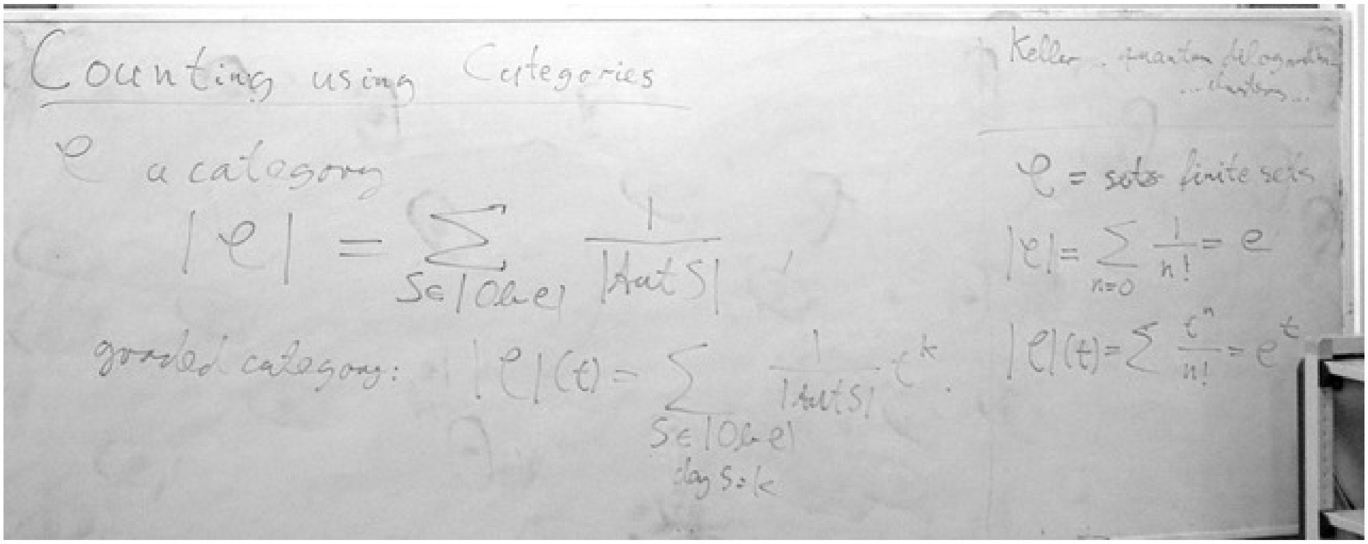
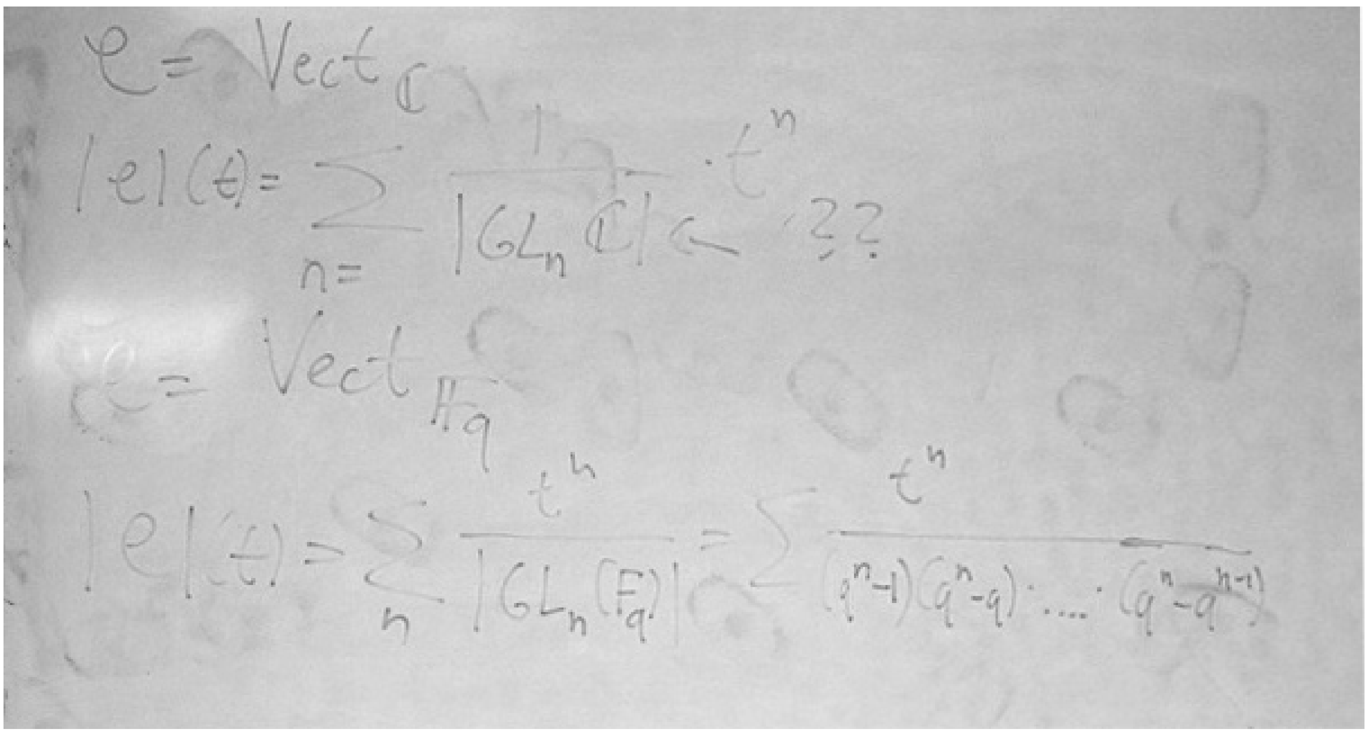
