OPERADS

I. BASICS

II. OPERADS IN HIGHER CATEGORY THEORY

RUNNING THEME

	View (i)	View (ii)
Groups	... as transformation groups	... as abstract structures
Lawvere theories	... as algebraic theories	... as abstract categories
Categories	... as domains of Set-valued functors	... as structures in their own right
Operads	... as algebraic theories	... as structures in their own right

I.1 PLAIN OPERADS

View (i): An operad is an algebraic theory for which you can take models in any monoidal category.

E.g. there is an operad for monoids, but not an operad for groups.

(Universal algebra approach: a **strongly regular** theory is one presentable by equations in which

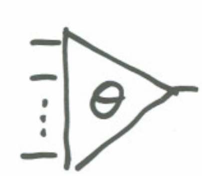
the same variables appear
in the same order
without repetition

on each side.)

Definition of operad

A (plain) operad consists of:

- a sequence $(P_n)_{n \in \mathbb{N}}$ of sets

  (whose elements θ can be thought of as "n-ary operations" and drawn )

- for each $n, k_1, \ldots, k_n \in \mathbb{N}$, a function

$$P_n \times P_{k_1} \times \cdots \times P_{k_n} \longrightarrow P_{k_1 + \cdots + k_n}$$
$$(\theta, \theta_1, \ldots, \theta_n) \mapsto \theta \circ (\theta_1, \ldots, \theta_n)$$

  (composition), e.g.

- an element $1 \in P_1$ (identity),

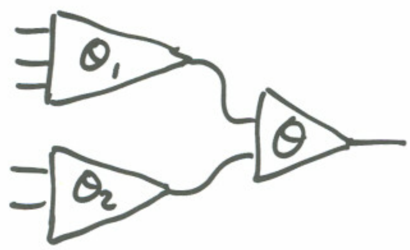
satisfying associativity & identity axioms, so that every tree of operations has a well-defined composite.

Definition of algebra ("model")

Let P be an operad. A P-algebra consists of

- a set A
- for each $n \in \mathbb{N}$ & $\theta \in P_n$, a function $\bar{\theta} : A^n \to A$

satisfying compatibility axioms, e.g. if



and $a_1, \ldots, a_5 \in A$

then

$$\overline{\theta \circ (\theta_1, \theta_2)}\,(a_1, \ldots, a_5) = \bar{\theta}\left(\bar{\theta}_1(a_1, a_2, a_3), \bar{\theta}_2(a_4, a_5)\right).$$

E.g. Take terminal operad, $P_n = 1\ \forall n$. Then a P-algebra is a set A with a function $*_n : A^n \to A$ for each n, satisfying axioms — i.e., a monoid.

E.g. Given a monoid M, define $P_n = \begin{cases} M & \text{if } n = 1 \\ \emptyset & \text{otherwise} \end{cases}$.

Then a P-algebra is an M-set.

View (ii):

Some operads not (usually) regarded as theories

- If $\mathcal{A}$ is a monoidal category and $A \in \mathcal{A}$, its endomorphism operad End(A) is given by

$$(\mathrm{End}(A))_n = \mathcal{A}(A^{\otimes n}, A)$$

and substitution.

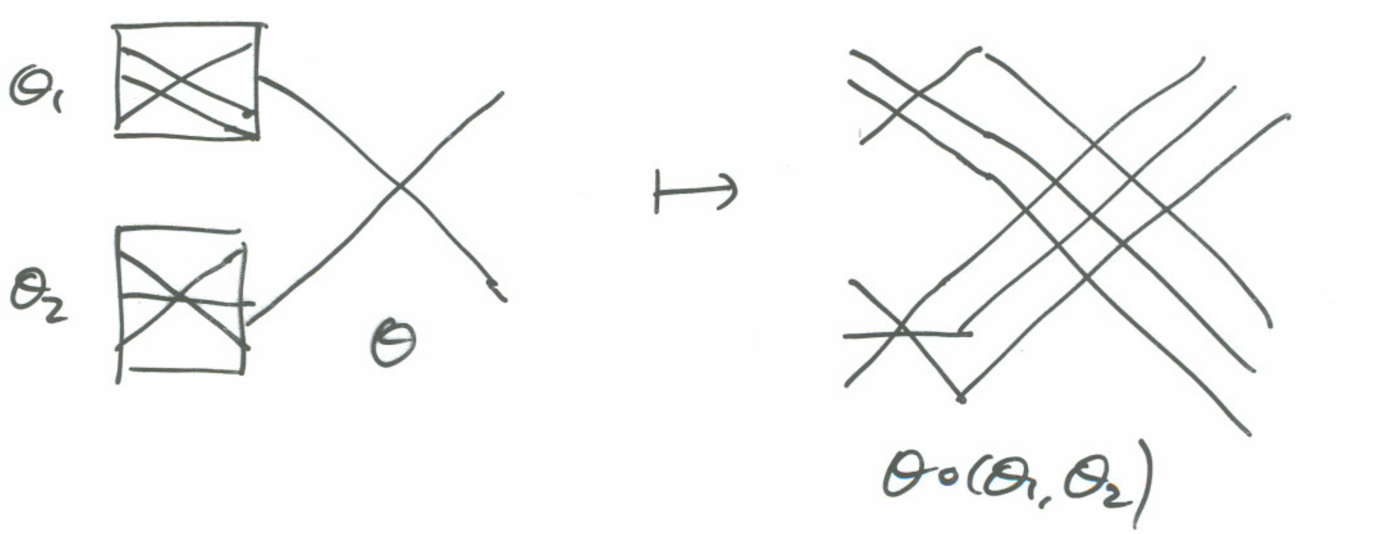
Remark: If P is an operad, a P-algebra is a set A plus a map $P \longrightarrow \mathrm{End}(A)$ of operads.

