The mathematics of biodiversity

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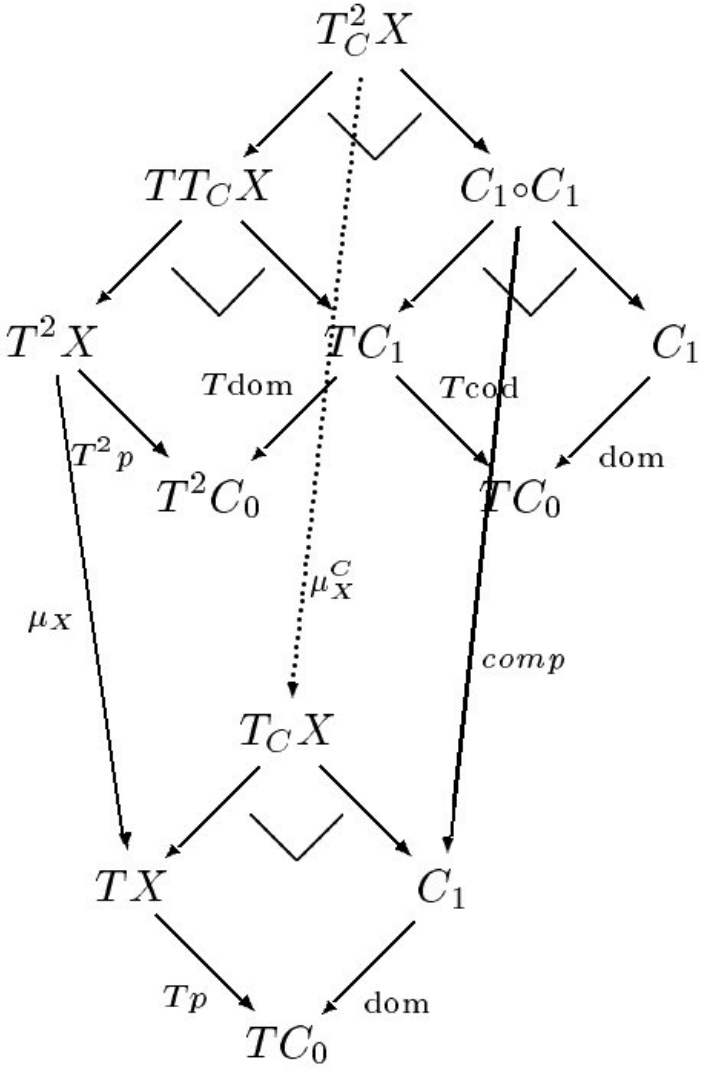
We will go on a journey. . .

. . . from here. . .

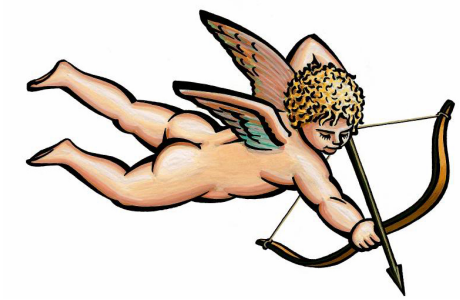


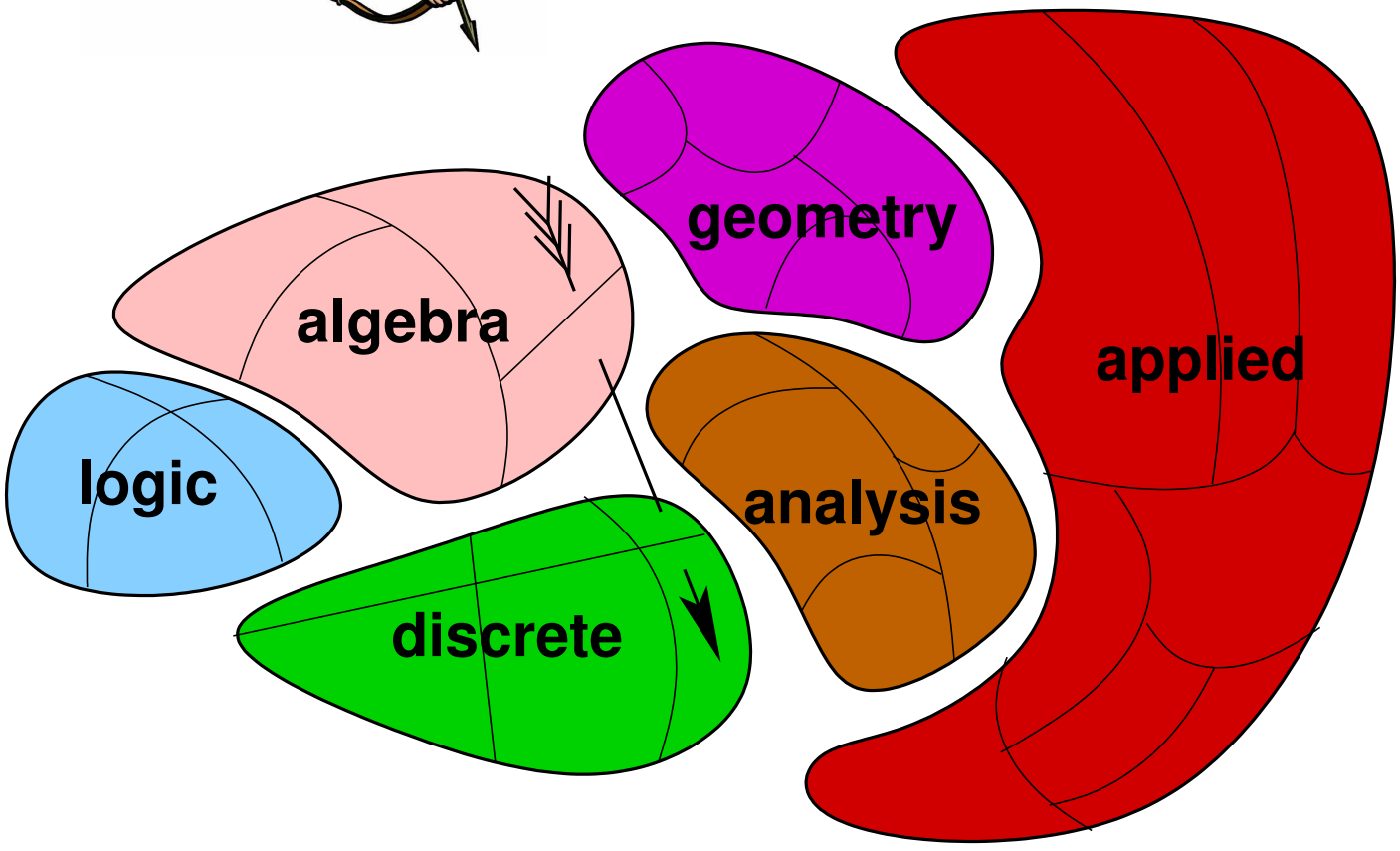
. . . to here.