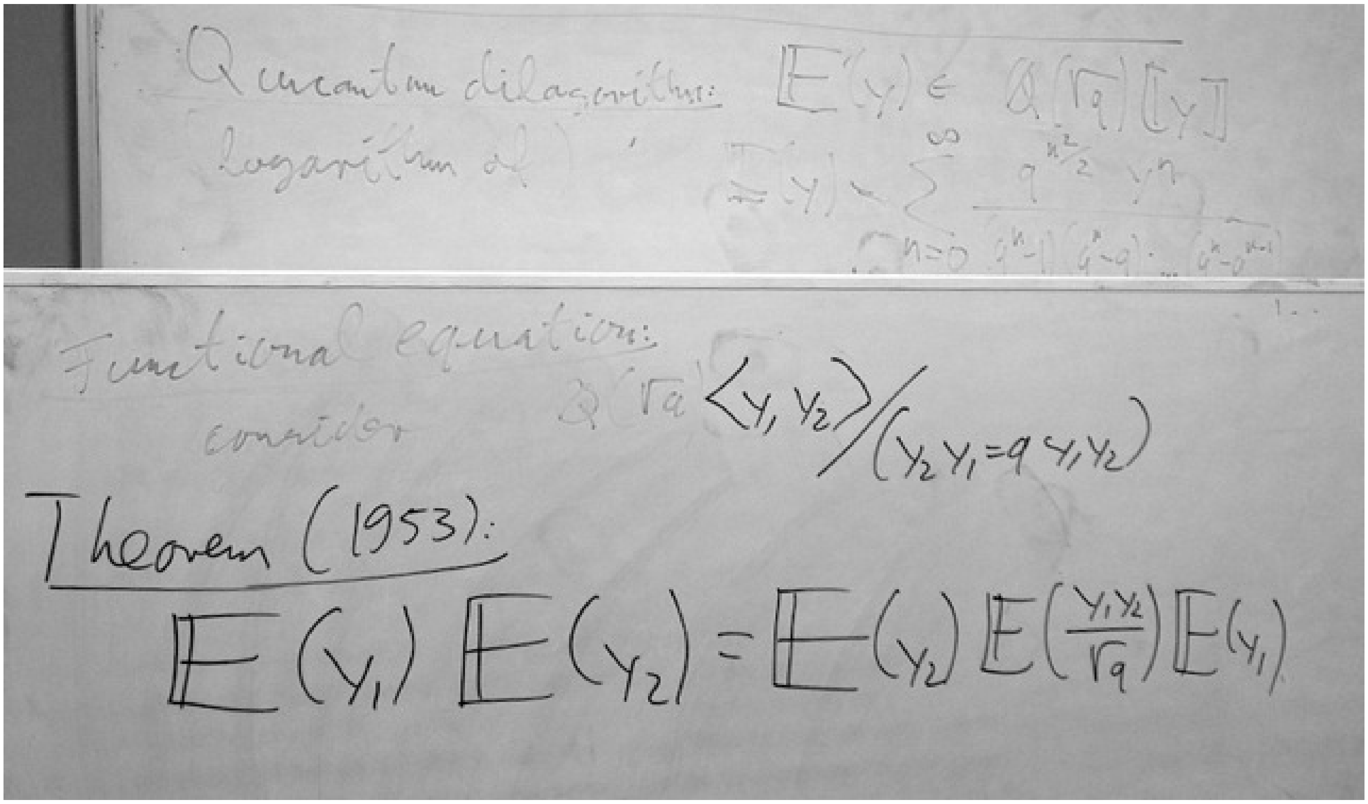
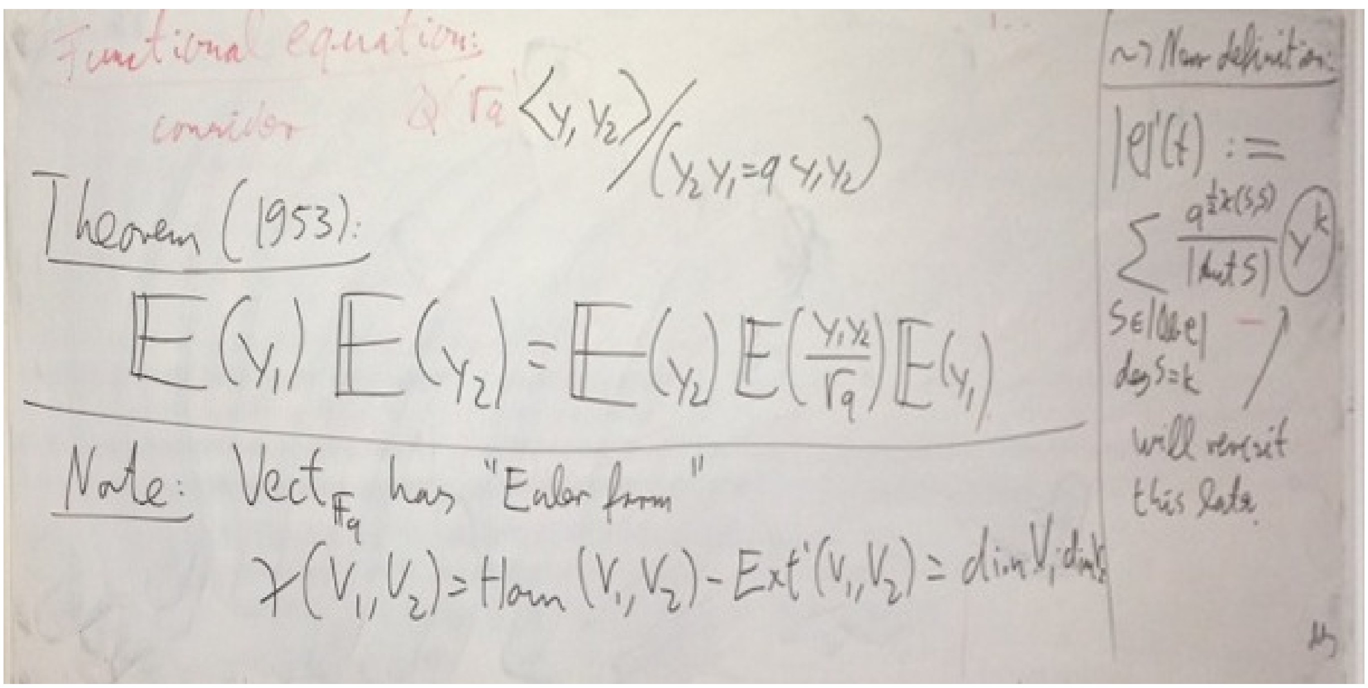