- There is an operad P given by

$$P_n = \mathbb{Z}[X_1, \ldots, X_n]$$

and substitution.

- The symmetric groups $(S_n)_{n \in \mathbb{N}}$ form an operad in an interesting way ("permute in blocks")



I.2 ELABORATIONS

Enrichment

Instead of taking the P_n's to be sets, can take them to be objects of a sym mon cat $\mathcal{V}$.

Instead of taking A to be a set, can take it to be an object of a cat $\mathcal{A}$ <u>either</u> enriched in <u>or</u> acted on by $\mathcal{V}$. (Often $\mathcal{A} = \mathcal{V}$.)

E.g. ($\mathcal{V} = \mathcal{A} = \mathrm{Top}_*$.) Given $Y \in \mathrm{Top}_*$, the loop space of Y is $\Omega Y = \underline{\mathrm{Top}_*}(S^1, Y)$.

Q. What algebraic structure does every loop space carry?

A. A P-algebra structure, where

$$P_n = \underline{\mathrm{Top}_*}(S^1, S^1 \vee \cdots \vee S^1).$$

Reason:

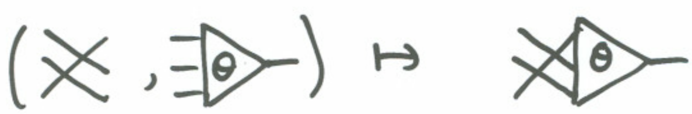
$$\begin{array}{c} (\Omega Y)^n \longrightarrow \Omega Y \quad \text{naturally in } Y \\ \hline \underline{\mathrm{Top}_*}(S^1 \vee \cdots \vee S^1, Y) \longrightarrow \underline{\mathrm{Top}_*}(S^1, Y) \quad \text{nat in } Y \\ \hline S^1 \longrightarrow S^1 \vee \cdots \vee S^1. \end{array}$$

Symmetry

A symmetric operad is an algebraic theory for which you can take models in any symmetric monoidal category.

Formally, each P_n has an S_n-action:

$$S_n \times P_n \longrightarrow P_n$$



E.g. The theory of Lie algebras is described by

$$[x,y] + \tau[x,y] = 0,$$

$$[[x,y],z] + \sigma[[x,y],z] + \sigma^2[[x,y],z] = 0$$

where $\tau = (12) \in S_2$ and $\sigma = (123) \in S_3$.

"Hence" there's a CommMon-enriched operad P whose algebras in Vect_k are k-Lie algebras.

I.3 PRECISE RELATION TO ALGEBRAIC THEORIES

There is a functor

$$\begin{array}{ccc} \mathbf{Operad} & \longrightarrow & (\text{monads on } \mathbf{Set}) \qquad \ldots (*) \\ P & \longmapsto & T_P = \sum_n P_n \times (-)^n \end{array}$$

and $P\text{-}\mathbf{Alg} = T_P\text{-}\mathbf{Alg}$.

Properties:

- Not injective on objects: $\exists P, Q$ with $P \not\cong Q$ but $T_P \cong T_Q$
- Image on objects is {strongly regular finitary theories}
- For all P, the monad T_P is finitary, cartesian, and preserves small connected limits (but not every such monad is of the form T_P)
- $(*)$ has a right adjoint, $T \mapsto (T(n))_{n \in \mathbb{N}}$ (e.g. if T = free ring, get $(\mathbb{Z}[x_1, \ldots, x_n])_{n \in \mathbb{N}}$).
(e.g. if $T = \mathrm{id}$, get $(n)_{n \in \mathbb{N}}$)

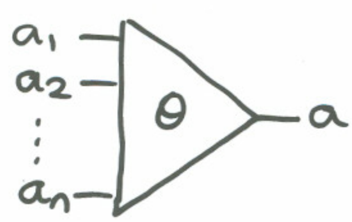
I.4 MULTICATEGORIES

... are $\left\{\begin{array}{l}\text{many-sorted} \\ \text{many-object}\end{array}\right\}$ operads.

