Enriched Categories

Hom sets $\longrightarrow$ Hom objects.

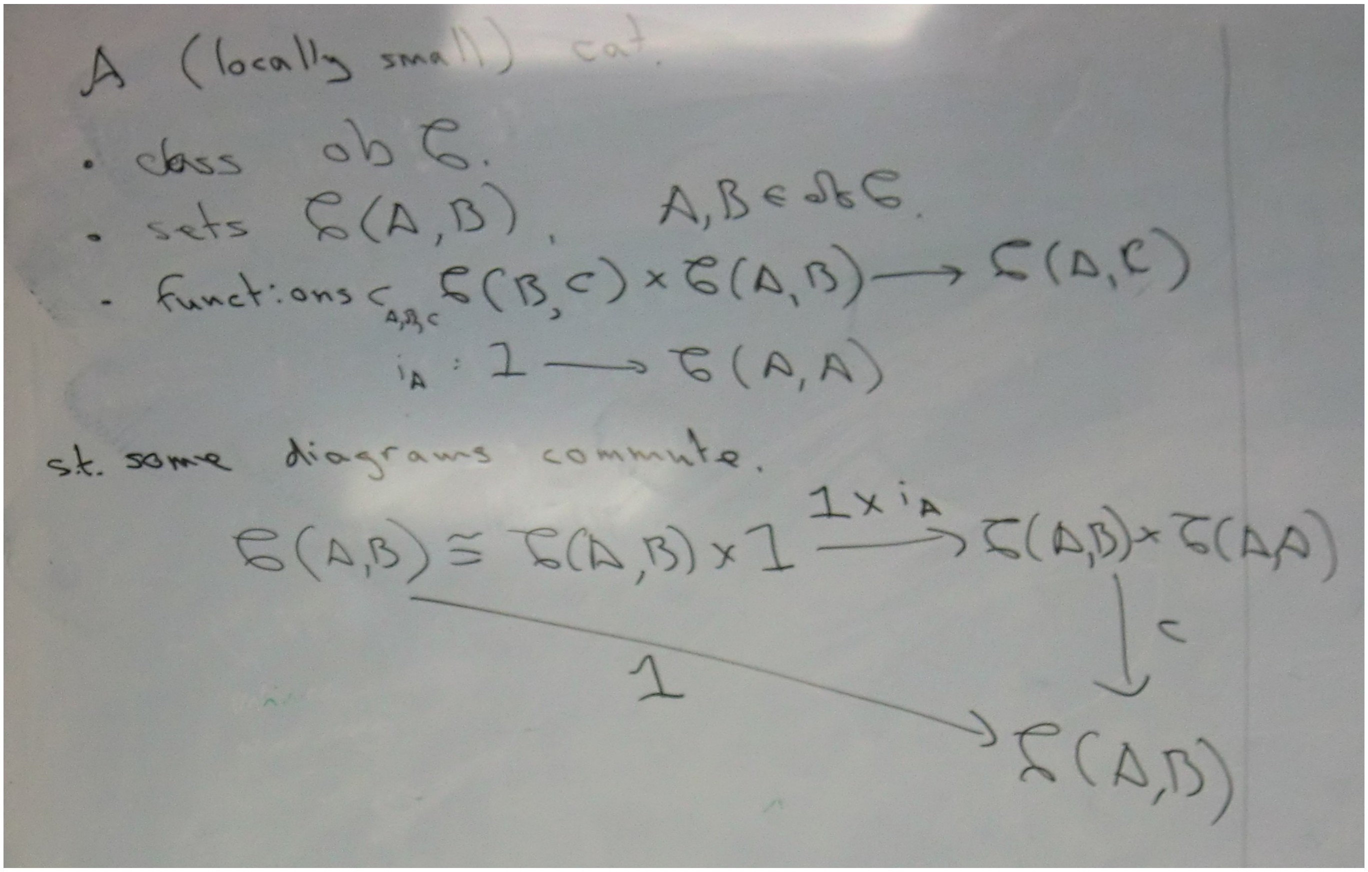
Monoidal categories.