Some
apparently pure
mathematics

A not-too-serious map of mathematics

Category theory



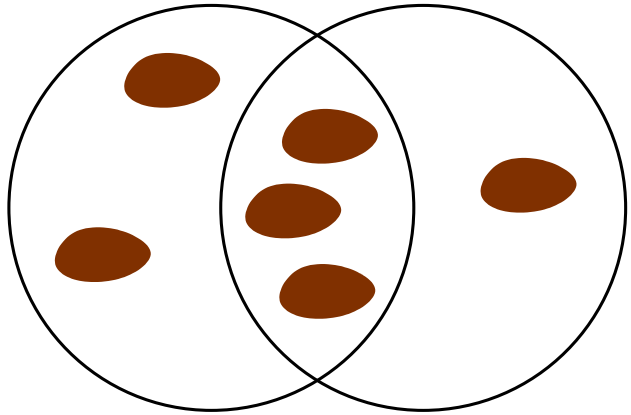


- Category theory is also *part* of mathematics.
- The map is constantly being redrawn.

Counting potatoes

The simplest notion of 'size' is the number of things.

Basic rule: the inclusion-exclusion principle:



On left: 5

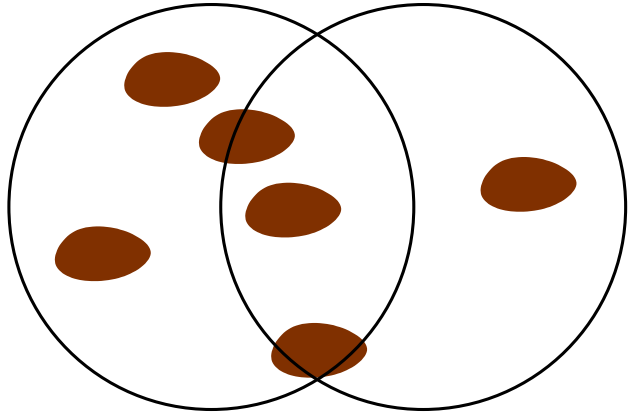
In middle: 3

On right: 4

Total: $5 + 4 - 3 = 6$ ✓

Counting potatoes

Less obvious: the inclusion-exclusion principle still holds if we allow overlaps!



On left: 5

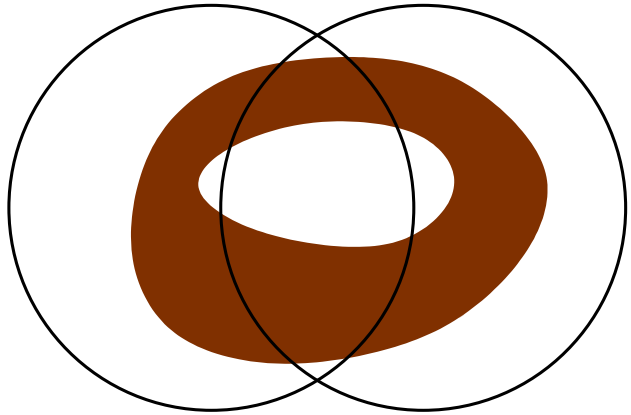
In middle: 3

On right: 4

Total: $5 + 4 - 3 = 6$ ✓

Counting potatoes

An unusually shaped potato:



On left: 1

In middle: 2

On right: 1

Total: $1 + 1 - 2 = 0$?!?

Can that really be right?

Yes! It's the story of Euler characteristic. . . which I won't tell.

