Monads & Barr-Beck:

$L : \mathcal{C} \rightleftarrows \mathcal{D} : R$

(L, R) form an adjoint pair.

(*) $\mathrm{Hom}_{\mathcal{D}}(Lc, d) \cong \mathrm{Hom}_{\mathcal{C}}(c, Rd)$.

e.g. L is an equivalence, $R \overset{?}{=} L^{-1}$!

Equiv to (*): $\mathrm{id}_{\mathcal{C}} \xrightarrow{\text{nat. trans.}} RL \in \mathrm{End}(\mathcal{C})$

$LR \longrightarrow \mathrm{id}_{\mathcal{D}} \in \mathrm{End}(\mathcal{D})$.

+ axioms...

$T = RL \in \mathrm{End}(\mathcal{C})$. ← monoidal category.

$\mathrm{id}_{\mathcal{C}} \to T$

$TT = R\underline{LR}L \to \underline{RL}$

$TT \to T$

$T \otimes T \to T$

T associative.

Def^{n} A monad T on $\mathcal{C}$ is an algebra object in $\mathrm{End}(\mathcal{C})$.

A module for T is an object $c \in \mathcal{C}$ ← T-algebra in $\mathcal{C}$

$T(c) \to$ ...

Given a monad T, $\mathcal{C}^T$ = T-modules in $\mathcal{C}$.

Fact: Every monad arises from an adjoint pair of functors $\underline{\mathcal{C}} \underset{R}{\overset{L}{\rightleftarrows}} \underline{\mathcal{D}} = \mathcal{C}^T$.

$T = RL$.

There are many choices of $\mathcal{D}$, L, R giving rise to a given monad.

$\mathcal{C}_T \longrightarrow \mathcal{D} \longrightarrow \mathcal{C}^T$

Kliesli category. Eilenberg-Moore cat.

Def^{n} A pair $\underline{\underline{(L, R)}}$ is monadic if the canonical map $\mathcal{D} \to \mathcal{C}^T$ is an equivalence.

$\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D} \to \mathcal{C}^T$

$R\underline{L}\,\underline{R(d)} \to R(d)$

1) Free : Sets $\rightleftarrows$ Monoids : Forget.

$\mathrm{id}_{\mathrm{Sets}}(X) \longrightarrow \mathrm{Forget}(\mathrm{Free}(X))$

$x \longmapsto$ one letter word

$\mathrm{Free}(\mathrm{For}(M)) \longrightarrow M$

$m_1 * \cdots * m_k \longmapsto m_1 \cdots m_k$.

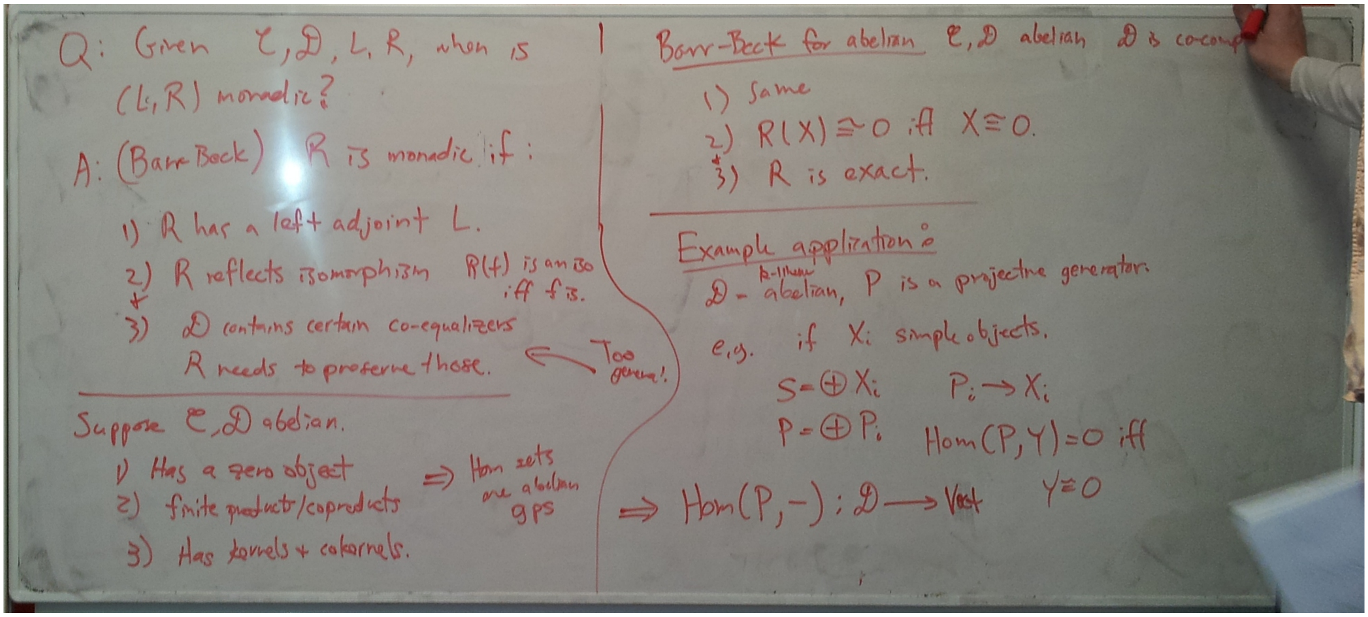
(Free, For) is monadic.