Analogies:

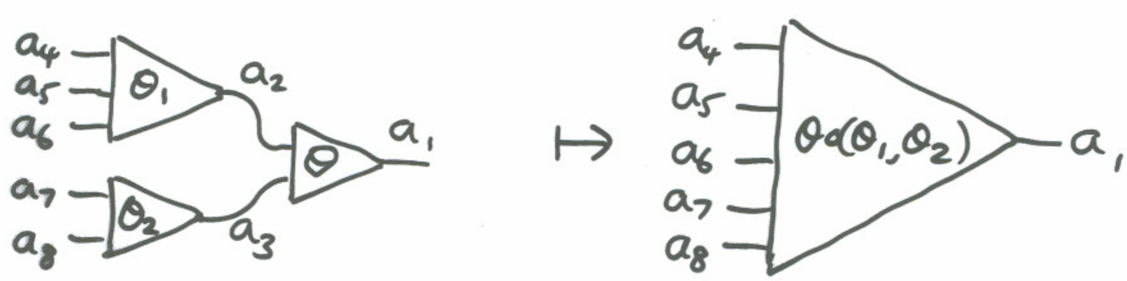
single-sorted theories	multi-sorted theories
monoids	categories
groups	groupoids
operads	multicategories

A multicategory is like a category, but arrows look like



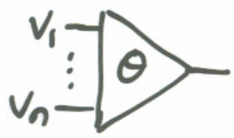
(a_i, a objects)

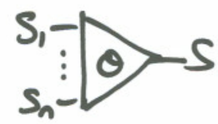
and composition looks like (e.g.)



An operad is just a one-object multicategory.

Examples of multicategories

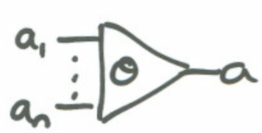
- Vector spaces + linear maps form a category.
  Vector spaces + multilinear maps form a multicategory.
  (Arrows  V are multilinear maps $V_1, \ldots, V_n \to V$.)

- Objects are sets; arrows  are functions $S_1 \times \cdots \times S_n \to S$.

- Similarly for any monoidal category.

- If $\mathcal{E}$ is a finite product category then $Ab(\mathcal{E})$ is a multicat (but not necessarily a mon cat).

- Lambek: objects are propositions, arrows are proofs.

- A category is a multicategory in which every arrow  has $n=1$.

What can you do with a multicategory?

- View (i): take their algebras. An algebra for a multicat C is a map $C \to \mathrm{Set}$ of multicats.
  - When C is an operad (= one-object multicat), this is the same as an operad-algebra
  - When C is a mon cat, this is a lax mon functor
  - Every operad P gives rise to a 2-object multicat Map_P whose algebras are maps of P-algebras.
- View (ii): enrich in them. (More fundamental than enrichment in mon cats?)
- View (ii): take monoids in them. (More fundamental again? E.g. consider quantales.)

II.1 THEORIES OF CATEGORIES-WITH-STRUCTURE

Q. What kind of theory is the theory of (weak) monoidal categories?

A. It's a Cat-operad (= operad enriched in Cat).

Why?

Let A be a mon cat. Then, e.g., we have functors

"" : $A^3 \longrightarrow A$

$$(a_1, a_2, a_3) \mapsto (a_1 \otimes a_2) \otimes a_3$$

"" : $A^3 \longrightarrow A$

$$(a_1, a_2, a_3) \mapsto a_1 \otimes (a_2 \otimes a_3)$$

and an iso between them.

Define a Cat-operad P by

P_n = indiscrete cat on
{n-leafed binary-or-nullary-branching trees}.

Then a P-algebra is exactly a mon cat.

Categorifying an ordinary theory

Intuitively,

theory of monoids

$\rightsquigarrow$ theory of (weak) mon cats.

In fact, there's a precise general process

plain operad P

$\mapsto$ Cat-operad $Wk(P)$,

and a weak P-category can be defined as a $Wk(P)$-algebra.

E.g. $P = 1$: a P-algebra is a monoid, and a weak P-cat is a (weak) mon cat.

Thm (M. Gould): Every weak P-category is equivalent to a strict P-category.