Other ways to measure a potato

- Its volume is 84cm^3.
- Its surface area is 91cm^2.
- Its mean width is 7cm.

Or simply:

- It's 1 potato.

All four measures obey the inclusion-exclusion principle.

A general notion of size

Using tools from category theory, it's possible to formulate a very broad definition of 'size', called magnitude, which:

- makes sense in lots of different branches of mathematics (geometry, algebra, . . .)
- appears to link together various old notions of size (volume, number of things, . . .)
- also produces some interesting new quantities.

The magnitude of a collection of points

The magnitude of a collection of points can be thought of as the 'effective number of points'.

Magnitude: 1

The magnitude of a collection of points

The magnitude of a collection of points can be thought of as the 'effective number of points'.

Magnitude: 1.01

The magnitude of a collection of points

The magnitude of a collection of points can be thought of as the 'effective number of points'.

Magnitude: 1.2

The magnitude of a collection of points

The magnitude of a collection of points can be thought of as the 'effective number of points'.

Magnitude: 1.6

The magnitude of a collection of points

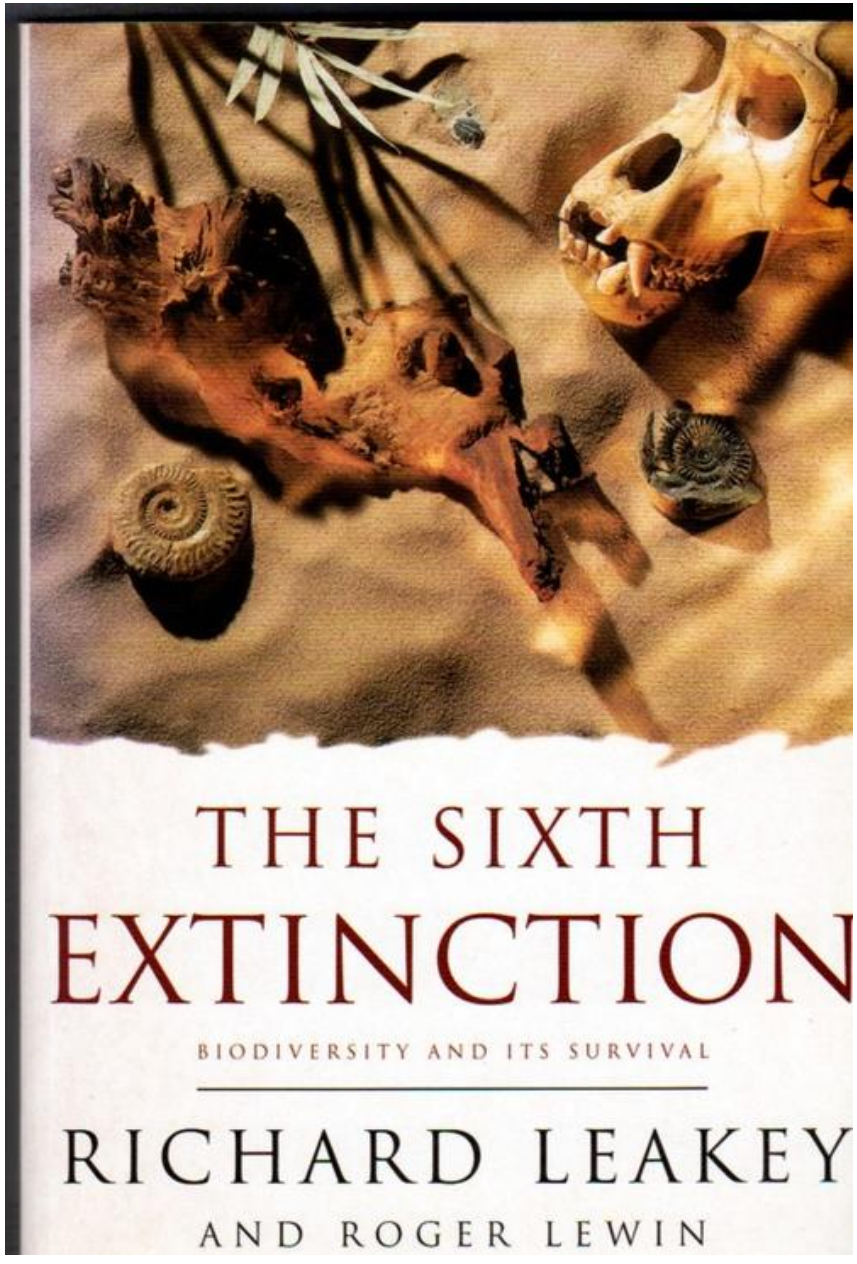
The magnitude of a collection of points can be thought of as the 'effective number of points'.

Magnitude: 2.3

As the points get further apart, the magnitude gets closer to 3.

Biological diversity

The global biodiversity crisis



It is an indication of misplaced priorities within the scientific community that relatively unimportant exercises such as the sequencing of the human genome can take priority over the assessment and preservation of Earth's irreplaceable botanical wealth.