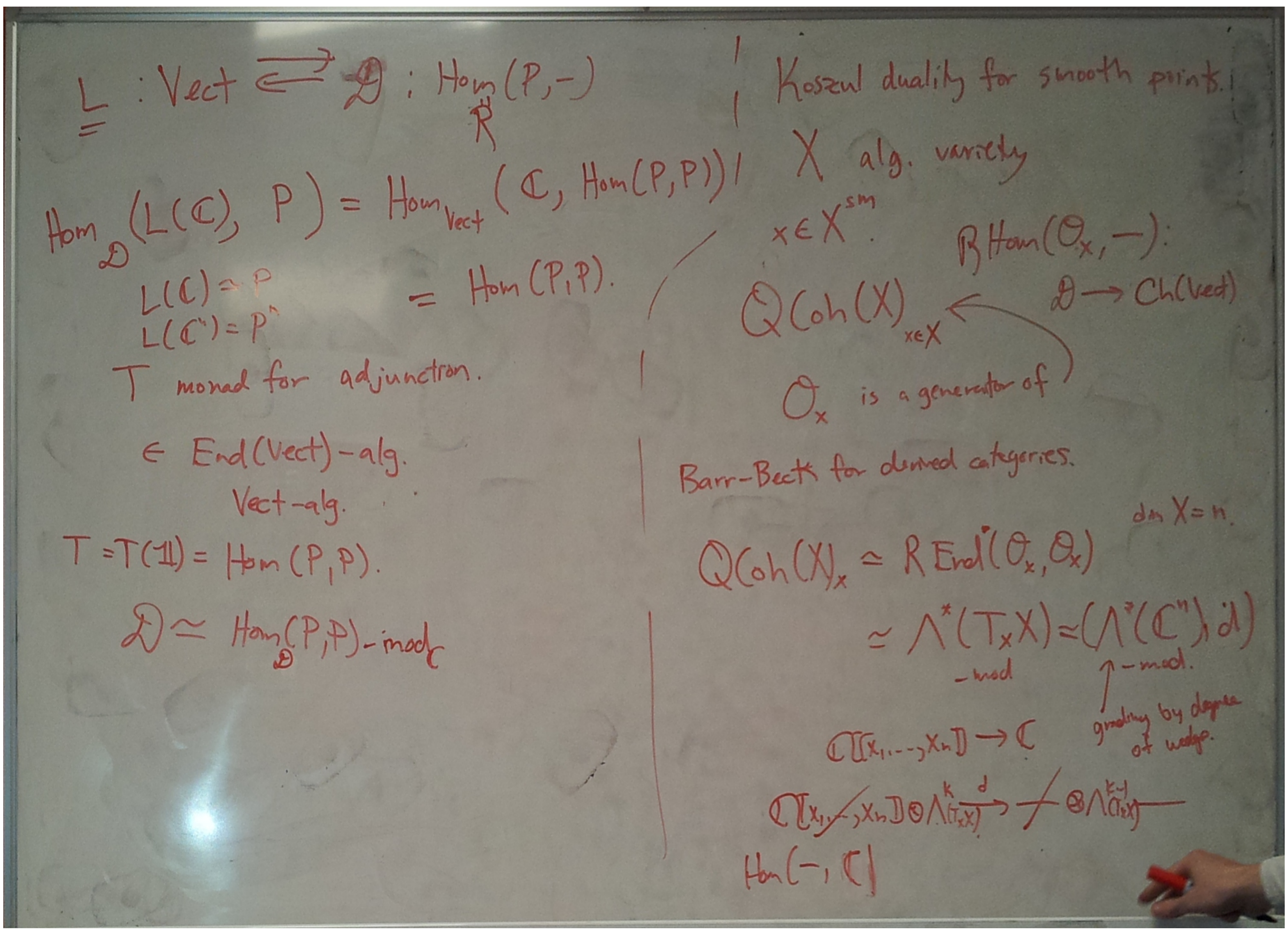
i.e. to give a monoid structure to a set X.

