathcal{R}(\mathcal{A})$ of $\mathcal{A}$ is the full subcategory of $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ consisting of the reflexive functors.

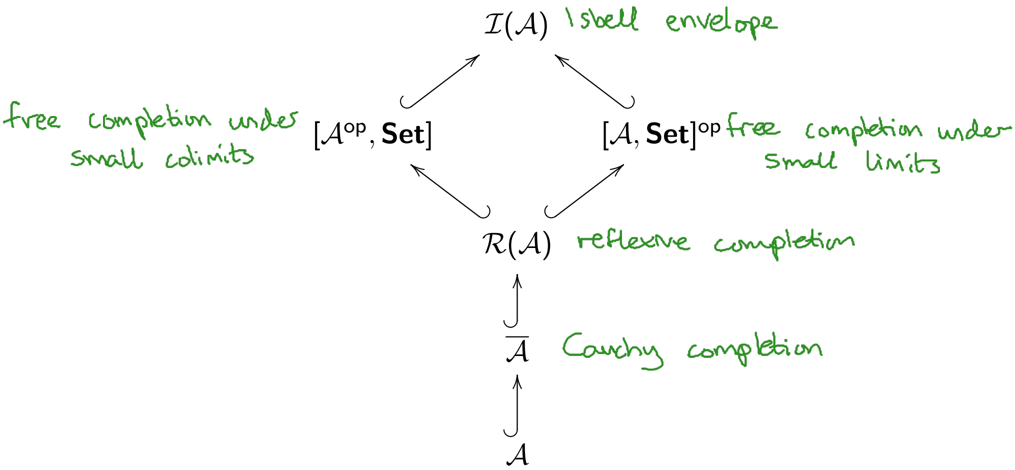
Equivalently, $\mathcal{R}(\mathcal{A})$ is the full subcategory of $[\mathcal{A}, \mathbf{Set}]^{\mathrm{op}}$ consisting of the reflexive functors.

That is, $\mathcal{R}(\mathcal{A})$ is the fixed category (invariant part) of the conjugacy adjunction

$$[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}] \underset{\vee^{\mathrm{op}}}{\overset{\vee}{\rightleftarrows}} [\mathcal{A}, \mathbf{Set}]^{\mathrm{op}}$$

E.g. Representables are reflexive, so there is a natural embedding $J_{\mathcal{A}} \colon \mathcal{A} \hookrightarrow \mathcal{R}(\mathcal{A})$.

The reflexive completion versus other kinds of completion



From small to locally small

Let $\mathcal{A}$ be any category, locally small but not necessarily small.

For $X \colon \mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Set}$, the formula

$$\text{“ } X^{\vee}(a) = [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}](X, \mathcal{A}(-, a)) \text{ ”}$$

doesn't necessarily define a functor $\mathcal{A} \longrightarrow \mathbf{Set}$, since the right-hand side need not be a set.

Small functors

A **Set**-valued functor is small if it is a small colimit of representables.

Examples

- If $\mathcal{A}$ is small then every functor $\mathcal{A}^{(\mathrm{op})} \longrightarrow \mathbf{Set}$ is small.
- If $\mathcal{A}$ is large and discrete then the terminal functor $\mathcal{A} \longrightarrow \mathbf{Set}$ is not small.

Write

$$\widehat{\mathcal{A}} = (\text{small functors } \mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Set}) \subseteq_{\text{full}} [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$$

(free completion of $\mathcal{A}$ under small colimits), and

$$\check{\mathcal{A}} = (\text{small functors } \mathcal{A} \longrightarrow \mathbf{Set})^{\mathrm{op}} \subseteq_{\text{full}} [\mathcal{A}, \mathbf{Set}]^{\mathrm{op}}$$

(free completion of $\mathcal{A}$ under small limits).

Conjugacy for general categories

Let $\mathcal{A}$ be a locally small category.

When $X : \mathcal{A}^{\text{op}} \longrightarrow \mathbf{Set}$ is small, we *can* define $X^\vee : \mathcal{A} \longrightarrow \mathbf{Set}$ by the usual formula,

$$X^\vee(a) = [\mathcal{A}^{\text{op}}, \mathbf{Set}](X, \mathcal{A}(-, a))$$

...although $X^\vee : \mathcal{A} \longrightarrow \mathbf{Set}$ needn't be small itself.

And dually.

We still have

$$\text{Hom}(X, Y^\vee) \cong \text{Hom}(Y, X^\vee)$$

and we still have unit maps $\eta_X : X \longrightarrow X^{\vee\vee}$ and $\eta_Y : Y \longrightarrow Y^{\vee\vee}$.