Paul Ehrlich, 1992

The global biodiversity crisis

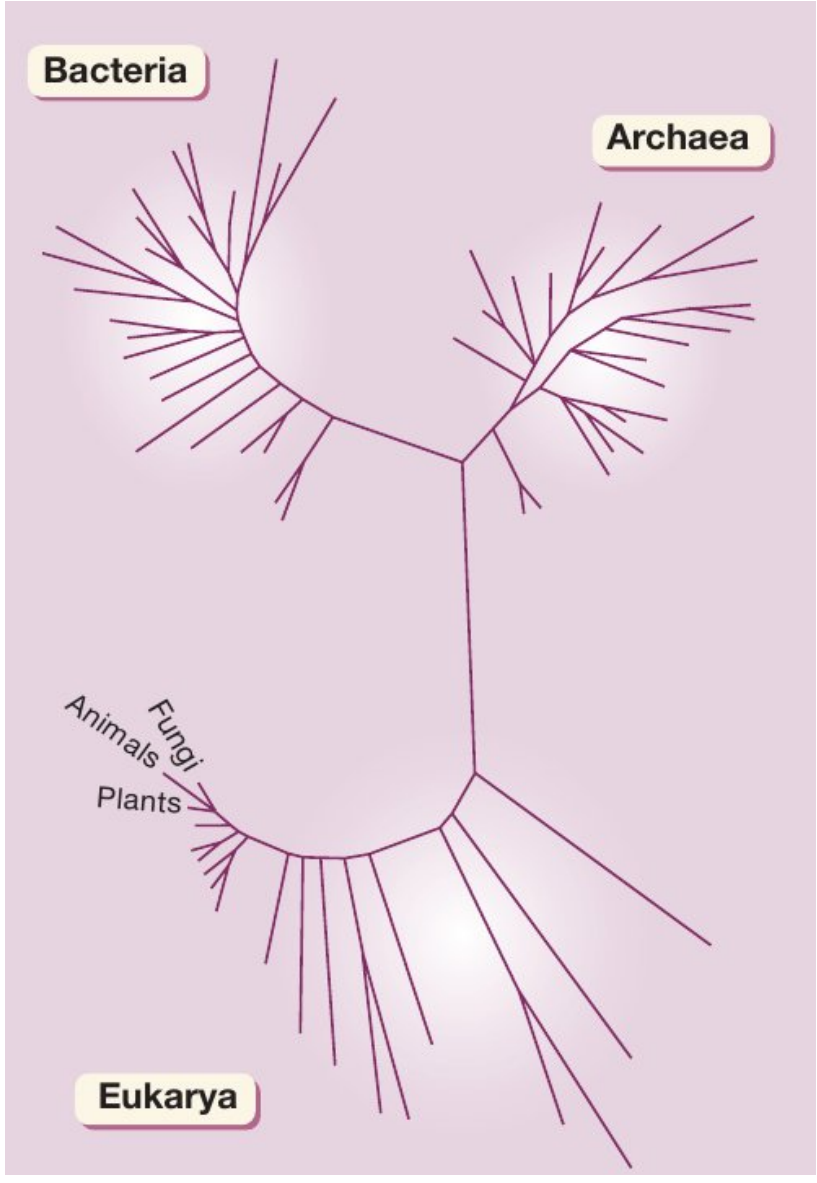
Photo by Ray Wiltshire/Rex Features

The global biodiversity crisis

Photo by Arthur Anker/Animal Earth/Thames & Hudson

The global biodiversity crisis

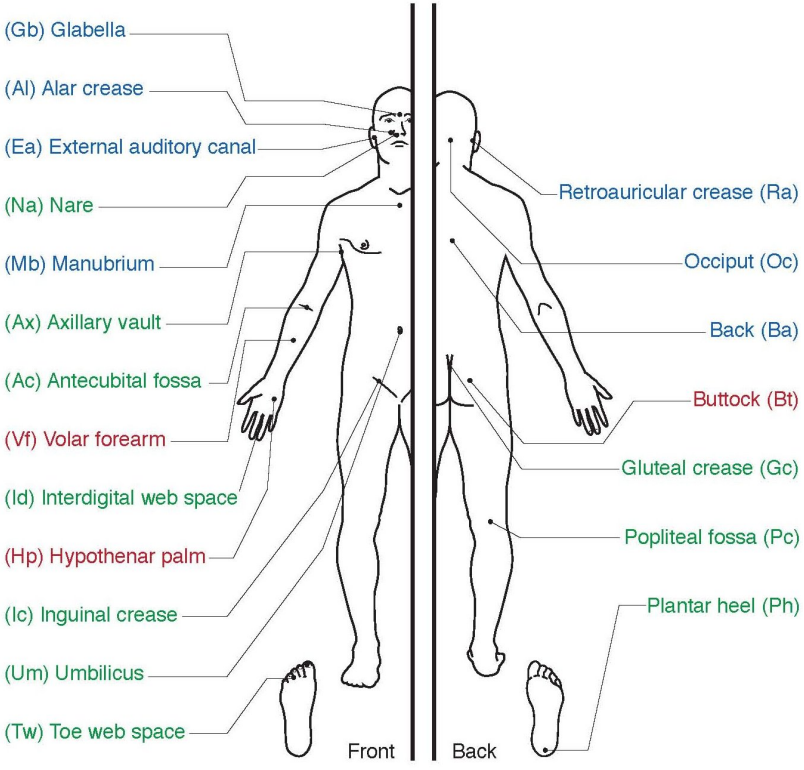
The tree of life



Most life is invisible!