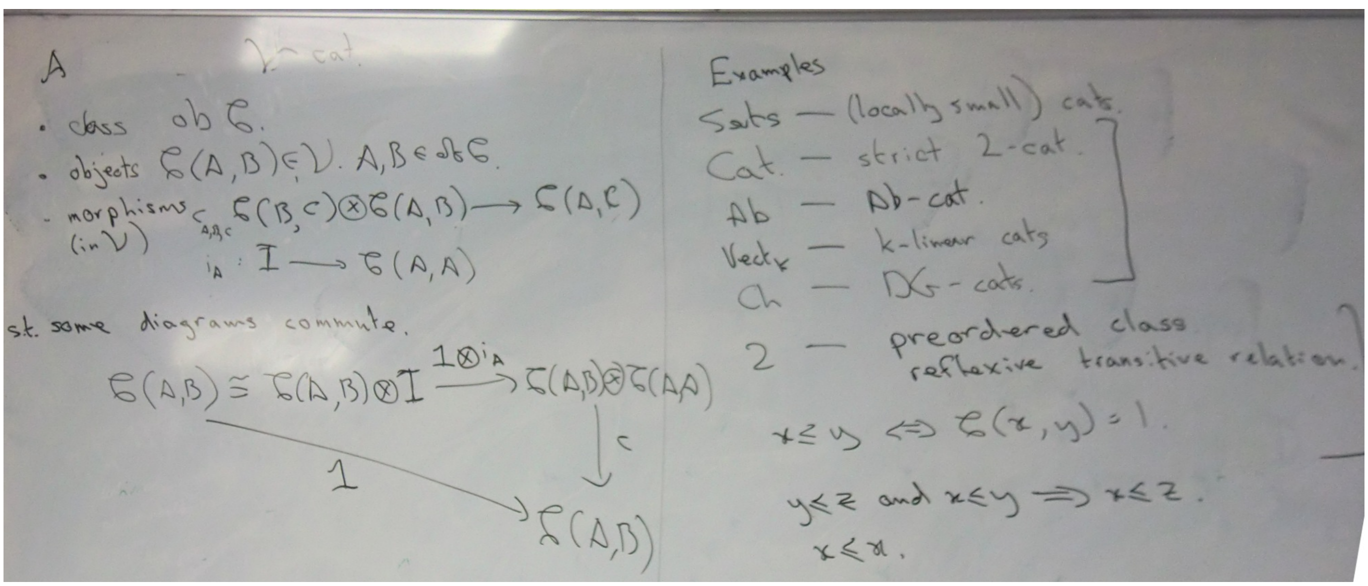
- A cat $\mathcal{V}$
- $\otimes : \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$.
- $I \in \mathcal{V}$
- isomorphisms. $\alpha : (A \otimes B) \otimes C \xrightarrow{\sim} A \otimes (B \otimes C)$
  $\lambda : I \otimes A \xrightarrow{\sim} A$
  $\rho : A \otimes I \xrightarrow{\sim} A$.
- coherence.

Examples

- Set. $\otimes = \times$, $I = 1 = \{*\}$.
- Cat. $\otimes = \times$, $I = 1 = \bullet$
- Any cat with $\times$ and 1.
- Ab abelian gps. $\otimes$, $I = \mathbb{Z}$
- Vect_k $\otimes$ $I = k$.
- Ch_k chain complexes $\otimes$. $I_n = \begin{cases} k & n = 0 \\ 0 & \text{o/w} \end{cases}$
- $2 = (0 \longrightarrow 1)$ $\otimes = \min$. $I = 1$
- $[0, \infty]$ objects real numbers ≥ 0 or ∞.

$$\big|[0,\infty](x,y)\big| = \begin{cases} 1 & x \geq y \\ \emptyset & \text{o/w}. \end{cases}$$


$\mathcal{A}$ (locally small) cat.
• class ob $\mathcal{C}$.
• sets $\mathcal{C}(A,B)$, $A,B \in \text{ob}\,\mathcal{C}$.
• functions $c_{A,B,C}$ $\mathcal{C}(B,C) \times \mathcal{C}(A,B) \longrightarrow \mathcal{C}(A,C)$
$i_A : 1 \longrightarrow \mathcal{C}(A,A)$
s.t. some diagrams commute.
$\mathcal{C}(A,B) \cong \mathcal{C}(A,B) \times 1 \xrightarrow{1 \times i_A} \mathcal{C}(A,B) \times \mathcal{C}(A,A)$
c
1
$\mathcal{C}(A,B)$