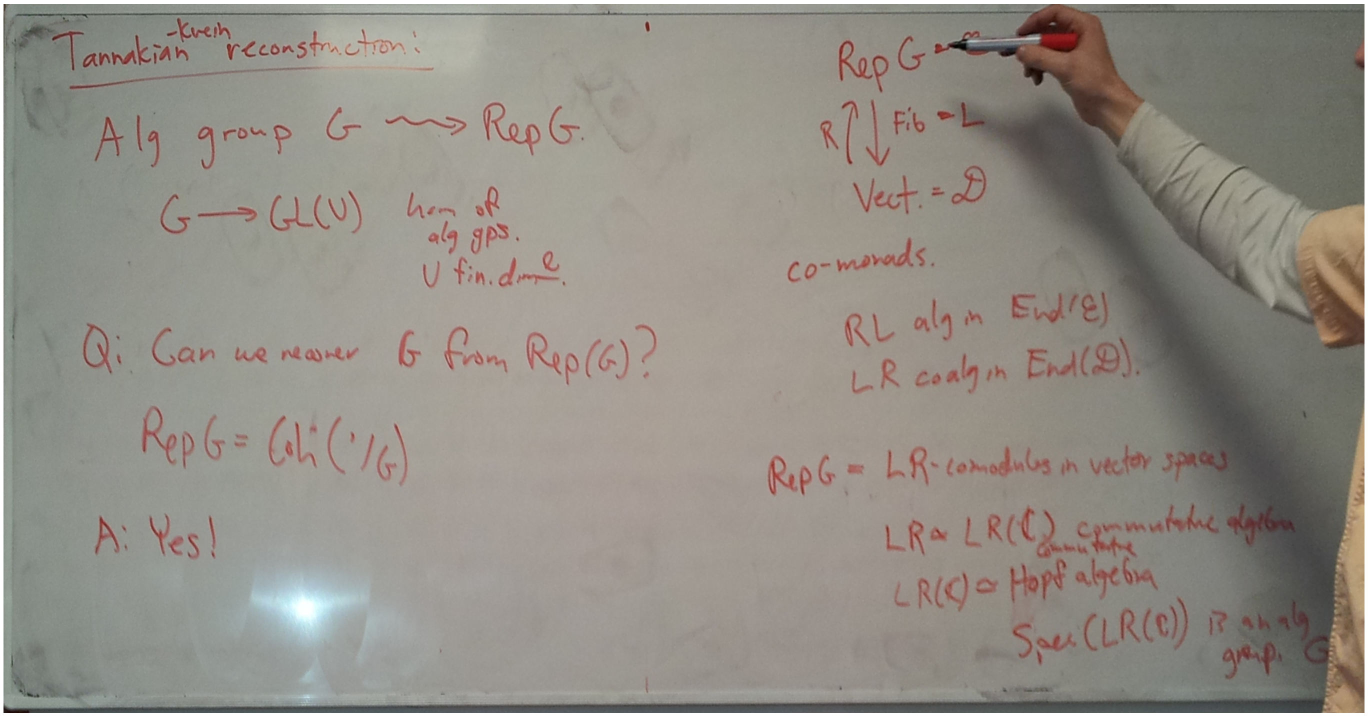
$\mathrm{Free}(X) \longrightarrow X$.