Your fellow travellers

Within the body of a healthy adult, microbial cells are estimated to outnumber human cells by a factor of ten to one. —BEI Resources



⟵these are just the ones on your skin

The diversity of this ecosystem matters! E.g. experiments suggest:

more diverse gut bacteria ⇜⇝ less likely to be obese.

Viruses and vaccines

Q. Why can't we have a single flu vaccination that lasts a lifetime?

A. Because the flu virus evolves *fast* — different strains appear every year.

The more diverse this year's collection of strains is, the more different vaccines you need.

Here, diversity is bad news (for us!)

What's the best measure of diversity?

There's been half a century of hot debate over this, leading to...

Despair:

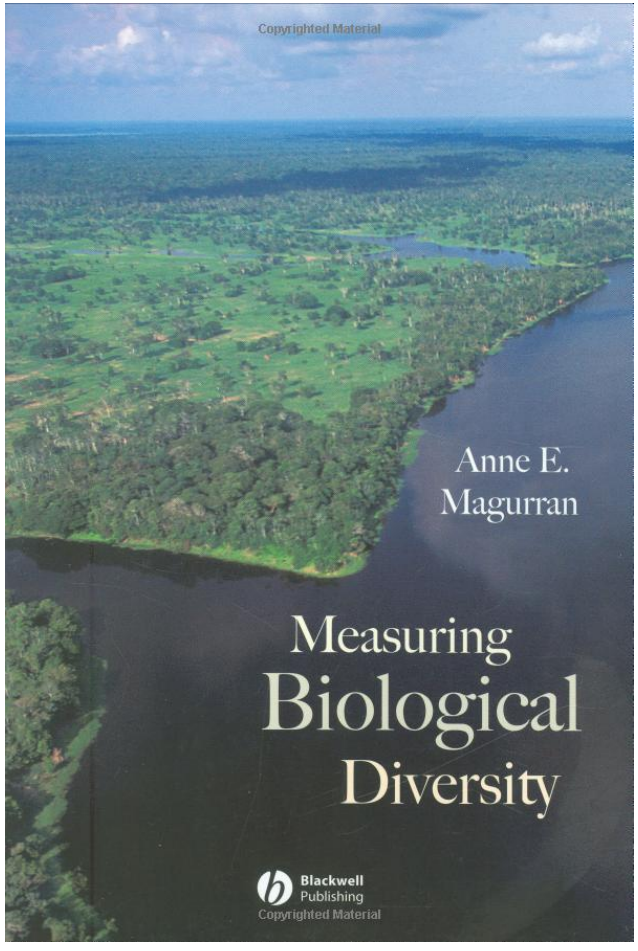
Stuart Hurlbert,
The nonconcept of species diversity,
1971.

Scorn:

The belief (or superstition) of some ecologists that a diversity index provides a basis (or talisman) for reaching a full understanding of community structure is totally unfounded.

—Evelyn Pielou, 1975.

Much scholarly effort:



What's the best measure of diversity?

The simplest notion of diversity is the number of species.

But sometimes things aren't so simple.

E.g. which of these two communities is more diverse?

more species, less balanced

fewer species, more balanced

Can't say, objectively: there are simply different viewpoints.

The mathematics of biodiversity

Making a connection, part 1

Barcelona, 2008: I gave a talk on magnitude ('effective number of points')...

André Joyal: *Don't you think magnitude is a bit like entropy?*

Me: *Huh?*

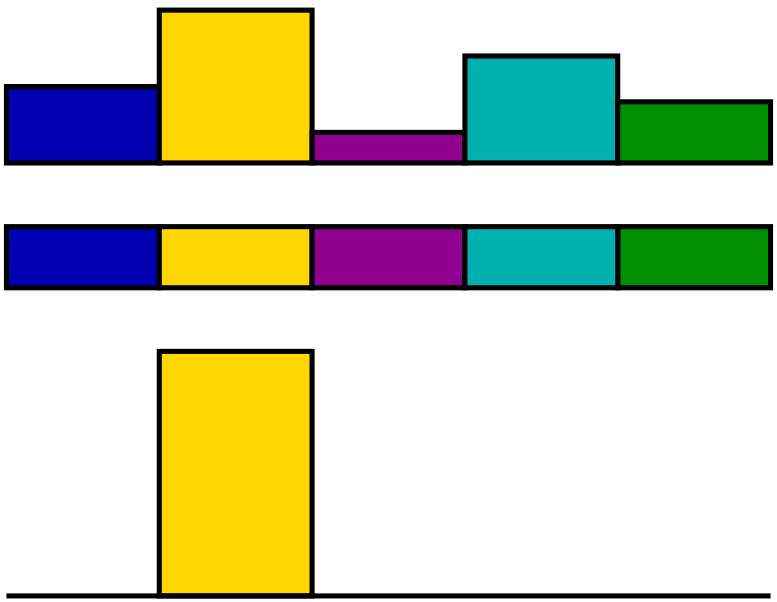
Entropy

Take numbers $p_1, p_2, \ldots, p_n$ between 0 and 1, adding up to 1.

Their entropy is

$$\frac{1}{p_1^{p_1} p_2^{p_2} \cdots p_n^{p_n}}.$$

Examples:



Entropy: about 3.8.