Counting using Categories
Keller ... quantum dilogarithm ... clusters ...
$\mathcal{C}$ a category
$|\mathcal{C}| = \sum_{S\in|\mathrm{Ob}\,\mathcal{C}|} \frac{1}{|\mathrm{Aut}\,S|}$
graded category: $|\mathcal{C}|(t) = \sum_{S\in|\mathrm{Ob}\,\mathcal{C}|,\ \deg S = k} \frac{1}{|\mathrm{Aut}\,S|} t^k$.
$\mathcal{C}$ = ~~sets~~ finite sets
$|\mathcal{C}| = \sum_{n=0} \frac{1}{n!} = e$
$|\mathcal{C}|(t) = \sum \frac{t^n}{n!} = e^t$


$\mathcal{C} = \mathrm{Vect}_{\mathbb{C}}$
$|\mathcal{C}|(t) = \sum_{n=} \frac{1}{|GL_n \mathbb{C}|} \cdot t^n$??
$\mathcal{C} = \mathrm{Vect}_{\mathbb{F}_q}$
$|\mathcal{C}|(t) = \sum_n \frac{t^n}{|GL_n(\mathbb{F}_q)|} = \sum \frac{t^n}{(q^n-1)(q^n-q)\cdot\ldots\cdot(q^n-q^{n-1})}$