2) Free : Sets $\rightleftarrows$ Grps : For...

monadic proof...


Q: Given $\mathcal{C}, \mathcal{D}, L, R$, when is (L,R) monadic?
A: (Barr Beck) R is monadic if:
1) R has a left adjoint L.
2) R reflects isomorphism
R(f) is an iso iff f is.
3) $\mathcal{D}$ contains certain co-equalizers R needs to preserve those.
Too general!
Suppose $\mathcal{C}, \mathcal{D}$ abelian.
1) Has a zero object
2) finite products/coproducts
⇒ Hom sets are abelian gps
3) Has kernels + cokernels.
Barr-Beck for abelian $\mathcal{C}, \mathcal{D}$ abelian $\mathcal{D}$ is cocomp
1) Same
2) $R(X) \cong 0$ iff $X \cong 0$.
3) R is exact.
Example application:
$\mathcal{D}$ - k-linear abelian, P is a projective generator.
e.g. if X_i simple objects.
$S = \oplus X_i$ $P_i \to X_i$
$P = \oplus P_i$ $Hom(P,Y)=0$ iff $Y \cong 0$
$\Rightarrow Hom(P,-) : \mathcal{D} \to Vect$


$L : Vect \rightleftarrows \mathcal{D} : Hom(P,-)$
R
$Hom_{\mathcal{D}}(L(\mathbb{C}), P) = Hom_{Vect}(\mathbb{C}, Hom(P,P))$
$L(\mathbb{C}) = P$
$L(\mathbb{C}^n) = P^n$
$= Hom(P,P)$.
T monad for adjunction.
$\in End(Vect)$-alg.
Vect-alg.
$T = T(\mathbb{1}) = Hom(P,P)$.
$\mathcal{D} \simeq Hom_{\mathcal{D}}(P,P)$-mod
Koszul duality for smooth points.
X alg. variety
$x \in X^{sm}$.
$R Hom(\mathcal{O}_x, -)$:
$QCoh(X)_{x \in X}$
$\mathcal{D} \to Ch(Vect)$
$\mathcal{O}_x$ is a generator of
Barr-Beck for derived categories.
dim X = n.
$QCoh(X)_x \simeq R End^{\bullet}(\mathcal{O}_x, \mathcal{O}_x)$
$\simeq \Lambda^*(T_x X) = (\Lambda^*(\mathbb{C}^n), d)$
-mod
-mod.
grading by degree of wedge.
$\mathbb{C}[[x_1, \dots, x_n]] \to \mathbb{C}$
$\mathbb{C}[[x_1, \dots, x_n]] \otimes \Lambda^k(T_x X) \xrightarrow{d} \otimes \Lambda^{k-1}(T_x X)$
$Hom(-, \mathbb{C})$


Tannakian-Krein reconstruction:
Alg group G ⇝ Rep G.
G ⟶ GL(V) hom of alg gps.
V fin. dim'l.
Q: Can we recover G from Rep(G)?
Rep G = Coh(·/G)
A: Yes!
Rep G
R Fib = L
Vect. = D
co-monads.
RL alg in End(E)
LR coalg in End(D).
Rep G = LR-comodules in vector spaces
LR ≃ LR(C) commutative algebra
LR(C) ≃ Hopf algebra
Spec(LR(C)) is an alg group. G