Entropy: exactly 5.

Entropy: exactly 1.

(Caveats: this is actually the *exponential* of entropy.
And it's just one of *many* quantities called entropy.)

Making a connection, part 2

Back home in Glasgow. . .

Christina Cobbold: *That's funny. Ecologists sometimes use entropy to measure biodiversity.*

Me: *Huh?*

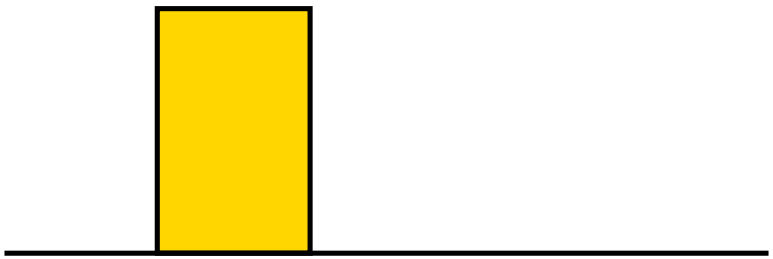
Entropy as a measure of biodiversity

Suppose we have an ecosystem consisting of n species, in proportions $p_1, p_2, \ldots, p_n$.

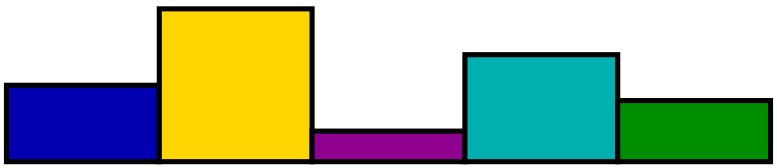
The entropy can be thought of as the 'effective number of species'.

Examples:

Effectively 5 species.



Effectively 1 species.



Effectively 3.8 species.

What's going on?

Ecologists sometimes consider 'effective numbers of species', taking into account how abundant they are.

In my 'pure' work, I'd considered 'effective numbers of points', taking into account how far apart they are.

Bringing it all together

A good measure of biodiversity should take two things into account:

- how abundant the various species are; but also
- how different the species are.