Quantum dilogarithm: $\mathbb{E}(y) \in \mathbb{Q}(\sqrt{q})[[y]]$
(logarithm of)
$\mathbb{E}(y) = \sum_{n=0}^{\infty} \frac{q^{n^2/2}\, y^n}{(q^n-1)(q^n-q)\cdots(q^n-q^{n-1})}$
Functional equation:
consider $\mathbb{Q}(\sqrt{q})\langle y_1, y_2\rangle / (y_2 y_1 = q\, y_1 y_2)$
Theorem (1953):
$\mathbb{E}(y_1)\,\mathbb{E}(y_2) = \mathbb{E}(y_2)\,\mathbb{E}\left(\frac{y_1 y_2}{\sqrt{q}}\right)\mathbb{E}(y_1)$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Note: $\mathrm{Vect}_{\mathbb{F}_q}$ has "Euler form"
$\chi(V_1, V_2) = \mathrm{Hom}(V_1, V_2) - \mathrm{Ext}^1(V_1, V_2) = \dim V_1 \dim V_2$
⇝ New definition:
$|e|'(t) := \sum_{S \in |\mathrm{Coh}\, e|,\ \deg S = k} \frac{q^{\frac{1}{2}\chi(S,S)}}{|\mathrm{Aut}\, S|}\, y^k$
will revisit this later.

$$\mathcal{A} = \mathrm{Mor}_{\mathrm{Vect}_{\mathbb{F}_q}} = \mathrm{Rep}_{\mathbb{F}_q}(\bullet \longrightarrow \bullet)$$

$V_1 \longrightarrow V_2 = \underline{V}$

$0 \longrightarrow k = S_2$

$k \longrightarrow 0 = S_1$

$S_2 \hookrightarrow (k \xrightarrow{\sim} k) \twoheadrightarrow S_1$

$$\chi(\underline{V}^a, \underline{V}^b) = v_1^a v_1^b + v_2^a v_2^b - v_1^a v_2^b$$

$$|\mathcal{A}|'(y_1, y_2) = \,?$$

Before computation, have to make sense of grading to get power series $A_Q = \mathbb{Q}(\sqrt{q})\langle y_1, y_2 \rangle / (y_2 y_1 = q y_1 y_2)$

spanned by $\underline{y}^{(\alpha_1, \alpha_2)}$, $\alpha_1, \alpha_2 \geq 0$

with relations $\underline{y}^{\underline{\alpha}} \cdot \underline{y}^{\underline{\beta}} = q^{\frac{1}{2}(\underline{\alpha}, \underline{\beta})} \underline{y}^{\underline{\alpha} + \underline{\beta}}$

$$\boxed{\begin{array}{l} y_1 y_2 = \frac{1}{\sqrt{q}} \underline{y}^{(1,1)} \\ y_2 y_1 = \sqrt{q}\, \underline{y}^{(1,1)} \end{array}}$$

$\Uparrow$

with $(\underline{\alpha}, \underline{\beta}) = \chi(\underline{\alpha}, \underline{\beta}) - \chi(\underline{\beta}, \underline{\alpha})$