The reflexive completion of a general category

Let $\mathcal{A}$ be a locally small category.

A functor $X\colon \mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is reflexive if X and $X^\vee$ are small and $\eta_X\colon X \longrightarrow X^{\vee\vee}$ is an isomorphism.

And dually for $Y\colon \mathcal{A} \longrightarrow \mathbf{Set}$.

The reflexive completion $\mathcal{R}(\mathcal{A})$ of $\mathcal{A}$ is the full subcategory of $\widehat{\mathcal{A}}$ consisting of the reflexive functors $\mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Set}$.

Equivalently, it is the full subcategory of $\check{\mathcal{A}}$ consisting of the reflexive functors $\mathcal{A} \longrightarrow \mathbf{Set}$.

E.g. Representables are still reflexive, so we still have an embedding $J_{\mathcal{A}}\colon \mathcal{A} \hookrightarrow \mathcal{R}(\mathcal{A})$.

The reflexive completion has a universal property, as follows. . .

Dense and adequate functors

A functor $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ is dense if its nerve functor

$$
\begin{array}{rccc}
N_F\colon & \mathcal{B} & \longrightarrow & [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}] \\
 & b & \mapsto & \mathcal{B}(F-, b)
\end{array}
$$

is full and faithful.

E.g. Yoneda: $\mathcal{A} \hookrightarrow [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ is dense.

F is small-dense if it is dense and $\mathcal{B}(F-, b)$ is small for each b. Then $N_F\colon B \longrightarrow \widehat{\mathcal{A}}$.

E.g. Yoneda: $\mathcal{A} \hookrightarrow \widehat{\mathcal{A}}$ is small-dense.

F is [small-]adequate if full, faithful, [small-]dense and [small-]codense.

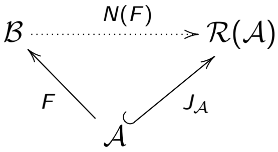
E.g. $J_{\mathcal{A}}\colon \mathcal{A} \longrightarrow \mathcal{R}(\mathcal{A})$ is small-adequate.

The universal property of the reflexive completion

Theorem (Isbell, more or less) $\mathcal{R}(\mathcal{A})$ *is the largest category containing* $\mathcal{A}$ *as a small-adequate subcategory.*

Analogy The completion of a metric space A is the largest space containing A as a dense subspace.

Precise statement: for any small-adequate functor $F\colon \mathcal{A} \longrightarrow \mathcal{B}$, there is a unique full and faithful functor $N(F)\colon \mathcal{B} \longrightarrow \mathcal{R}(\mathcal{A})$ such that



commutes. Moreover, $N(F)$ is adequate.

2. How is the reflexive completion functorial?

A no-go theorem

Most kinds of completion are functorial. Is reflexive completion?

Not in the obvious way:

Theorem *There is no functor (or even pseudofunctor)* $\mathbf{CAT} \longrightarrow \mathbf{CAT}$ *or* $\mathbf{CAT}^{op} \longrightarrow \mathbf{CAT}$ *acting as* $\mathcal{A} \mapsto \mathcal{R}(\mathcal{A})$ *on objects.*

How the reflexive completion is functorial

But $\mathcal{R}$ is functorial in a more restricted way: any *small-adequate* functor $F\colon \mathcal{A} \longrightarrow B$ induces an adequate functor

$$\begin{array}{ccc} \mathcal{R}(\mathcal{A}) & \xleftarrow{\mathcal{R}(F)} & \mathcal{R}(\mathcal{B}) \\ {\scriptstyle J_{\mathcal{A}}} \uparrow & & \uparrow {\scriptstyle J_{\mathcal{B}}} \\ \mathcal{A} & \xrightarrow[F]{} & \mathcal{B}, \end{array}$$

which is unique such that the square commutes. This follows from the universal property.

So we have a pseudofunctor

$$\mathcal{R}\colon (\text{cats} + \text{small-adequate functors})^{\text{op}} \longrightarrow (\text{cats} + \text{adequate functors}).$$

If $\mathcal{R}(\mathcal{A})$ is seen as $\subseteq [\mathcal{A}^{\text{op}}, \mathbf{Set}]$, and similarly $\mathcal{R}(\mathcal{B})$, then $\mathcal{R}(F) = - \circ F$.

3. A mystery

The difficulty of giving examples

We know that every small-adequate functor $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ induces an adequate functor $\mathcal{R}(F)\colon \mathcal{R}(\mathcal{B}) \longrightarrow \mathcal{R}(\mathcal{A})$.

But it's hard to find any nontrivial examples!