0.5

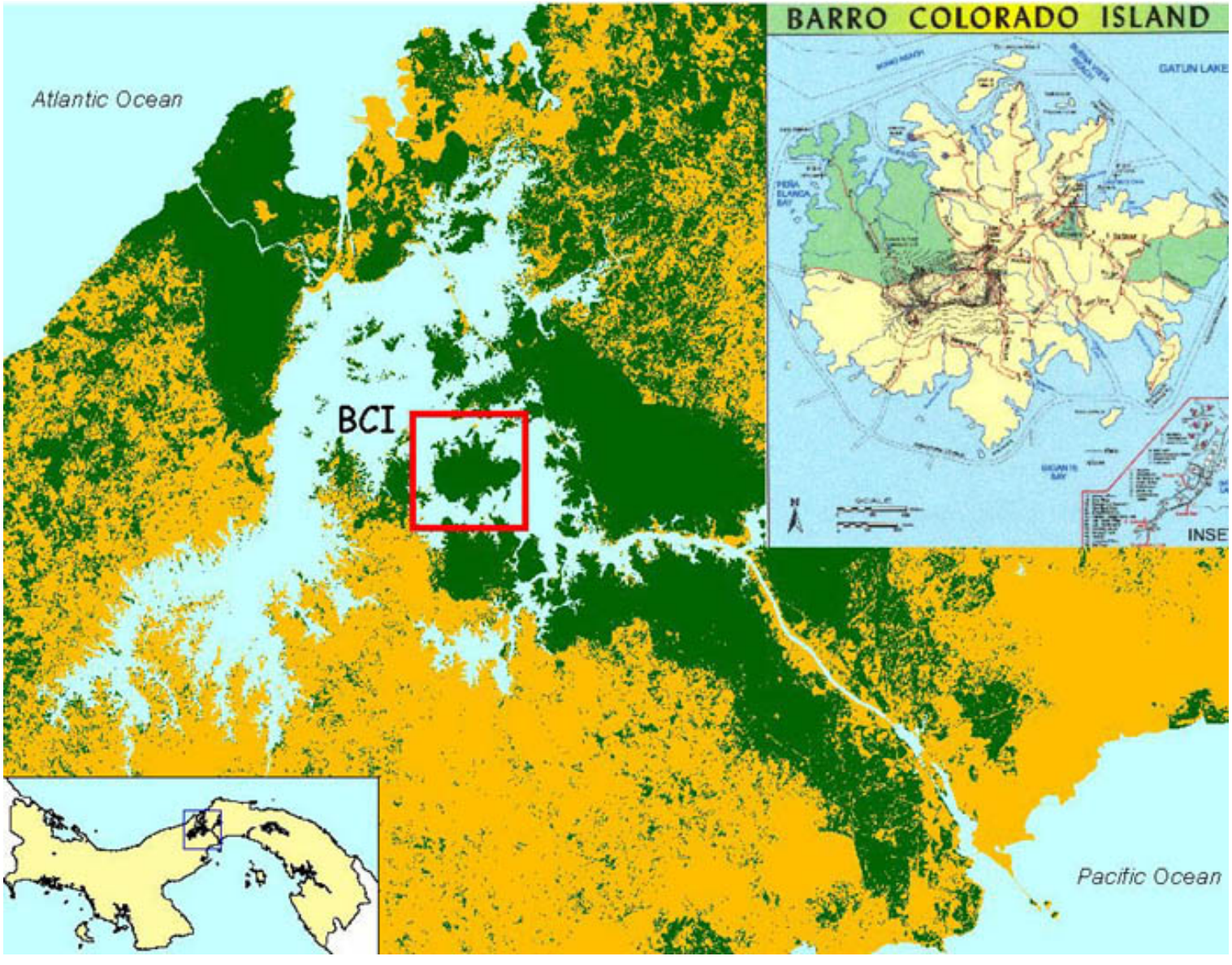
0.3

0.2

Almost no existing measures of diversity did that!

Christina Cobbold and I combined two ideas — entropy and magnitude — to make a diversity measure that reflects both abundance and difference. It seems to work well...

Barro Colorado Island, Panama



The island is mostly tropical forest.

Barro Colorado Island, Panama

A common problem:

Money for conservation is scarce.

How do we decide where to spend it?

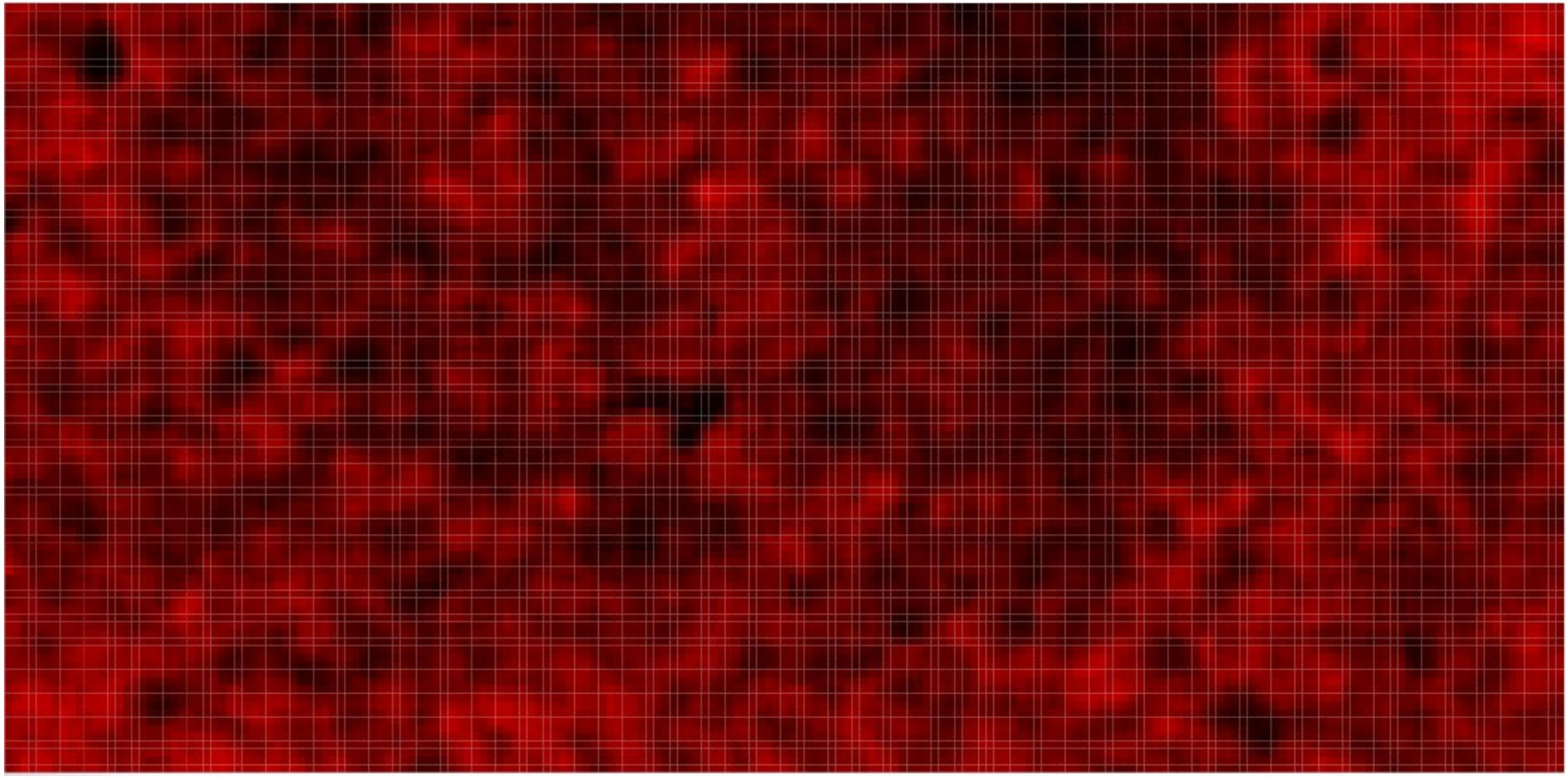
We need quantitative tools to identify areas of high diversity — or unusual areas.

Richard Reeve and Louise Matthews have developed some. . .