$y_1 = \underline{y}^{(1,0)}$, $y_2 = \underline{y}^{(0,1)}$

<u>Def.:</u> $$|\mathcal{A}|'(y_1, y_2) = \sum_S \frac{q^{\frac{1}{2}\chi(S,S)}}{|\mathrm{Aut}\, S|} \cdot \underline{y}^{\underline{\dim}\, S}$$

<u>Computation 1:</u>

$$|\mathcal{A}|'(y_1, y_2) = \sum_{\underline{\alpha} = (\alpha_1, \alpha_2)} \frac{q^{\alpha_1 \cdot \alpha_2} \cdot q^{\frac{1}{2}(\alpha_1^2 + \alpha_2^2 - \alpha_1 \alpha_2)}}{(q^{\alpha_1} - 1)(\dots)(q^{\alpha_2} - 1)\dots(q^{\alpha_2} - q^{\alpha_2 - 1})} \cdot \underline{y}^{\underline{\alpha}}$$

$$= \mathbb{E}(y_2) \cdot \mathbb{E}(y_1)$$

$\mathcal{A}$ has 3 indecomposable objects:

$S_1, S_2, \quad M = (\mathbb{F}_q \xrightarrow{\sim} \mathbb{F}_q)$

$V_1 \longrightarrow V_2 = \underline{V}$

$0 \longrightarrow \mathbb{F}_q = S_1$

$\mathbb{F}_q \quad n$

Any object in $\mathcal{A}$ is of the form

$S_1^{\oplus a_1} \oplus S_2^{\oplus a_2} \oplus M^{\ell} \leftarrow (a_1+\ell, a_2+\ell)$

$$|\mathcal{A}|'(y_1,y_2) = \sum_{a_1,a_2,\ell} \frac{q^{\frac12\left((a_1+\ell)^2+(a_2+\ell)^2-(a_1+\ell)(a_2+\ell)\right)}}{|GL_{a_1}(\mathbb{F}_q)|\,|GL_{a_2}(\mathbb{F}_q)|\,|GL_{\ell}(\mathbb{F}_q)|\, q^{a_1\ell} q^{a_2\ell}} \underline{y}^{(a_1+\ell,a_2+\ell)}$$

$$= \sum \frac{q^{\frac12\left(a_1^2+a_2^2+\ell^2-a_1\ell-a_2\ell-a_1a_2\right)}}{|\;||\;||\;|} \underline{y}^{(a_1+\ell,a_2+\ell)}$$

$$= \mathbb{E}(y_1)\,\mathbb{E}\left(\frac{y_2y_1}{\sqrt{q}}\right)\mathbb{E}(y_2)$$

<u>More general background:</u>

$\mathcal{A} \to$ Hall-algebra: $\mathbb{Q}^{|\mathrm{Ob}\text{-}\mathrm{el}|}$

<u>Product:</u> $\left([M]*[N]\right)(A) = \#\,|0 \hookrightarrow M \hookrightarrow A \twoheadrightarrow N \to 0|$

$\rightsquigarrow$ Hall algebra $H(\mathcal{A})$.

$\mathcal{A} = Q$-Rep, Q quiver, <u>no relations</u>, then

<u>Reineke:</u> algebra homom. $H(\mathcal{A}) \longrightarrow \mathbb{A}_Q$, $[M] \mapsto \dfrac{q^{\frac12\chi(M,M)}}{|\mathrm{Aut}\, M|}\underline{y}^{\underline{\dim M}}$

Stability + HN filtrations give identities in $H(\mathcal{A})$:

In previous example:

$$\mathbb{1}_{\mathcal{A}} = \sum [S_2^{\oplus k}] * \sum [S_1^{\oplus k}]$$

$\boxed{\text{Reineke}}$

$$= \sum_k [S_1^{\oplus k}] * \sum_k [M^k] * \sum_k [S_2^{\oplus k}]$$

$$S_2^{\oplus a_2} \hookrightarrow A \to S_1^{\oplus a_1}$$

$$\begin{tikzcd} S_1^{\oplus a_1} \arrow[r, hook] & A \arrow[r, two heads] & Z \arrow[r, two heads] & S_2^{\oplus a_2} \\ & & M^{\oplus l} \arrow[u, hook] & \end{tikzcd}$$