II.2 HIGHER DIMENSIONS

Q. What kind of theory is the/a theory of n-categories?

In ordinary, Set-based, algebra, an operation takes an <u>n-tuple</u> of elements as input, e.g.

$$(a_1, a_2, a_3) \mapsto a_1 \cdot a_2 \cdot a_3 .$$

In higher-dimensional algebra, an operation may take a <u>diagram</u> of data as input, e.g.

$$\begin{tikzcd}[column sep=large]
a \arrow[r, bend left=70, "f_1"] \arrow[r, "f_2" description] \arrow[r, bend right=70, "f_3"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\alpha_1"] \arrow[r, phantom, shift right=3ex, "\Downarrow\alpha_2"] & b \arrow[r, bend left=50, "g_1"] \arrow[r, bend right=50, "g_2"'] \arrow[r, phantom, "\Downarrow\beta"] & c
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}[column sep=huge]
a \arrow[r, bend left=60, "g_1 \circ f_1"] \arrow[r, bend right=60, "g_2 \circ f_3"'] \arrow[r, phantom, "\Downarrow \beta * (\alpha_2 \circ \alpha_1)"] & c
\end{tikzcd} .$$

One possible A. A generalized or "higher-dimensional" operad, capable of handling such shapes.

The point: sequences (one-dimensional) are replaced by higher-dimensional shapes.

II.3 GENERALIZED OPERADS & MULTICATEGORIES
(Burroni)

A category C consists of sets & functions

$$\begin{tikzcd} & C_1 \arrow[dl, "dom"'] \arrow[dr, "cod"] & \\ C_0 & & C_0 \end{tikzcd} \qquad \begin{array}{c} C_1 \times_{C_0} C_1 \xrightarrow{comp} C_1 \\ C_0 \xrightarrow{ids} C_1 \end{array}$$

satisfying associativity & identity axioms.

Write T for the free monoid monad on **Set**.
A multicategory C consists of sets & functions

$$\begin{tikzcd} & C_1 \arrow[dl, "dom"'] \arrow[dr, "cod"] & \\ TC_0 & & C_0 \end{tikzcd} \qquad \left.\begin{array}{c} C_1 \times_{TC_0} TC_1 \xrightarrow{comp} C_1 \\ C_0 \xrightarrow{ids} C_1 \end{array}\right\} (\dagger)$$

satisfying associativity & identity axioms.
(Here comp: $(\theta, (\theta_1, \ldots, \theta_n)) \mapsto \theta \circ (\theta_1, \ldots, \theta_n)$.)

Let $\mathcal{E}$ be a cat with pullbacks and $T = (T, \eta, \mu)$ a cartesian monad on $\mathcal{E}$ (i.e. T preserves pullbacks, and the naturality squares of η and μ are pullbacks). A T-multicategory consists of data $(\dagger)$ ($C_0, C_1 \in \mathcal{E}$) satisfying associativity & identity axioms.

A T-operad is a T-multicat C with C_0 terminal.

Examples of generalized operads & multicategories

- $\mathcal{E} = \mathbf{Set}$, $T = \mathrm{id}$: a T-multicategory is a category.
  A T-operad is a monoid.

- $\mathcal{E} = \mathbf{Set}$, $T =$ free monoid: a T-multicat is a multicat.
  A T-operad is a plain operad.

There is a definition of an algebra for a T-operad.

$$X_2 \underset{t}{\overset{s}{\rightrightarrows}} X_1 \underset{t}{\overset{s}{\rightrightarrows}} X_0$$

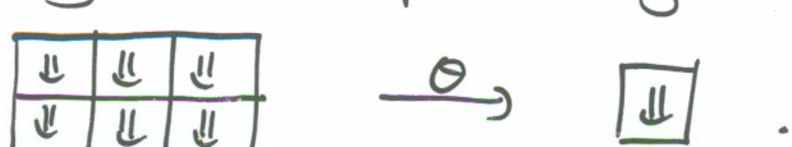
- $\mathcal{E} = 2\text{-}\mathbf{Graph}$, $T =$ free strict 2-cat:
  a T-operad consists of "operations" of shapes like

There are T-operads whose algebras are
  - strict 2-categories
  - bicategories
  - unbiased bicategories

respectively. (Batanin)

- Similarly for n- and ∞-categories

- Similarly for n-tuple categories:



View (ii): an example

Consider $\mathcal{E} = \mathbf{DirGph}$, T = free category.

As it turns out:

$$\begin{tikzcd}
\mathbf{Bicat} \arrow[dr, hook] & \\
\mathbf{DblCat} \arrow[r, hook] & T\text{-}\mathbf{Multicat} \\
\mathbf{Multicat} \arrow[ur, hook] &
\end{tikzcd}$$

You can:

- speak of categories enriched in a T-multicategory (generalizing enrichment in bicats & in multicats)
- do the bimodules construction on T-multicats (without technical restrictions).

Moral: Generalized operads & multicategories are good for more than just being "theories of something else".

Meta-moral: Higher-dimensional category theory is much more than the theory of higher-dimensional categories.