Barro Colorado Island, Panama

Red: areas of high variation.

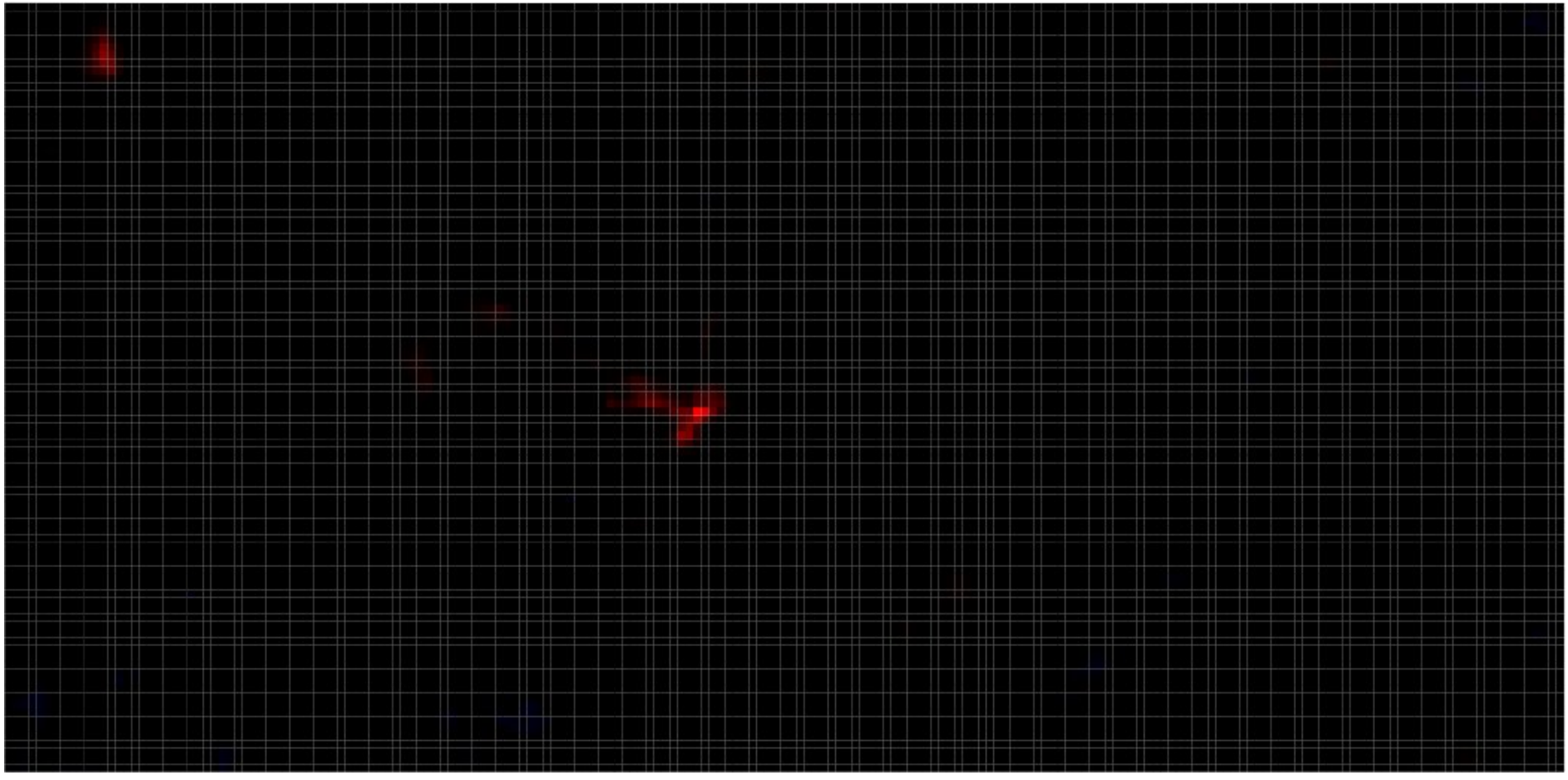
Black: areas of low variation.



Barro Colorado Island, Panama

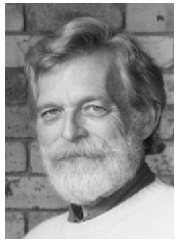
Red: areas most different from rest of forest.

Black: areas most similar to rest of forest.



Thanks

Christina Cobbold



Steve Schanuel

Louise Matthews

Simon Willerton

Sean Nee

Chris Quince

Richard Reeve

The Barro Colorado Island project

(The BCI forest dynamics research project was made possible by National Science Foundation grants to Stephen P. Hubbell: DEB-0640386, DEB-0425651, DEB- 0346488, DEB-0129874, DEB-00753102, DEB-9909347, DEB-9615226, DEB- 9615226, DEB-9405933, DEB-9221033, DEB-9100058, DEB-8906869, DEB- 8605042, DEB-8206992, DEB-7922197, support from the Center for Tropical Forest Science, the Smithsonian Tropical Research Institute, the John D. and Catherine T. MacArthur Foundation, the Mellon Foundation, the Small World Institute Fund, and numerous private individuals, and through the hard work of over 100 people from 10 countries over the past two decades. The plot project is part of the Center for Tropical Forest Science, a global network of large-scale demographic tree plots.)