$\mathcal{A}$ $\mathcal{V}$-cat.
• class ob $\mathcal{C}$.
• objects $\mathcal{C}(A,B) \in \mathcal{V}$. $A,B \in \text{ob}\,\mathcal{C}$.
- morphisms (in $\mathcal{V}$) $c_{A,B,C}$ $\mathcal{C}(B,C) \otimes \mathcal{C}(A,B) \longrightarrow \mathcal{C}(A,C)$
$i_A : I \longrightarrow \mathcal{C}(A,A)$
s.t. some diagrams commute.
$\mathcal{C}(A,B) \cong \mathcal{C}(A,B) \otimes I \xrightarrow{1 \otimes i_A} \mathcal{C}(A,B) \otimes \mathcal{C}(A,A)$
c
1
$\mathcal{C}(A,B)$
Examples
Sets — (locally small) cats.
Cat — strict 2-cat.
Ab — Ab-cat.
Vect_k — k-linear cats
Ch — DG-cats.
2 — preordered class
reflexive transitive relation.
$x \leq y \iff \mathcal{C}(x,y) = 1$.
$y \leq z$ and $x \leq y \implies x \leq z$.
$x \leq x$.

Examples

$\mathcal{C}(y,z) + \mathcal{C}(x,y) \geq \mathcal{C}(x,z)$

$0 \geq \mathcal{C}(x,x)$

Generalised metric space.

allow $\mathcal{C}(x,y) = \infty$

$\mathcal{C}(x,y) = 0$

$\mathcal{C}(x,y) \neq \mathcal{C}(y,x)$.

Ordinary.

$f: A \to B$. "a way of getting from A to B"

$\mathcal{C}(A,B)$ "the set of ways..."

Enriched

$\mathcal{C}(A,B)$ "some info about getting from A to B"

2 – "can you get from A to B"

$[0,\infty]$ "effort"

A $\mathcal{V}$-functor $F: \mathcal{C} \to \mathcal{D}$.

For each $A \in \mathrm{ob}\,\mathcal{C}$, $FA \in \mathcal{D}$.

morphisms.

$F_{A,B}: \mathcal{C}(A,B) \to \mathcal{D}(FA,FB)$

conditions.

Sets – Functors

Cats – Strict 2-Functors.

Ab, Vect_k, Ch — usual notions.

2 — $x \leq y \Rightarrow Fx \leq Fy$.

order preserving maps.

$[0,\infty]$. $\mathcal{C}(x,y) \geq \mathcal{D}(Fx,Fy)$

Lipshitz, maps.

$1 \to \mathcal{C}(A,B)$

A $\mathcal{V}$-nat trans $\alpha: F \to G: \mathcal{C} \to \mathcal{D}$

consists of

morphisms

$\alpha_A : I \to \mathcal{D}(FA,GA)$

in $\mathcal{V}$.

s.t.

$$\begin{tikzcd}
& I \otimes \mathcal{C}(A,B) \arrow[r, "\alpha_B \otimes F_{A,B}"] & \mathcal{D}(FB,GB) \otimes \mathcal{D}(FA,FB) \arrow[dr, "c"] & \\
\mathcal{C}(A,B) \arrow[ur, "\sim"] \arrow[dr, "\sim"'] & & & \mathcal{D}(FA,GB) \\
& \mathcal{C}(A,B) \otimes I \arrow[r, "G_{A,B} \otimes \alpha_A"'] & \mathcal{D}(GA,GB) \otimes \mathcal{D}(FA,GA) \arrow[ur, "c"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
FA \arrow[r, "\alpha"] \arrow[d, "Ff"'] & GA \arrow[d, "Gf"] \\
FB \arrow[r, "\alpha"'] & GB
\end{tikzcd}$$

Set, Cat, Ab, Vect, Ch — ordinary nat trans between underlying.