For almost every small-adequate F you can think of, $\mathcal{R}(F)$ is an equivalence.

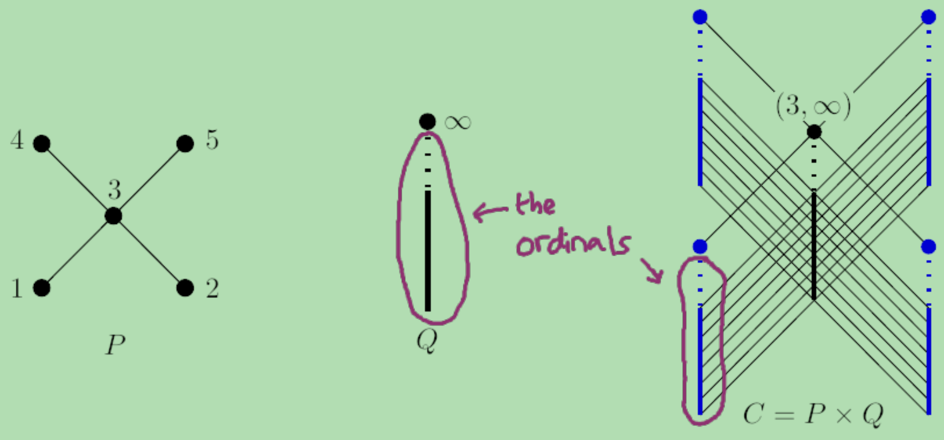
A confession. . .

induces an equivalence $\mathscr{R}(\mathscr{A}) \simeq \mathscr{R}(\mathscr{B})$:

$$\begin{array}{ccc} \mathscr{R}(\mathscr{A}) & \overset{\simeq}{\dashrightarrow} & \mathscr{R}(\mathscr{B}) \\ \uparrow & & \uparrow \\ \mathscr{A} & \longrightarrow & \mathscr{B}. \end{array}$$

$\mathcal{R}(F)$ is not always an equivalence

...but the simplest known counterexample is complicated (Avery and Leinster, TAC, 2021):



So it seems that $\mathcal{R}(F)$ is *very often* an equivalence.

What explains this?

4. The mystery resolved

Adequacy versus small-adequacy

Recall how the functorial action of $\mathcal{R}$ was defined...

But $\mathcal{R}$ is functorial in a more restricted way: any *small-adequate* functor $F\colon \mathcal{A} \longrightarrow B$ induces an adequate functor

$$\begin{array}{ccc} \mathcal{R}(\mathcal{A}) & \xleftarrow{\mathcal{R}(F)} & \mathcal{R}(\mathcal{B}) \\ {\scriptstyle J_{\mathcal{A}}}\uparrow & & \uparrow{\scriptstyle J_{\mathcal{B}}} \\ \mathcal{A} & \xrightarrow[F]{} & \mathcal{B}, \end{array}$$

which is unique such that the square commutes. This follows from the universal property.

So we have a pseudofunctor

$$\mathcal{R}\colon (\text{cats} + \text{small-adequate functors})^{\text{op}} \longrightarrow (\text{cats} + \text{adequate functors}).$$

...and recall that small-adequacy is stronger than adequacy.

Lemma $\mathcal{R}(F)$ is *small*-adequate $\iff$ $\mathcal{R}(F)$ is an equivalence.

Moral Whether $\mathcal{R}(F)$ is an equivalence comes down to a *size condition*.

Why $\mathcal{R}(F)$ is so often an equivalence

Call a category $\mathcal{A}$ gentle if:

- its free cocompletion $\widehat{\mathcal{A}}$ is complete, and
- its free completion $\check{\mathcal{A}}$ is cocomplete.

Very many categories are gentle! Examples:

- Small categories are gentle, since $\widehat{\mathcal{A}} = [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ etc.
- Theorem of Day and Lack: complete & cocomplete $\Rightarrow$ gentle.

Take a small-adequate functor $F \colon \mathcal{A} \longrightarrow \mathcal{B}$.

Although $\mathcal{R}(F) \colon \mathcal{R}(\mathcal{B}) \longrightarrow \mathcal{R}(\mathcal{A})$ is not *always* an equivalence, we do have:

Theorem *If $\mathcal{B}$ is gentle then $\mathcal{R}(F)$ is an equivalence.*

Since so many categories are gentle, this explains why $\mathcal{R}(F)$ is so often an equivalence.

References

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- Tom Avery and Tom Leinster. Isbell conjugacy and the reflexive completion. *Theory and Applications of Categories* 36 (2021), 306–347.