2 – $F, G: \mathcal{C} \to \mathcal{D}$

$F \to G$ iff. $Fx \leq Gx$.

$1 \to \mathcal{D}(Fx,Gx)$

$[0,\infty]$ $F \to G$. iff. $\mathcal{D}(Fx,Gx) \leq 0$.

Ordinary.

$f: A \to B$. "a way of getting from A to B"

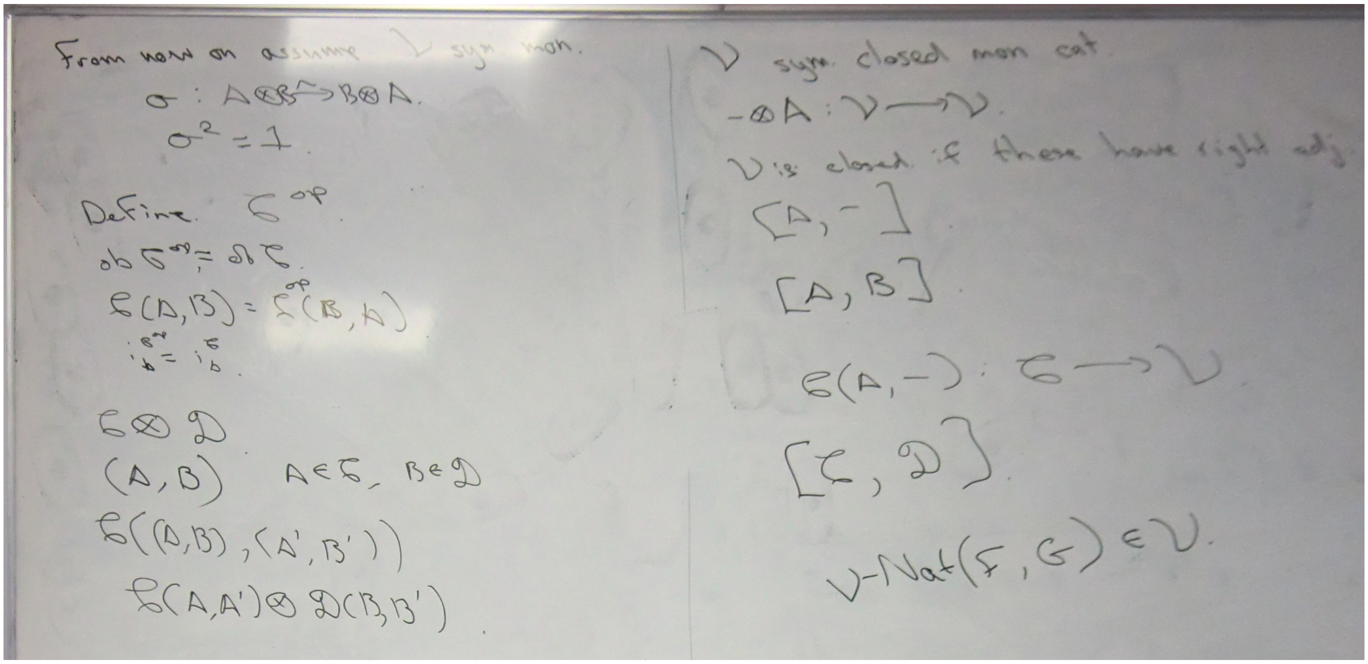
$\mathcal{C}(A,B)$ "the set of ways..."

Enriched

$\mathcal{C}(A,B)$ "some info about getting from A to B"

2 – "can you get from A to B"

"effort"


From now on assume V sym mon.
σ : A⊗B ⥲ B⊗A.
σ² = 1.
Define. C^op.
ob C^op = ob C.
C(A,B) = C^op(B,A)
j_A^{C^op} = j_A^C.
C ⊗ D
(A,B) A ∈ C, B ∈ D
C((A,B), (A′,B′))
C(A,A′) ⊗ D(B,B′)
V sym. closed mon cat.
− ⊗ A : V ⟶ V.
V is closed if these have right adj.
[A, −]
[A, B].
C(A, −) : C ⟶ V.
[C, D].
V-Nat(F, G) ∈ V.